MATH 211 EXAM ONE
Fall 2005
September 29

Name: KEY

The test consists of 2 pages. Justify your work when necessary.

(1) Find a simplifying expression for the following sets. Here $\mathcal{U}$ is the universe, A and B are two sets such that $B \subseteq A$.

a) $A \cup \emptyset = A$ b) $A \cup \mathcal{U} = \mathcal{U}$
c) $A - \mathcal{U} = \phi$ d) $A \oplus A = \phi$
e) $\emptyset - A = \phi$ f) $A \cap B = B$
g) $A \cup B = A$ h) $\overline{A} \cap B = \phi$
i) $A \oplus B = A - B$ j) $\mathcal{P}(\emptyset) = \{\phi\}$

(2) Using only p, q, r, $\neg$ and/or the connective $\wedge$, write a proposition equivalent to each of the following

(a) $(p \to q) \to r \equiv (\neg p \vee q) \to r \equiv \neg(p \wedge \neg q) \to r$
$\equiv \neg(\neg(p \wedge \neg q)) \vee r \equiv (p \wedge \neg q) \vee r \equiv \neg((p \wedge \neg q) \wedge \neg r)$

(b) $p \to (q \to r) \equiv \neg p \vee (q \to r) \equiv \neg p \vee (\neg q \vee r) \equiv \neg(p \wedge \neg(\neg q \vee r))$
$\equiv \neg(p \wedge (q \wedge \neg r))$

(3) Write the contrapositive and converse of the statement: "You sleep late if it is Saturday".

the statement itself is "if it is Sat. then you sleep well"
so contra. is "if you don't sleep well then it is not Sat."
converse is: "if you sleep well then it is Sat."

(4) In the following, $P(x, y)$ means "$x + 2y = xy$". Where x and y are integers. Determine the truth value of the statement.

(a) **T** **F** $\exists y P(x, 3)$ this is not even a proposition; y is free.
(b) **T** **F** $\forall x \exists y P(x, y)$. when $x = 2$ we get $2 = 0$!
(c) **T** **F** $\exists x \forall y P(x, y)$ when $x = 0$: $2y = 0$ which is true $\forall y$ certainly.
You can argue the same way for any choice of x.

(5) Suppose the variable x represents students and the variable y represents courses, and $A(y)$: y is an advanced course $S(x)$: x is a sophomore $F(x)$: x is a freshman $T(x, y)$: x is taking y. Write the following statements using these predicates and any needed quantifiers.

(a) There is a course that every freshman is taking.

$\exists y \forall x (F(x) \to T(x, y))$

(b) No freshman is a sophomore.

$\forall x(F(x) \rightarrow \neg S(x))$ or $\neg \exists x(F(x) \wedge S(x))$

(c) Some freshman is taking an advanced course.

$\exists x(F(x) \wedge T(x,y) \wedge A(y))$

(d) There are at least two freshman students taking the exact same courses.

$\exists x_1 \exists x_2 (F(x_1) \wedge F(x_2) \wedge \forall y(T(x_1,y) \leftrightarrow T(x_2,y)))$

(6) Determine whether the following argument is valid.

$p \rightarrow r$
$q \rightarrow r$
$\neg(p \vee q)$
$\therefore \neg r$

not valid: the assignment

$p = F$, $q = F$ & $r = T$

leaves $p \rightarrow r$, $q \rightarrow r$ & $\neg(p \vee q)$ all true

but $\neg r$ false.

(7) Determine whether the following argument is valid.

1) She is a Math Major or a Computer Science Major.
2) If she does not know discrete math, she is not a Math Major.
3) If she knows discrete math, she is smart.
4) She is not a Computer Science Major.
5) Therefore, she is smart.

6): (1) & (4) mean she is a MATH major (disjunction syllogism)
7) If she is a math major then she knows discrete (contrapositive of (2))
8) (6) & (7) & modus ponens mean she knows discrete math.
9) (3), (8) & modus ponens give (5)
so it is a valid argument

(8) Determine whether the rule describes a function. If your answer is no say why.

(a) $f : \mathbb{N} \rightarrow \mathbb{N}$ where $f(n) = \sqrt{n}$.

no because $\sqrt{n}$ need not be in $\mathbb{N}$, e.g. $n = 2$.

(b) $g : \mathbb{N} \to \mathbb{N}$ where $g(n) =$ any integer $> n$.

no. The definition is not unique, e.g. $g(2)$ could be $3, 4, 5, \ldots$

(9) Give an example of a function from $\mathbb{Z}$ to $\mathbb{N}$ that is both one-to-one and onto.

$f : \mathbb{Z} \to \mathbb{N}$

$$f(z) = \begin{cases} 2z - 1 & \text{if } z > 0 \\ -2z & \text{if } z < 0 \end{cases}$$

So
$0 \to 0$
$1 \to 1$
$-1 \to 2$
$2 \to 3$
$-2 \to 4$
$3 \to 5$
$-3 \to 6$
$\vdots$

(10) Give an example of a function from $\mathbb{Z}$ to $\mathbb{N}$ that is onto but NOT one-to-one.

$f(z) = |z|$ this takes every two opposite values to one value, hence <u>not</u> one-to-one but surely onto (every positive value is the absolute value of itself.

(11) Let $f : A \to B$. Let $B' \subset B$. Show that $f(f^{-1}(B')) \subseteq B'$. WHAT condition is needed for the containment in the other direction?

let $y \in f(f^{-1}(B')) \Rightarrow \exists x \in f^{-1}(B')$ such that $y = f(x)$. But since $x \in f^{-1}(B')$ we get that $f(x) \in B'$. Gathering the last two pieces we get $y \in B'$.

For the other containment (i.e. $B' \subseteq f(f^{-1}(B'))$) we need every element of B' to be an image of some element of A. This is guaranteed when <u>f is onto.</u>