17.1 Use the defining integral to find the Fourier transform of the following functions:

a)
$$f(t) = A \sin \frac{\pi}{2} t, \qquad -2 \leq t \leq 2;$$
$$f(t) = 0, \qquad \text{elsewhere.}$$

b)
$$f(t) = \frac{2A}{\tau} t + A, \qquad -\frac{\tau}{2} \leq t \leq 0;$$
$$f(t) = -\frac{2A}{\tau} t + A, \quad 0 \leq t \leq \frac{\tau}{2};$$
$$f(t) = 0, \qquad \text{elsewhere.}$$

P 17.1 **[a]** $F(\omega) = \int_{-2}^{2} \left[A \sin\left(\frac{\pi}{2}\right) t\right] e^{-j\omega t}\, dt = \frac{-j4\pi A}{\pi^2 - 4\omega^2} \sin 2\omega$

[b] $F(\omega) = \int_{-\tau/2}^{0} \left(\frac{2A}{\tau}t + A\right) e^{-j\omega t}\, dt + \int_{0}^{\tau/2} \left(\frac{-2A}{\tau}t + A\right) e^{-j\omega t}\, dt$

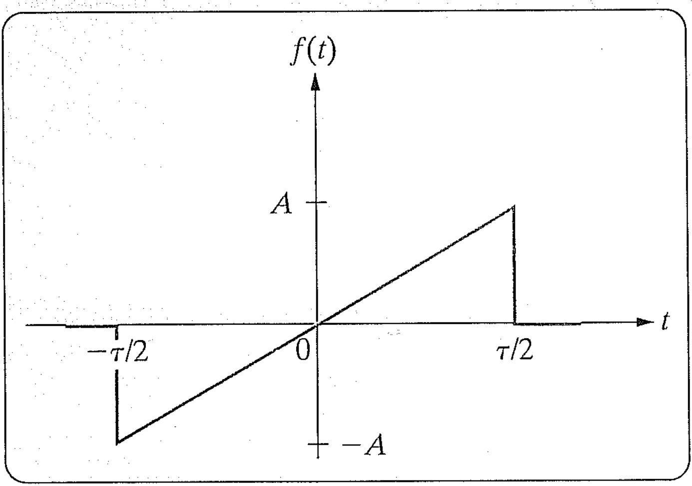
$$= \frac{4A}{\omega^2\tau}\left[1 - \cos\left(\frac{\omega\tau}{2}\right)\right]$$

17.2 a) Find the Fourier transform of the function shown in Fig. P17.2.

b) Find $F(\omega)$ when $\omega = 0$.

c) Sketch $|F(\omega)|$ versus ω when $A = 1$ and $\tau = 1$. *Hint:* Evaluate $|F(\omega)|$ at $\omega = 0, 2, 4, 6, 8, 9, 10, 12, 14,$ and 15. Then use the fact that $|F(\omega)|$ is an even function of ω.

Figure P17.2



P 17.2 **[a]** $F(\omega) = \int_{-\tau/2}^{\tau/2} \frac{2A}{\tau} te^{-j\omega t}\,dt$

$$= \frac{2A}{\tau}\left[\frac{e^{-j\omega t}}{-\omega^2}(-j\omega t - 1)\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{\omega^2\tau}\left[e^{-j\omega\tau/2}\left(\frac{j\omega\tau}{2}+1\right) - e^{j\omega\tau/2}\left(\frac{-j\omega\tau}{2}+1\right)\right]$$

$$F(\omega) = \frac{2A}{\omega^2\tau}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2} + j\frac{\omega\tau}{2}\left(e^{-j\omega\tau/2}+e^{j\omega\tau/2}\right)\right]$$

$$F(\omega) = j\frac{2A}{\tau}\left[\frac{\omega\tau\cos(\omega\tau/2) - 2\sin(\omega\tau/2)}{\omega^2}\right]$$

[b] Using L'Hopital's rule,

$$F(0) = \lim_{\omega\to 0} j2A\left[\frac{\omega\tau(\tau/2)(-\sin\omega\tau/2) + \tau\cos\omega(\tau/2) - 2(\tau/2)\cos(\omega\tau/2)}{2\omega\tau}\right]$$

$$= \lim_{\omega\to 0} j2A\left[\frac{-\omega\tau(\tau/2)\sin(\omega\tau/2)}{2\omega\tau}\right]$$

$$= \lim_{\omega\to 0} j2A\left[\frac{-\tau\sin(\omega\tau/2)}{4}\right] = 0$$

$$\therefore \qquad F(0) = 0$$

[c] When $A = 1$ and $\tau = 1$

$$F(\omega) = j2\left[\frac{\omega\cos(\omega/2) - 2\sin(\omega/2)}{\omega^2}\right]$$

$$|F(\omega)| = \left|\frac{2\omega\cos(\omega/2) - 4\sin(\omega/2)}{\omega^2}\right|$$

$$F(0) = 0$$

$$|F(2)| = \left|\frac{4\cos 1 - 4\sin 1}{4}\right| = 0.30$$

$$|F(4)| = \left|\frac{8\cos 2 - 4\sin 2}{16}\right| = 0.44$$

$$|F(6)| = \left|\frac{12\cos 3 - 4\sin 3}{36}\right| = 0.35$$

$$|F(8)| = \left|\frac{16\cos 4 - 4\sin 4}{64}\right| = 0.12$$

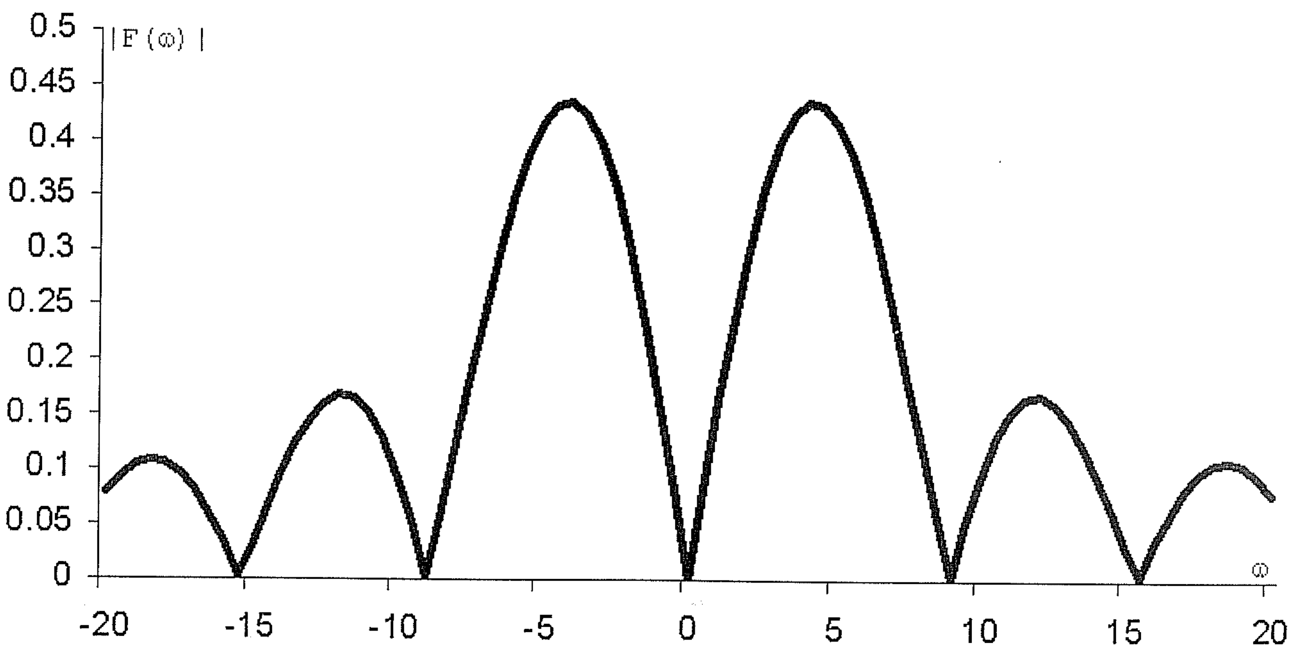
$$|F(9)| = \left|\frac{18\cos 4.5 - 4\sin 4.5}{81}\right| \cong 0$$

$$|F(10)| = \left|\frac{20\cos 5 - 4\sin 5}{100}\right| = 0.10$$

$$|F(12)| = \left|\frac{24\cos 6 - 4\sin 6}{144}\right| = 0.17$$

$$|F(14)| = \left|\frac{28\cos 7 - 4\sin 7}{196}\right| = 0.09$$

$$|F(15.5)| = \left|\frac{31\cos 7.75 - 4\sin 7.75}{240.25}\right| \cong 0$$



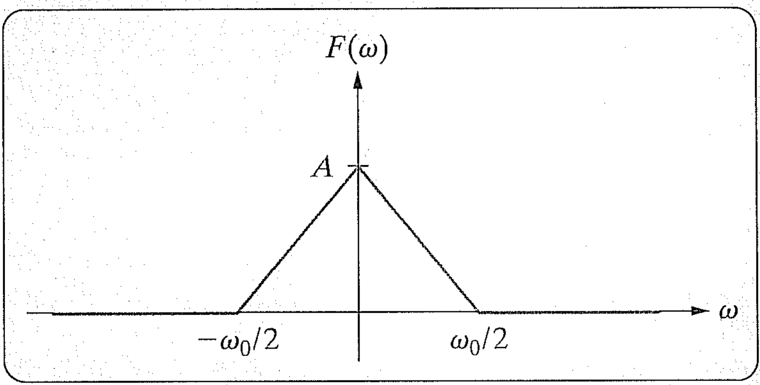
17.3 The Fourier transform of $f(t)$ is shown in Fig. P17.3.

a) Find $f(t)$.

b) Evaluate $f(0)$.

c) Sketch $f(t)$ for $-10 \leq t \leq 10$ s when $A = 20\pi$ and $\omega_0 = 2$ rad/s. *Hint:* Evaluate $f(t)$ at $t = 0, 1, 2, 3, \ldots,\ 10$ s, and then use the fact that $f(t)$ is even.

Figure P17.3



P 17.3 **[a]** $F(\omega) = A + \frac{2A}{\omega_o}\omega, \quad -\omega_o/2 \leq \omega \leq 0$

$$F(\omega) = A - \frac{2A}{\omega_o}\omega, \quad 0 \leq \omega \leq \omega_o/2$$

$$F(\omega) = 0 \qquad \text{elsewhere}$$

$$f(t) = \frac{1}{2\pi}\int_{-\omega_o/2}^{0}\left(A + \frac{2A}{\omega_o}\omega\right)e^{jt\omega}\,d\omega$$

$$+\frac{1}{2\pi}\int_{0}^{\omega_o/2}\left(A - \frac{2A}{\omega_o}\omega\right)e^{jt\omega}\,d\omega$$

$$f(t) = \frac{1}{2\pi}\left[\int_{-\omega_o/2}^{0} Ae^{jt\omega}\,d\omega + \int_{-\omega_o/2}^{0}\frac{2A}{\omega_o}\omega e^{jt\omega}\,d\omega\right.$$

$$\left.+\int_{0}^{\omega_o/2} Ae^{jt\omega}\,d\omega - \int_{0}^{\omega_o/2}\frac{2A}{\omega_o}\omega e^{jt\omega}\,d\omega\right]$$

$$= \frac{1}{2\pi}[\,\text{Int1} + \text{Int2} + \text{Int3} - \text{Int4}\,]$$

$$\text{Int1} = \int_{-\omega_o/2}^{0} Ae^{jt\omega}\,d\omega = \frac{A}{jt}(1 - e^{-jt\omega_o/2})$$

$$\text{Int2} = \int_{-\omega_o/2}^{0}\frac{2A}{\omega_o}\omega e^{jt\omega}\,d\omega = \frac{2A}{\omega_o t^2}(1 - j\frac{t\omega_o}{2}e^{-jt\omega_o/2} - e^{-jt\omega_o/2})$$

$$\text{Int3} = \int_{0}^{\omega_o/2} Ae^{jt\omega}\,d\omega = \frac{A}{jt}(e^{jt\omega_o/2} - 1)$$

$$\text{Int4} = \int_{0}^{\omega_o/2}\frac{2A}{\omega_o}\omega e^{jt\omega}\,d\omega = \frac{2A}{\omega_o t^2}(-j\frac{t\omega_o}{2}e^{jt\omega_o/2} + e^{jt\omega_o/2} - 1)$$

$$\text{Int1} + \text{Int3} = \frac{2A}{t}\sin(\omega_o t/2)$$

$$\text{Int2} - \text{Int4} = \frac{4A}{\omega_o t^2}[1 - \cos(\omega_o t/2)] - \frac{2A}{t}\sin(\omega_o t/2)$$

$$\therefore \quad f(t) = \frac{1}{2\pi}\left[\frac{4A}{\omega_o t^2}(1 - \cos(\omega_o t/2))\right]$$

$$= \frac{2A}{\pi\omega_o t^2}\left[2\sin^2(\omega_o t/4)\right]$$

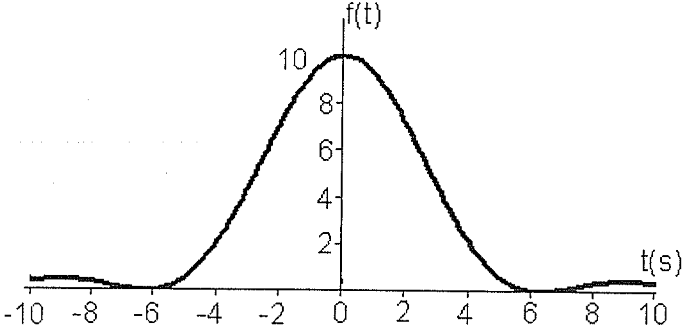
$$= \frac{4\omega_o A}{\pi\omega_o^2 t^2}\sin^2(\omega_o t/4)$$

$$= \frac{\omega_o A}{4\pi}\left[\frac{\sin(\omega_o t/4)}{(\omega_o t/4)}\right]^2$$

[b] $f(0) = \frac{\omega_o A}{4\pi}(1)^2 = 79.58 \times 10^{-3}\omega_o A$

[c] $A = 20\pi; \qquad \omega_o = 2$ rad/s

$$f(t) = 10\left[\frac{\sin(t/2)}{(t/2)}\right]^2$$



17.4 Find the Fourier transform of each of the following functions. In all of the functions, a is a positive real constant and $-\infty \leq t \leq \infty$.

a) $f(t) = |t|e^{-a|t|}$;

b) $f(t) = t^3 e^{-a|t|}$;

c) $f(t) = e^{-a|t|} \cos \omega_0 t$;

d) $f(t) = e^{-a|t|} \sin \omega_0 t$;

e) $f(t) = \delta(t - t_0)$.

P 17.4 **[a]** $F(s) = \mathcal{L}\{te^{-at}\} = \dfrac{1}{(s+a)^2}$

$$F(\omega) = F(s)\Big|_{s=j\omega} + F(s)\Big|_{s=-j\omega}$$

$$F(\omega) = \left[\frac{1}{(a+j\omega)^2}\right] + \left[\frac{1}{(a-j\omega)^2}\right]$$

$$= \frac{2(a^2-\omega^2)}{(a^2-\omega^2)^2+4a^2\omega^2} = \frac{2(a^2-\omega^2)}{(a^2+\omega^2)^2}$$

[b] $F(s) = \mathcal{L}\{t^3e^{-at}\} = \dfrac{6}{(s+a)^4}$

$$F(\omega) = F(s)\Big|_{s=j\omega} + F(s)\Big|_{s=-j\omega}$$

$$F(\omega) = \frac{6}{(a+j\omega)^4} - \frac{6}{(a-j\omega)^4} = -j48a\omega\frac{a^2-\omega^2}{(a^2+\omega^2)^4}$$

[c] $F(s) = \mathcal{L}\{e^{-at}\cos\omega_0 t\} = \dfrac{s+a}{(s+a)^2+\omega_0^2} = \dfrac{0.5}{(s+a)-j\omega_0} + \dfrac{0.5}{(s+a)+j\omega_0}$

$$F(\omega) = F(s)\Big|_{s=j\omega} + F(s)\Big|_{s=-j\omega}$$

$$F(\omega) = \frac{0.5}{(a+j\omega)-j\omega_0} + \frac{0.5}{(a+j\omega)+j\omega_0}$$

$$+ \frac{0.5}{(a-j\omega)-j\omega_0} + \frac{0.5}{(a-j\omega)+j\omega_0}$$

$$= \frac{a}{a^2+(\omega-\omega_0)^2} + \frac{a}{a^2+(\omega+\omega_0)^2}$$

[d] $F(s) = \mathcal{L}\{e^{-at}\sin\omega_0 t\} = \dfrac{\omega_0}{(s+a)^2+\omega_0^2} = \dfrac{-j0.5}{(s+a)-j\omega_0} + \dfrac{j0.5}{(s+a)+j\omega_0}$

$$F(\omega) = F(s)\Big|_{s=j\omega} - F(s)\Big|_{s=-j\omega}$$

$$F(\omega) = \frac{-ja}{a^2+(\omega-\omega_0)^2} + \frac{ja}{a^2+(\omega+\omega_0)^2}$$

[e] $F(\omega) = \displaystyle\int_{-\infty}^{\infty} \delta(t-t_o)e^{-j\omega t}\,dt = e^{-j\omega t_o}$

(Use the sifting property of the Dirac delta function.)

17.19 Suppose that $f(t) = f_1(t) f_2(t)$, where

$$f_1(t) = \cos \omega_0 t,$$

$$f_2(t) = 1, \quad -\tau/2 < t < \tau/2;$$

$$f_2(t) = 0, \quad \text{elsewhere.}$$

a) Use convolution in the frequency domain to find $F(\omega)$.

b) What happens to $F(\omega)$ as the width of $f_2(t)$ increases so that $f(t)$ includes more and more cycles on $f_1(t)$?

P 17.19 **[a]** $f_1(t) = \cos\omega_0 t, \qquad F_1(u) = \pi[\delta(u+\omega_0) + \delta(u-\omega_0)]$

$f_2(t) = 1, \quad -\tau/2 < t < \tau/2, \qquad$ and $f_2(t) = 0$ elsewhere

Thus $\quad F_2(u) = \dfrac{\tau\sin(u\tau/2)}{u\tau/2}$

Using convolution,

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u)F_2(\omega-u)\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi[\delta(u+\omega_0) + \delta(u-\omega_0)]\tau\frac{\sin[(\omega-u)\tau/2]}{(\omega-u)(\tau/2)}\,du$$

$$= \frac{\tau}{2}\int_{-\infty}^{\infty} \delta(u+\omega_0)\frac{\sin[(\omega-u)\tau/2]}{(\omega-u)(\tau/2)}\,du$$

$$+ \frac{\tau}{2}\int_{-\infty}^{\infty} \delta(u-\omega_0)\frac{\sin[(\omega-u)\tau/2]}{(\omega-u)(\tau/2)}\,du$$

$$= \frac{\tau}{2}\cdot\frac{\sin[(\omega+\omega_0)\tau/2]}{(\omega+\omega_0)(\tau/2)} + \frac{\tau}{2}\cdot\frac{\sin[(\omega-\omega_0)\tau/2]}{(\omega-\omega_0)\tau/2}$$

[b] As τ increases, the amplitude of $F(\omega)$ increases at $\omega = \pm\omega_0$ and at the same time the duration of $F(\omega)$ approaches zero as ω deviates from $\pm\omega_0$. The area under the $[\sin x]/x$ function is independent of τ, that is

$$\frac{\tau}{2}\int_{-\infty}^{\infty}\frac{\sin[(\omega-\omega_0)(\tau/2)]}{(\omega-\omega_0)(\tau/2)}\,d\omega = \int_{-\infty}^{\infty}\frac{\sin[(\omega-\omega_0)(\tau/2)]}{(\omega-\omega_0)(\tau/2)}[(\tau/2)\,d\omega] = \pi$$

Therefore as $t\to\infty$,

$f_1(t)f_2(t) \to \cos\omega_0 t \quad$ and $\quad F(\omega) \to \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$