Ch1- 1.5: Rules of Inference

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Definition:

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is *valid* if the truth of all its premises implies that the conclusion is true.

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Note:

A sequence of compound arguments is called an argument form.

From the definition, an argument with premises $p_1, p_2, \dots, p_n$ and conclusion q is valid if $(p_1 \wedge p_2 \wedge \dots p_n) \rightarrow q$ is a tautology.

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Sometimes we may have many propositional variables so we can not draw a truth table to determine the validity of an argument since it will need $2^{10} = 1024$ rows.

We will now prove the validity of some arguments, call them particular names, and use them to prove the validity of an argument form! We will call them Rules of Inference.

Ch1- 1.5: Rules of Inference

- An *Inference Rule* is
  - A pattern establishing that if we know that a set of *premises* are all true, then we can deduce that a certain *conclusion* statement is true.
- *premise 1*
  premise 2 …
  $\therefore$ *conclusion*   "$\therefore$" means "therefore"

Ch1- 1.5: Rules of Inference

Some Rules of Inference:

- $\dfrac{p}{\therefore\ p \vee q}$ Rule of Addition
- $\dfrac{p \wedge q}{\therefore\ p}$ Rule of Simplification
- $\dfrac{p \quad q}{\therefore\ p \wedge q}$ Rule of Conjunction

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- p
$\underline{p \rightarrow q}$
$\therefore q$

  Rule of *modus ponens* (*law of detachment*) *mode of affirming*

- $\neg q$
$\underline{p \rightarrow q}$
$\therefore \neg p$

  Rule of *modus tollens* *mode of denying*

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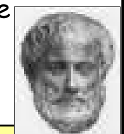
- $p \rightarrow q$
$\underline{q \rightarrow r}$
$\therefore p \rightarrow r$

  Rule of hypothetical syllogism (transitivity)

- $p \vee q$
$\underline{\neg p}$
$\therefore q$

  Rule of disjunctive syllogism



Aristotle (384-322 B.C.)

Ch1- 1.5: Rules of Inference

- $p \vee q$
$\underline{\neg p \vee r}$
$\therefore q \vee r$

  Resolution

Ch1- 1.5: Rules of Inference

- For each of these rules, we can easily prove their *validity*: **If their premises are true than their conclusion must be true.**
- Example: Rule of disjunctive syllogism: "If $p \vee q$ and $\neg p$ then q" Use truth tables to show this rule is valid.

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Refer to worksheet 1

Ch1- 1.5: Rules of Inference

- Each **valid** logical inference rule corresponds to an implication that is a tautology.
- *premise 1*
premise 2 …
$\therefore$ *conclusion*

  Inference rule

- Corresponding tautology:

  ((premise 1) ∧ (premise 2) ∧ …) → conclusion

Ch1- 1.5: Rules of Inference

More generally, if there is a valid argument

premise 1,.., premise n $\therefore$ *conclusion*

then the implication

((premise 1) $\wedge$ .. $\wedge$ *(premise n))* $\rightarrow$ *conclusion*

is a tautology.

Ch1- 1.5: Rules of Inference

Incorrect arguments involve several fallacies. Some common ones are:

1) Fallacy of affirming the conclusion:

$$[(p \rightarrow q) \wedge q] \rightarrow p$$

1) Fallacy of denying the hypothesis:

$$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$$

Ch1- 1.5: Rules of Inference

- $\forall x\, P(x)$
  $\therefore P(a)$ (substitute any object *a*)

Universal instantiation

Ch1- 1.5: Rules of Inference

- $P(g)$
  $\therefore \forall x\, P(x)$? Universal Generalization!!!
- This is not a valid inference of course. But suppose you can prove $P(g)$ without using any information about g ...
- ... then the inference to $\forall x\, P(x)$ is valid!
- In other words ...

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Probably the trickiest of the four rules.

- What do you think of this inference:
- $\exists x\, P(x)$
  $\therefore P(c)$?

Existential instantiation

Ch1- 1.5: Rules of Inference

- $\forall x\, P(x)$
  $\therefore P(a)$ universal instantiation
- $P(g)$ for g an *arbitrary* element
  $\therefore \forall x\, P(x)$ universal generalization
- $\exists x\, P(x)$ existential instantiation
  $\therefore P(b)$ for some element b
- $P(c)$ for some object c
  $\therefore \exists x\, P(x)$ existential generalization

Ch1- 1.5: Rules of Inference

- Unproblematic rules:
  - Univ. instantiation
  - Existential generalisation
- Be more careful with:
  - Universal generalisation
    ("*any* object" in premise)
  - Existential instantiation
    ("*new* object" in conclusion)

Next

Ch1- 1.6: Introduction to Proofs

Ch1- 1.6: Introduction to Proofs

Definition:
A proof is a valid argument that establishes the truth of a mathematical statement

Ch1- 1.6: Introduction to Proofs

Terminology:

- A theorem is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered somewhat important.
- Less important theorems are sometimes called propositions.

(both can be referred to as facts or results)

Ch1- 1.6: Introduction to Proofs

- We demonstrate that a theorem is true using a proof.
- A less important statement that is helpful in the proof of other results is called a lemma.
- A corollary is a theorem that can be established directly from a theorem that has been proved.

Ch1- 1.6: Introduction to Proofs

- An axiom (or postulate) is a statement that is assumed true without proof.
- A conjecture is a statement that is proposed to be a true statement usually on the basis of some partial evidence.
- A theory is the set of all theorems that can be proven from a given set of axioms.

Ch1- 1.6: Introduction to Proofs

- Methods for Proofs:
  - Direct Proof
  - Proof by contraposition
  - Proof by contradiction
  - Exhaustive proof or proof by cases (section 1.7)

Refer to worksheet 1

Ch1- 1.7: Proof methods and Strategy

You may also want to know terms like: wrong

-begging the question or circular reasoning
(one or more step of the proof are based on the truth of the statement being proved. In other words, a statement is proved using itself or a statement equivalent to it)

-without loss of generality or WLOG Use carefully
(after proving one case of a theorem, wlog means that no additional argument is needed for the other cases. I.e, other cases follow in a similar way)

Ch1- 1.7: Proof methods and Strategy

-existence proof (can be constructive or non-constructive)
(show that an element x with a desired property exists i.e., $\exists x\, P(x)$)

-uniqueness proof
(shows that if y ≠ x then y does not have the desired property)

Ch1- 1.7: Proof methods and Strategy

-forward reasoning (start from the premises use certain steps to reach the conclusion)

-backward reasoning (find a statement that will lead to the conclusion and move backward in the same way to reach the premises)

WARNING: You must never start from the conclusion or assume that it is true.

Ch1- 1.7: Proof methods and Strategy

- Exercise:
  Use backward reasoning to show that the arithmetic mean of two positive real numbers x and y, is greater than their geometric mean, that is

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Ch1- 1.7: Proof methods and Strategy

- Other proof methods and strategies will be seen later in this course!