Worksheet

I- Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$?

a) $f(x) = \frac{1}{x}$

b) $f(x) = \pm\sqrt{x^2+1}$

a) $f(0)$ is not defined

b) an element in $\mathbb{R}$ has two images

II- Are the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ one-to-one?

a) $f(n) = n - 1$

b) $f(n) = n^2 + 1$

a) yes since if $f(n_1) = f(n_2)$

$n_1 - 1 = n_2 - 1$

$n_1 = n_2$

b) no since $f(-1) = (-1)^2 + 1 = 1^2 + 1$

$= f(1)$

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Let f be a function from the set A to the set B, and let S and T be two subsets of A. Show that:

a) $f(S \cup T) = f(S) \cup f(T)$.

b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

a) Let $y \in f(S \cup T)$

so $\exists x \in S \cup T$ such that $f(x) = y$

$x \in S \cup T$

so $x \in S$ or $x \in T$

so $f(x) \in f(S)$ or $f(x) \in f(T)$

$f(x) = y \in f(S) \cup f(T)$

therefore $\boxed{f(S \cup T) \subseteq f(S) \cup f(T)}$

Moreover,

Let $y' \in f(S) \cup f(T)$

$y' \in f(S)$ or $y' \in f(T)$

so $\exists x' \in S, f(x') = y'$ or $\exists x'' \in T, f(x'') = y'$

so $\exists x \in S \cup T$ such that $f(x) = y'$

this means that $y' \in f(S \cup T)$

$\boxed{f(S) \cup f(T) \subseteq f(S \cup T)}$

b) Let $y \in f(S \cap T)$

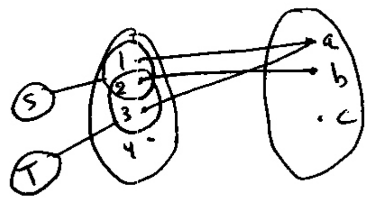
so $\exists x \in S \cap T$ such that $f(x) = y$

$x \in S$ and $x \in T$

$f(x) \in f(S)$ and $f(x) \in f(T)$

so $f(x) = y \in f(S) \cap f(T)$

Note: Take



$S = \{1, 2\}$

$T = \{2, 3\}$

$f(S) = \{a, b\}$

$f(T) = \{a, b\}$

$f(S \cap T) = \{b\}$

so $f(S) \cap f(T) \not\subseteq f(S \cap T)$

IV- Given two mappings f and g from $\mathbb{N}$ to $\mathbb{N}$ defined by:

$$f\colon \mathbb{N} \longrightarrow \mathbb{N} \qquad\qquad g\colon \mathbb{N} \longrightarrow \mathbb{N}$$
$$n \longmapsto n+1 \qquad\qquad n \longmapsto \max\{0, n-1\}$$

(a) Calculate $f(n)$ and $g(n)$ for $n = 0, 1, 2, 3, 4, 73$.

(b) Is f an injection? a surjection?

(c) Is g an injection? a surjection?

(d) Show that $f \circ g \neq id_{\mathbb{N}}$ but $g \circ f = id_{\mathbb{N}}$.

a) $f(0) = 1$ $\quad g(0) = 0$
$f(1) = 2$ $\quad g(1) = 0$
$f(3) = 4$ $\quad g(2) = 1$
$f(4) = 5$ $\quad g(3) = 2$
$f(73) = 74$ $\quad g(4) = 3$
$f(2) = 3$ $\quad g(73) = 72$

b) f is injective (proved earlier)
f is surjective since given any $y \in \mathbb{N}$ we can find an $x \in \mathbb{N}$ such that $f(x) = y$ in the following way.
$x + 1 = y$
$x = y - 1$
that is $f(y-1) = y$

c) g is not injective $g(0) = g(1)$
g is surjective since $g(y+1) = y$ for any $y \in \mathbb{N}$.

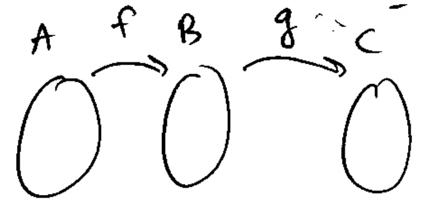
d) $f\circ g(0) = f(g(0)) = f(0) = 1$ so $f\circ g \neq id_{\mathbb{N}}$

$g\circ f(x) = g(x+1) = \max\{0, x+1-1\}$
$= \max\{0, x\} = x$ since $x \geq 0$
$= id_{\mathbb{N}}$.

V- Show that if $g \circ f$ is a bijection then g is surjective and f is injective.

To show that g is surjective,

<u>want:</u> $\forall c \in C \quad \exists b \in B$ such that $g(b) = c$



So take $c \in C$

since $g \circ f$ is surjective $\exists a \in A$ such that $g \circ f(a) = c$

so $g(f(a)) = c$

i.e, $g(b) = c$ where $b = f(a)$

this means we could find $b \in B$ such that $g(b) = c$.

so g is surjective.

To show that f is injective,

<u>want:</u> if $f(a_1) = f(a_2)$ then $a_1 = a_2$

say $f(a_1) = f(a_2)$

$g(f(a_1)) = g(f(a_2))$ why?

$g \circ f(a_1) = g \circ f(a_2)$

$a_1 = a_2$ since $g \circ f$ is injective.