CMPS 211
Fall 2010-2011
Homework 1

Section 1.1

10- *p*: You get an A on the final exam.

q: You do every exercise in this book.

r: You get an A in this class.

a) You get an A in this class, but you do not do every exercise in this book.
$r \wedge \neg q$

b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

$p \wedge q \wedge r$

c) To get an A in this class, it is necessary for you to get an A on the final.

$r \rightarrow p$

d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

$p \wedge \neg q \wedge r$

e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

$(p \wedge q) \rightarrow r$

f) You will get an A in this class if an only if you either do every exercise in this book or you get an A on the final.

$r \leftrightarrow (q \vee p)$

28- a) $p \to \neg p$

p	$\neg p$	$p \to \neg p$
0	1	1
1	0	0

b) $p \leftrightarrow \neg p$

p	$\neg p$	$p \leftrightarrow \neg p$
0	1	0
1	0	0

c) $p \oplus (p \vee q)$

p	q	$p \vee q$	$p \oplus (p \vee q)$
0	0	0	0
0	1	1	1
1	0	1	0
1	1	1	0

d) $(p \wedge q) \to (p \vee q)$ ***Tautology***

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

e) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$q \to \neg p$	$p \leftrightarrow q$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	1	0	0
1	1	0	0	1	0

f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$ ***Tautology***

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
0	0	1	1	0	1
0	1	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	1

32- a) $(p \vee q) \vee r$

p	q	r	$p \vee q$	$(p \vee q) \vee r$
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

b) $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

c) $(p \wedge q) \vee r$

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1

1	1	0	1	0
1	1	1	1	1

d) $(p \wedge q) \wedge r$

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1	0
1	1	1	1	1

e) $(p \vee q) \wedge \neg r$

p	q	r	$\neg r$	$p \vee q$	$(p \vee q) \wedge \neg r$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	0

f) $(p \wedge q) \vee \neg r$

p	q	r	$\neg r$	$p \wedge q$	$(p \wedge q) \vee \neg r$
0	0	0	1	0	1
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	1	1

50- p: The system software is being upgraded.

q: Users can access the file system.

r: Users can save new files.

The given specifications:

1. Whenever the system software is being upgraded, users cannot access the file system. $p \rightarrow \neg q$

2. If users can access the file system, then they can save new files. $q \rightarrow r$

3. If users cannot save new files, then the system software is not being upgraded. $\neg r \rightarrow \neg p$

To check the consistency of the specifications, we must find an assignment of truth values for the propositions *p, q, r* which makes the specifications true.

Starting with specification 1, we assign p to be true, then $\neg q$ must be true, implying that q must be false. If q is false, then we cannot conclude anything in specification 2 about the value of r. But specification 3 involves both r and p, and we want p to be true in this case, so $\neg p$ must be false, and thus $\neg r$ cannot be true (otherwise $\neg p$ would be true). We assign $\neg r$ to false, and consequently r is true. Therefore, the assignments of *p, q, r* to true, false, true respectively satisfies the specifications, and so the system is consistent.

Section 1.2

10- a) $[\neg p \wedge (p \vee q)] \rightarrow q$

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

			A	B	C	D	
p	q	r	$p \rightarrow q$	$q \rightarrow r$	$A \wedge B$	$p \rightarrow r$	$C \rightarrow D$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

c) $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

			A	B	C	D	
p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$A \wedge B \wedge C$	$D \rightarrow r$
0	0	0	0	1	1	0	1
0	0	1	0	1	1	0	1
0	1	0	1	1	0	0	1
0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1

12- We show that if the hypothesis is true, then the conclusion is true too.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$

$\neg p \wedge (p \vee q) \equiv (\neg p \wedge p) \vee (\neg p \wedge q)$ by the distributive law

$\equiv 0 \vee (\neg p \wedge q)$

$\equiv \neg p \wedge q$

If $\neg p \wedge q$ is true, then by the definition of conjunction, q is true, so the conclusion is true. Therefore, this is a tautology.

b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

If $(p \rightarrow q) \wedge (q \rightarrow r)$ is true, then by the definition of conjunction,

$(p \rightarrow q)$ is true and $(q \rightarrow r)$ is true. If $p \rightarrow q$ and $q \rightarrow r$, then by transitivity,

$p \rightarrow r$ is also true. Therefore, this is tautology.

c) $[p \wedge (p \rightarrow q)] \rightarrow q$

If $p \wedge (p \rightarrow q)$ is true, then by the definition of conjunction, p is true and

$p \rightarrow q$ is true. Since p is true and the implication $p \rightarrow q$ is true, then q must be true, and the conclusion is true. Therefore, this is tautology.

d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

If $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$, then by the definition of conjunction, each of $(p \vee q)$ and $(p \rightarrow r)$ and $(q \rightarrow r)$ is true. Since $p \vee q$ is true, then either p or q or both is true. If p is true and $p \rightarrow r$ is true, then r is true. If q is true and $q \rightarrow r$ is true, then r is true. So, in all cases, the conclusion r is true. Therefore, this is a tautology.

22- $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

			A	**B**			
p	**q**	**r**	**$p \rightarrow q$**	**$p \rightarrow r$**	**$A \wedge B$**	**$q \wedge r$**	**$p \rightarrow (q \wedge r)$**
0	0	0	1	1	1	0	1
0	0	1	1	1	1	0	1
0	1	0	1	1	1	0	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	1	0	0	0
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

46- *p NAND q* is true when either *p* or *q* or both are false, and it is false when both *p* and *q* are true.

p	q	p NAND q
0	0	1
0	1	1
1	0	1
1	1	0

Section 1.3

10- *C(x)* is "*x* has a cat" *D(x)* is "*x* has a dog" *F(x)* is "*x* has a ferret"

domain: all students in your class

a) A student in your class has a cat, a dog, and a ferret.

$\exists x\ (C(x) \wedge D(x) \wedge F(x))$

b) All students in your class have a cat, a dog, or a ferret.

$\forall x\ (C(x) \vee D(x) \vee F(x))$

c) Some student in your class has a cat and a ferret, but not a dog.

$\exists x\ (C(x) \wedge F(x) \wedge \neg D(x))$

d) No student in your class has a cat, a dog, and a ferret.

$\forall x \neg\ (C(x) \wedge D(x) \wedge F(x))$

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals as a pet.

$\exists x\ (C(x)) \wedge \exists y\ (D(y)) \wedge \exists z\ (F(z))$

40- a) When there is less than 30 megabytes free on the hard disk a warning message is sent to all users.

$(\exists x\ M(30)) \rightarrow \forall x\ W(x)$

M(x) is "the hard disk has less than *x* megabytes free"

W(x) is "a warning message is sent to user *x*"

b) No directories in the file system can be opened and no files can be closed when system errors have been detected.

$(\exists x\ E(x)) \rightarrow (\forall x \neg D(x)) \wedge (\forall x \neg F(x))$

D(x) is "directory *x* in the file system can be opened"

F(x) is "file *x* in the file system can be closed"

E(x) is "system error *x* has been detected"

c) The file system cannot be backed up if there is a user currently logged on.

$(\exists x\ L(x)) \rightarrow \neg F$

L(x) is "user *x* is currently logged on"

F is "the file system can be backed up"

d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.

$M(8) \wedge C(56) \rightarrow V$

M(x) is "memory available is at least *x* megabytes"

C(x) is "connection speed is at least *x* kilobits per second"

V is "video on demand can be delivered"

Section 2.1

18- Set Cardinality:

a) | Ø | = 0

b) | { Ø } | = 1

c) | { Ø, { Ø } } | = 2

d) | { Ø, { Ø }, { Ø, { Ø } } } | = 3

22- Determine whether it's a power set of some set:

a) Ø — **Not** a power set, since any power set must at least contain the element Ø.

b) { Ø, {a} } **Yes**. This is P({a}).

c) { Ø, {a}, {Ø,a} } **Not** a power set, since the power set of the set {Ø, a} must also contain the element {Ø}, so the correct P({Ø,a}) = { Ø, {Ø}, {a}, {Ø,a} }.

d) { Ø, {a}, {b}, {a,b}} **Yes**. This is P({a,b}).

28- A = {a, b, c} B = {x, y} C = {0, 1}

a) A × B × C = {(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)}.

b) C × B × A = {(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)}.

c) C × A × B = {(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1 c, y)}.

d) B × B × B = {(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)}.

34- a) There exists a real number such that its cube is -1. True

b) There exists an integer when incremented results in an integer greater than the original integer. True

c) Any integer when decremented remains integer. True

d) Any integer when squared remains an integer. True

36- a) {1, 2, 3, …} or x Є Z^{+*}

b) Ø

c) {…, -3, -2, -1, 2, 3, …} or Z – {0, 1}

Section 2.2

18- A, B and C are sets.

a) Show: **$(A \cup B) \subseteq (A \cup B \cup C)$**

$$x \in A \cup B \Rightarrow x \in A \vee x \in B$$

$$\Rightarrow x \in A \vee x \in B \vee x \in C$$

$$\Rightarrow x \in A \cup B \cup C$$

$\therefore$ $(A \cup B) \subseteq (A \cup B \cup C)$

c) Show: **$(A - B) - C \subseteq A - C$**

$$x \in (A - B) - C \Rightarrow x \in (A - B) \wedge x \notin C$$

$$\Rightarrow (x \in A \wedge x \notin B) \wedge x \notin C$$

$$\Rightarrow x \in A \wedge x \notin C$$

$$\Rightarrow x \in A - C$$

$\therefore$ $(A - B) - C \subseteq A - C$

e) Show: **$(B - A) \cup (C - A) = (B \cup C) - A$**

$$(B - A) \cup (C - A) = \{x \mid (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A)\}$$

$$= \{x \mid (x \in B \vee x \in C) \wedge x \notin A\}$$

$$= \{x \mid x \in (B \cup C) \wedge x \notin A\}$$

$$= \{x \mid x \in (B \cup C) - A\}$$

$$= (B \cup C) - A$$

38- A and B are sets.

a) Show: **$A \oplus B = B \oplus A$**

$$A \oplus B = \{x \mid (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$$

$= \{x \mid (x \in B \vee x \in A) \wedge (x \notin B \cap A)\}$ (by commutative law)

$= \{x \mid x \in B \oplus A\}$

$= B \oplus A$

b) Show: **(A ⊕ B) ⊕ B= A**

This can be done by set implications or by Venn diagrams.

Using set implications:

$(A \oplus B) \oplus B = \{ x \mid (x \in (((A\oplus B) \cup B) - ((A\oplus B) \cap B)) \}$

$= \{ x \mid (x \in ((((A \cup B) - (A \cap B)) \cup B) - (((A \cup B) - (A \cap B)) \cap B)) \}$

$= \{ x \mid (x \in (A \cup B) \wedge x \notin (A \cap B) \vee x \in B) - (x \in (A \cup B) \wedge x \notin (A \cap B) \wedge x \in B) \}$

$= \{ x \mid (x \in A \vee x \in B) - (x \in B \wedge x \notin (A \cap B)) \}$

$= \{ x \mid (x \in A \cup B) - (x \in B - A) \}$

$= \{ x \mid (x \in A \cup B) \wedge (x \notin B - A) \}$

$= \{ x \mid (x \in A \cup B) \wedge (x \notin B) \}$

$= \{ x \mid x \in A \}$

$= A$

<u>54</u> - The bit in the i-th position of the bit string of the symmetric difference of two sets is 1 if the i-th bit of the first string is different from that of the second string in the same position.

Section 2.3

12 – a) $f(n) = n - 1$ one-to-one (4 pts)

$f(a)=f(b)$, a & $b \in Z$

$a - 1 = b - 1$ (add 1 to both sides)

$a = b$

b) $f(n) = n^2+1$ not one-to-one

$f(-1)=f(1) = 2$ but $-1 \neq 1$

c) $f(n)=n^3$ one-to-one

same argue as a)

d) $f(n) = \lceil n/2 \rceil$ not one-to-one

$f(2) = 1$

$f(1) = 1$

$f(2)=f(1)$ but $1 \neq 2$

30- *f* is one-to-one *f o g* is one-to-one

$f \circ g(a) = f \circ g(b) \Rightarrow a = b$

AND $f(a) = f(b) \Rightarrow a = b$

But $f \circ g(a) = f \circ g(b) \Rightarrow a = b$

*i*mplies that $f(g(a)) = f(g(b)) \Rightarrow a = b$

meaning that $g(a) = g(b) \Rightarrow a = b$ (because *f* is one-to-one)

Thus, *g* is one-to-one since $g(a) = g(b) \Rightarrow a = b$.

34- $f(x) = ax + b$ $g(x) = cx + d$

$f \circ g = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b$

$g \circ f = g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d$

$f \circ g = g \circ f$

$\Rightarrow \quad acx + ad + b = acx + bc + d$

$\Rightarrow \quad ad + b = bc + d$ (subtract acx)

The constants must fulfill the relation $ad + b = bc + d$.

38- $f(x) = x^2$

a) $f^{-1}(\{1\}) = \{-1, 1\}$

b) $f^{-1}(\{x \mid 0 < x < 1\}) = \{x \mid -1 < x < 0\} \cup \{x \mid 0 < x < 1\}$

c) $f^{-1}(\{x \mid x > 4\}) = \{x \mid x < -2\} \cup \{x \mid x > 2\}$

66- $f : Y \to Z$ is an invertible function

$g : X \to Y$ is an invertible function

$f \circ g(x) = f(g(x)) = f(y) = z$ where $x \in X$, $y \in Y$, $z \in Z$

$\Rightarrow \quad (f \circ g)^{-1}(z) = x$

$g^{-1} \circ f^{-1}(z) = g^{-1}(f^{-1}(z)) = g^{-1}(y) = x$

$\therefore \quad (f \circ g)^{-1} = g^{-1} \circ f^{-1}.$

72- Show $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$

x is a real number, so $x = n + \in$, where n is the largest integer in x and $0<\in<1$

Take cases according to the value of $\in$:

Case 1: $\in < \frac{1}{3}$

Then, $\lfloor x \rfloor = n$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor n + \in + \frac{1}{3} \rfloor = n$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor n + \in + \frac{2}{3} \rfloor = n$

$\lfloor 3x \rfloor = \lfloor 3\,(n + \in) \rfloor = 3n$

Thus, $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = 3n$

Case 2: $\frac{1}{3} \le \in < \frac{2}{3}$

Then, $\lfloor x \rfloor = n$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor n + \in + \frac{1}{3} \rfloor = n$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor n + \in + \frac{2}{3} \rfloor = n + 1$

$\lfloor 3x \rfloor = \lfloor 3\,(n + \in) \rfloor = 3n + 1$

Thus, $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = 3n + 1$

Case 3: $\frac{2}{3} \le \in < 1$

Then, $\lfloor x \rfloor = n$

$$\lfloor x + \frac{1}{3} \rfloor = \lfloor n + \in + \frac{1}{3} \rfloor = n + 1$$

$$\lfloor x + \frac{2}{3} \rfloor = \lfloor n + \in + \frac{2}{3} \rfloor = n + 1$$

$$\lfloor 3x \rfloor = \lfloor 3 (n + \in) \rfloor = 3n + 2$$

Thus, $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = 3n + 2$