AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 211
FINAL EXAMINATION
FALL 2005-2006
Closed Book, 2 HOURS

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points)

   (a) (3 points) Prove without using truth tables that the implication

   $$(\neg q \wedge (p \to q)) \to \neg p,$$

   is a tautology.

   (b) (5 points) Let a, b, c be any three consecutive positive integers. Prove that $a^3 + b^3 + c^3$ is divisible by 9

   (c) (2 points) Let $P(x)$ and $Q(x)$ be predicates on a universal set $U = \{x_1, x_2\}$. Prove the equivalence :

   $$(\forall x\,(P(x) \wedge Q(x))) \equiv (\forall x\,P(x)) \wedge (\forall x\,Q(x)).$$

2. (10 points) Let $M = 2^p - 1$, ($p \in \mathbb{N}$).

   (a) (3 points) Show that M prime implies that p is prime.

   (b) (3 points) Let $N = (2^p - 1)2^{p-1}$, where p is prime. Give the elements of the set S_N that consists of the divisors of N, **others than** N :

$$S_N = \{x_1, x_2, ..., x_n\},$$

   with $\{\forall i = 1, 2...n : \ x_i | N, \ x_i \neq N\}$. What is the cardinality of S_N.

(c) (2 points) Show that

$$\sum_{i-1}^{n} x_i = N.$$

(d) (2 points) Give N and S_N for $p = 2, 5$

3. (10 points) Consider the recurrence relation :

(1) $$a_n - 5a_{n-1} + 6a_{n-2} = 2^n + 3n,\ a_0 = 0,\ a_1 = 1.$$

(a) (4 points) Find the general solution of the recurrence equation :

$$a_n - 5a_{n-1} + 6a_{n-2} = 0.$$

(b) (4 points) Find a particular solution to :

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n + 3n.$$

(Hint : find first particular solutions for each of 2^n and $3n$).

(c) (3 points) Find the solution to the recurrence relation (1)

4. (10 points) Given a sequence of $n+1$ real numbers $\{a(0), a(1), ..., a(n)\}$ and a real number c, consider the following algorithm :

```
y :=a(0)
p :=1
for i from 1 to n
    p :=p*c
    y :=y+a(i)*p
end
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(a) (4 points) Give $f(n)$ the number of artithmetic operations needed to execute this algorithm.

(b) (2 points) Give α in $f(n) = \Theta(n^\alpha)$.

(c) (4 points) (independent from (a) and (b)) Assume that $f(n) = \Theta(n^3)$ and $g(n) = \Theta(n)$. Give $\Theta(f(n) + g(n))$ and $\Theta(\frac{f(n)}{g(n)})$.

5. (10 points) Let S be a set with n elements. Answer the following justifying your answer.

(a) (2 points) How many ordered pairs are there in $S \times S$?

(b) (3 points) How many relations are there on S (subsets of $S \times S$)?

(c) (5 points) Let A, B, C be 3 sets. Suppose $|A| = |B| = |C| = 100$, $|A \cap B| = 70$, $|A \cap C| = 50$, $|B \cap C| = 45$ and $|A \cup B \cup C| = 175$. How many elements are there in $A \cap B \cap C$?

6. (10 points) Let $n \in \mathbb{N}_+$.

(a) (3 points) Let S_n be the set of all non-empty subsets, $\{a_1, a_2, ..., a_k | 1 \leq k \leq n\}$ of the set of first n integers $\{1, 2, ..., n\}$. Give a recursive definition of the sets S_n. Indicate the basis and recursive steps.
Hint : The recursive step should include 3 cases.

(b) (2 points) Let

$$s_n = \Sigma_{\{a_1,a_2,...,a_k\} \in S_n} (\frac{1}{a_1.a_2...a_{k-1}.a_k}).$$

Give the expressions and values of s_1, s_2 and s_3.

(c) (5 points) Prove by Mathematical induction that $s_n = n$.

Additional space for answers