Math 211 — Fall 2006–07
Discrete Structures
Final exam, January 18 — Duration: 2 hours 15 minutes

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	7	8	9	10	TOTAL/120

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation W 4	Recitation W 11	Recitation Th 12:30	Recitation Th 3:30
Professor Makdisi	Ms. Karam	Ms. Karam	Ms. Karam

The usual instructions:

1. **Write your NAME and AUB ID number, and circle your SECTION above.**
2. **Solve the problems inside the booklet. Explain your reasoning precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.**
3. **You may use the back of each page for scratchwork OR for solutions. There are four blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.**
4. **Open book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.**
5. **You do not need to simplify your answers to counting problems — for example, if the answer is $\binom{211}{30} + (12!)7 \cdot 6 \cdot 5$, just leave it that way.**

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points. Take a minute to look at all the questions, THEN solve each problem on its corresponding page INSIDE the booklet.

1. Show that $(p \to r) \wedge (q \to r)$ is logically equivalent to $(p \vee q) \to r$ in two different ways, (a) and (b+c):
   a) Use a truth table.
   b) Give a direct proof that if $(p \to r) \wedge (q \to r)$, then $(p \vee q) \to r$.
   c) Give a direct proof of the converse of the statement in (b).

2. a) Show that the following statement is TRUE:

$$\forall n \in \mathbf{Z}, \quad \left[n^2 \not\equiv 0 \pmod{12}\right] \to \left[n \not\equiv 0 \pmod{12}\right].$$

   b) Show that the converse of the above statement is FALSE.

3. Given a function $f : X \to Y$.
   a) Show that for all subsets $S \subseteq X$, we have $S \subseteq f^{-1}(f(S))$.
   b) Show that f is injective **if and only if** for all $S \subseteq X$, we have $S = f^{-1}(f(S))$.

4. Find the cardinality $|X|$ of the following set (recall that 0 is included in $\mathbf{N}$):

$$X = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{N}^5 \mid x_1 + \cdots + x_5 = 24,$$
$$\text{exactly TWO of the variables } x_1, \ldots, x_5 \text{ are equal to } 6,$$
$$\text{and the others are all } \leq 5\}.$$

5. a) Show directly from the definition that 5^n is $O(n!)$. (Make sure to find explicit constants C and k.)
   b) (UNRELATED) Define a function $f : \mathbf{N} \to \mathbf{N}$ recursively by

$$f(0) = 211, \qquad f(1) = 201, \qquad f(2) = 230, \qquad \forall n \geq 3, \quad f(n) = f(n-3) + 7.$$

Find a general formula for $f(n)$ (hint: the formula depends on $n \bmod 3$), and use it to show that $f(n)$ is $\Theta(n)$.

6. We wish to write an algorithm to compute the function $\mathtt{f(n)}$, defined by

$$\mathtt{f(n)} = \text{ the product of all positive odd numbers } \mathtt{k} \text{ such that } \mathtt{k} \leq \mathtt{n}.$$

For example, `f(10)` $= 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$ and `f(11)` $= 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$. Note that to tell if a number is odd, you can use the function `IsOdd(n)`, which return `true` or `false` depending on whether `n` is odd or not. You can also use a similar function `IsEven(n)`.

a) Write an algorithm (in GAP or pseudocode) that computes `f(n)` WITHOUT any recursion. (Use a loop.)

b) Write a RECURSIVE algorithm to compute `f(n)`.

7. a) Write the number $42 (= 42_{10})$ in base 3.

b) (UNRELATED) A **ternary string** is a string made from the symbols $0, 1, 2$. For example, 010122011 is a ternary string. Define

$$a_n = \text{the number of ternary strings of length } n \text{ that do NOT contain 00 anywhere.}$$

Find a recurrence for a_n, and specify the initial conditions. Do NOT solve the recurrence for the general value of a_n.

8. a) Use the (extended) Euclidean algorithm to find the inverse of 30 modulo 43. In other words, find $x \in \mathbf{Z}$ such that $30x \equiv 1 \pmod{43}$.

b) (UNRELATED) Show that the `while` loop below is repeated $\lceil \log_2 \log_2 x \rceil$ times:

```
<< x is some number greater than 2 from earlier in the program >>
        a := 2;
        while a < x do
            a := a*a;
        od;
<< program continues >>
```

9. Write a RECURSIVE algorithm `totalsum(ll)` (in GAP or pseudocode) that adds up all the numbers in the list `ll` and all its sublists. For instance,

$$\texttt{totalsum}([3,[1,4,1],[5,[9,2],[6,5,3]]]) = 3+1+4+1+5+9+2+6+5+3 = 31.$$

You may use the GAP function `IsList` to tell if you have a list or not. Hint: you will probably want to keep track of the sum in a variable `ans`, and use a loop of the form `for x in ll do`... to compute something involving each sublist `x`.

10. a) Find the general solution of the recurrence

$$(*) \qquad a_k = 6a_{k-1} - 8a_{k-2}, \quad \text{for } k \geq 2.$$

b) In a certain "divide-and-conquer" algorithm with complexity $f(n)$, we obtain the recurrence

$$(**) \qquad f(n) = 4f(\lceil n/2 \rceil) + n \quad \text{for } n \geq 2, \qquad f(1) = 1.$$

Define $a_k = f(2^k)$, and show that a_k satisfies the recurrence $(*)$. (Hint: from $(**)$, deduce a first recurrence for a_k that you must then use twice to prove $(*)$.)

c) Find the exact value of $f(2^k)$. (Caution: $a_0 = f(2^0) = 1$.)