**MATH 346/Spring 2005** **FINAL EXAM**

**NAME:** KEY

**Instructions:**

a. There are sixteen problems on this test. All of them are required.
b. All questions are equally weighted. The Final Exam grade accounts for 25% of the total grade of the course.
b. Justify your answers for all the problems below in order to get full credit.
c. If a wrong answer is given without any justification or explanation then no credit will be given for that question.

(1) Suppose the variable $x$ represents people, and $F(x)$ : $x$ is friendly, $T(x)$ : $x$ is tall, $A(x)$ : $x$ is angry. Write the following statements using the above predicates and any needed quantifiers.

(a) some tall angry people are friendly

$\exists x\,(T(x) \wedge A(x) \wedge F(x))$

(b) if a person is friendly, then that person is not angry

$\forall x\,(F(x) \rightarrow \neg A(x))$

(2) Let the sequence $\{a_n\}$ be defined recursively as $a_n = a_{n-1}^3 + 2$; $a_0 = 0$. Find the term $a_3$ of the sequence.

$a_1 = 0 + 2$; $a_2 = 2^3 + 2 = 10$, $a_3 = 10^3 + 2 = 1002$.

(3) A communication channel can transmit two types of signals. One signal takes one microsecond and the other signal takes two microseconds to be transmitted. How many different messages

made of these two types of signals can be transmitted in 50 microseconds if a signal is started right after another is finished?

1) write a recurrence relation:

a message can start with either kind of signals, if it starts with signal 1, we have $(n-1)$ more micro-seconds, it it starts with signal 2, we have $(n-2)$ ms.

so $a_n = a_{n-1} + a_{n-2}$; $a_0 = 0$; $a_1 = 1$.

2) charact. eq. is $r^2 = r + 1 \Rightarrow r^2 - r - 1 = 0$

$r_1 = \frac{1 + \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$.

3) general solution: $\alpha_1 r_1^n + \alpha_2 r_2^n$.

4) when $n = 0$: $\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = -\alpha_1$; when $n = 1$:

$\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \Rightarrow (\alpha_1 + \alpha_2) + \sqrt{5}(\alpha_1 - \alpha_2) = 2$

but $\alpha_1 + \alpha_2 = 0$ & $\alpha_1 = -\alpha_2$ so $\sqrt{5}(2\alpha_1) = 2 \Rightarrow \alpha_1 = \frac{1}{\sqrt{5}}$ & $\alpha_2 = \frac{-1}{\sqrt{5}}$

so we get

$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

(4) Get the general solution for the inhomogeneous recurrence equation $a_n = 3a_{n-1} + 4a_{n-2} + 2^n$.

Similar to (3), solve the homog. problem:

1) char. eq. $r^2 - 3r - 4 = 0$

2) $r_1 = 4$; $r_2 = -1$ so $a_n^{(h)} = \alpha_1 4^n + \alpha_2 (-1)^n$.

Next find a particular solution of inhomog. problem (one of the form $a_n^{(p)} = c \cdot 2^n$: plug it in recurrence eq.:

$c \cdot 2^n = 3 \cdot c\, 2^{n-1} + 4 \cdot c \cdot 2^{n-2} + 2^n$; divide this by $2^{n-2}$:

$4c = 6c + 4c + 4 \Rightarrow -6c = 4 \Rightarrow \boxed{c = -2/3}$

so general solution is

$a_n = \alpha_1 4^n + \alpha_2 (-1)^n - \frac{2}{3} \cdot 2^n$.

(5) Represent the number 0.71717171 ⋯ as a fraction. Hint: use a geometric sum.

$$0.7171\cdots = 71\times\frac{1}{100} + 71\times\left(\frac{1}{100}\right)^2 + \cdots = \sum_{i=1}^{\infty} (71)\cdot\left(\frac{1}{100}\right)^i = \frac{71/100}{1-\frac{1}{100}}$$

$$= \frac{71/100}{99/100} = \frac{71}{99}$$ using the geometric sum formula.

(6) Determine whether the two propositions $p \to (q \to r)$ and $p \to (q \wedge r)$ are equivalent

first

second

$$p \to (q \to r) \equiv p \to (\neg q \vee r) \equiv \neg p \vee \neg(\neg q \vee r)$$

$$\equiv \neg p \vee (q \wedge \neg r) \equiv p \to (q \wedge \neg r)$$

so they are not equivalent: use the assignment $p = T$; $q = T$ & $r = T$: the first is $T \to (T \wedge F) \equiv F$ but the second is $T \to T$ which is $T$.

(7) Let $f(n) = \lfloor \frac{n}{2} \rfloor$ be a function from $\mathbb{N}$ to $\mathbb{N}$. Answer the following two questions by giving either a proof or a counter example.

(a) is $f$ one to one?

no: $f(0) = f(1) = 0$

(b) is $f$ onto

Yes given an integer $n$, consider $2n$:

$$f(2n) = \left\lfloor \frac{2n}{2} \right\rfloor = \lfloor n \rfloor = n.$$

(8) How many ways are there to choose 12 cookies if there are 5 varieties, including chocolate chips if at least four chocolate chip cookies must be chosen?

four of them are dictated (chocolate chips) each of the other 8 cookies can be selected from 5 varieties so by the product rule we have: $\underbrace{5\times\cdots\times 5}_{8} = 5^8$ ways.

(9) How many four letter words can be made using the letters $\{a, b, c, d, e, f, g, h\}$ if the words must consist of different letters? (a word here is just a string of characters, meaning that it need not be meaningful)

Since 2 words are different if they have the same letters but shuffled, we are looking at the # ways to arrange 4 objects out of 8 objects so we have $P(8,4) = 8!/4! = 8 \times 7 \times 6 \times 5$ words.

(10) Prove that the sum of two even integers is even.

consider two even integers $m$ & $n$, since 2 divides $m$ & $n$, we have $m = 2K_1$ & $n = 2K_2$ for some integers $K_1$ & $K_2$, now

$m+n = 2K_1 + 2K_2 = 2(K_1+K_2)$, hence

$2 \mid (m+n)$ i.e., $m+n$ is even

(11) What is the coefficient of $x^2y^{18}$ in the expansion of $(x+y)^{20}$?

By the binomial theorem this is

$\binom{20}{2} = \frac{20 \times 19}{2} = 190.$

(12) How many students must be in class to guarantee that at least five were born on the same day of the week?

If we have $4 \times 7$ students, then in the "worst" case, every 4 students were born on the same day, if we add one student, then by the pigeon hole principle one day will have 5 students, so Answer is 29.

(13) How many bit strings of length 10 have at least two zeroes in them?

This is total − # bitstrings with one zero or no zeroes $= 2^{10} - 10 - 1 = 1024 - 11$

$= 1013$

(14) Use mathematical induction to prove that $n! \geq 2^{n-1}$ whenever $n$ is a positive integer.

Base case: $n=1$ : $1! = 1$; $2^{1-1} = 1$ & $1 \geq 1$.

Ind. hyp. suppose for some $k \geq 1$, $k! \geq 2^{k-1}$.

Ind. step: we need to show that $(k+1)! \geq 2^k$

But $(k+1)! = (k+1)\, k!$;

$k \geq 1 \Rightarrow (k+1) \geq 2$ & by the ind. hyp.: $k! \geq 2^{k-1}$

so $(k+1) \cdot k! \geq 2 \cdot 2^{k-1} = 2^k$, i.e, $(k+1)! \geq 2^k$.

It follows by the principle of math. ind. that $\forall n \geq 1$, $n! \geq 2^{n-1}$!

(15) Give a combinatorial proof that $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$ where $r$ does not exceed $m$ and does not exceed $n$.

i) we know $\binom{m+n}{r}$ is the # ways to choose $r$ objects out of $(m+n)$.

ii) Think of choosing $r$ objects from $m+n$ where we have $m$ of one categories & $n$ from another category: since $r \leq m$ & $r \leq n$, the $r$ objects chosen may include $i$ objects from the first category, where $i = 0, \ldots, r$ & the remaining $r-i$ then come from the other category

so we have $\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0}$.

Now since the two methods count the same number, the proposed equality follows.

(16) Use the method of the generating function to find the solution to the recurrence relation $a_n = 3a_{n-1} + 2^n$ with the initial condition $a_0 = 1$.

Let $G(x)$ be the generating function of $a_n$, that is $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

over

Let us multiply the recurrence equation by $x^n$ & sum from $n=1$ to $\infty$:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n + \sum_{n=1}^{\infty} 2^n \cdot x^n$$

so $-a_0 + G(x) = 3x\,G(x) + \dfrac{2x}{1-2x}$

Using $a_0 = 1$:

$$G(x) - 3x\,G(x) = 1 + \frac{2x}{1-2x} = \frac{1-2x+2x}{1-2x}$$

so $G(x) = \dfrac{1}{(1-2x)(1-3x)} = \dfrac{A}{1-2x} + \dfrac{B}{1-3x}$

$$= \frac{A(1-3x) + B(1-2x)}{(1-2x)(1-3x)} = \frac{(A+B)-(3A+2B)x}{(1-2x)(1-3x)}$$

So $A+B = 1$;

$3A + 2B = 0 \Longrightarrow A = \dfrac{-2B}{3}$

so $-\dfrac{2}{3}B + B = 1 \Rightarrow \dfrac{1}{3}B = 1 \Rightarrow B = 3$

& $A = -2/3 \times 3 = -2$, so $G(x) = \dfrac{-2}{1-2x} + \dfrac{3}{1-3x}$

$$= -2\sum_{n=0}^{\infty}(2x)^n + 3\sum_{n=0}^{\infty}(3x)^n =$$

$$\sum_{n=0}^{\infty} (-2)\,2^n \cdot x^n + 3 \cdot 3^n \cdot x^n = \sum_{n=0}^{\infty}\left(3^{n+1} - 2^{n+1}\right)x^n$$

So $a_n = 3^{n+1} - 2^{n+1}$