Math 211 — Fall 2007–08
Discrete Structures
Final exam, January 31, 2008 — Duration: 2 hours

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	7	8	9	10	11	TOTAL/132

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation W 4	Recitation W 11	Recitation Th 12:30	Recitation Th 3:30
Professor Makdisi	Mrs. Karam	Mrs. Karam	Mrs. Karam

INSTRUCTIONS:

1. **Write your NAME and AUB ID number, and circle your SECTION above.**
2. **Solve the problems inside the booklet. Explain your reasoning precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points. EACH PART OF A GIVEN PROBLEM CARRIES EQUAL WEIGHT.**
3. **You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.**
4. **Open book and notes. CALCULATORS ARE ALLOWED. Turn OFF and put away any cell phones.**
5. **You do not need to simplify your answers to counting problems — for example, if the answer is $\binom{211}{30} + (12!)7 \cdot 6 \cdot 5$, just leave it that way.**

NEW! 6. The problems are ordered ROUGHLY from shortest to longest, but the difficulty of the parts varies even though they have equal weight. EVEN IF YOU CANNOT DO AN EARLIER PART OF A PROBLEM, YOU MAY ASSUME THE RESULT FOR THE LATER PARTS OF THE SAME PROBLEM.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points.
Each part of a problem is weighted equally.
Take a few minutes to look at all the questions, BEFORE you start solving.

1. In how many ways may one distribute a standard deck of 52 (distinguishable) playing cards to four (distinguishable) players, each player receiving 13 cards, in such a way that each player receives exactly one ace?

2. We are given three statements: (i) $\neg s \to \neg p$, (ii) $s \to r$, and (iii) $q \wedge \neg s \to r$. Use the given statements to show that $(p \vee q) \to r$.

3. Let $X = \{0, 1, 2, \ldots, 9\}$, and consider the function $f : X \to X$ given by $f(x) = (7x + 1) \bmod 10$. (For example, $f(2) = 15 \bmod 10 = 5$.) Show that f is a bijection and give a simple description of the inverse function f^{-1}.

4. Find a simple formula for a_n given that $a_0 = a_1 = 1$ and that for $n \geq 2$, we have $a_n = -3a_{n-1} + 10a_{n-2}$.

5. a) How many times does the following loop repeat?

```
a := x;
while a > 1 do
    a := a/2;
od;
```

b) Show that the following nested loop repeats a total of $\Theta(n \log n)$ times:

```
for x in [1..n] do
    a := x;
    while a > 1 do
        a := a/2;
    od;
od;
```

6. Let $f : X \to Y$ be a function and consider a subset $A \subseteq X$.
a) Show that $f(X) - f(A) \subseteq f(X - A)$.
b) Show that if f is injective, then in fact $f(X) - f(A) = f(X - A)$.

7. a) Find the prime factorizations of 444 and 120. Use them to find

$$d = \gcd(444, 120).$$

b) Find $x_0, y_0 \in \mathbf{Z}$ such that $d = 444x_0 + 120y_0$. (Use the extended Euclidean algorithm.)
c) Find $x_1, y_1 \in \mathbf{Z}$ such that $100d = 444x_1 + 120y_1$. (Your answer can be in terms of x_0, y_0 from part (b).)
d) Show that the following two sets A and B are equal:

$$A = \{n \in \mathbf{Z} \mid \exists x, y \in \mathbf{Z} \text{ s.t. } n = 444x + 120y\}, \qquad B = \{n \in \mathbf{Z} \mid n \text{ is a multiple of } d\}.$$

8. Given the following recursively defined function $f : \mathbf{N} \to \mathbf{N}$:

$$f(0) = 1; \qquad \text{for } n \geq 1,\ f(n) = f(n-1) + f(\lfloor n/2 \rfloor).$$

a) Find $f(n)$ for $1 \leq n \leq 6$.

b) Write a recursive program (in GAP or pseudocode) to compute $f(n)$. You may use a function `floor()` as though it were built in to GAP (or pseudocode).

c) Use strong induction to show that $\forall n \in \mathbf{N},\ f(n) \leq 2^n$. (In fact $f(n)$ is $o(2^n)$, but do not try to show this.)

9. For $n \in \mathbf{N}$ we consider the set $S(n)$ of positive integers $\leq n$ that are divisible by 2, 3, or 5:

$$S(n) = \{k \in \mathbf{N} \mid (1 \leq k \leq n) \wedge (2|k \ \vee \ 3|k \ \vee \ 5|k)\}.$$

a) Write a function (in GAP or in pseudocode) that returns a list whose elements are the elements of $S(n)$. For example, `S(12) = [2,3,4,5,6,8,9,10,12]`. (Either write a simple loop, or use `Filtered()`.)

b) Using inclusion-exclusion and further reasoning, find a formula for the cardinality $|S(n)|$ in terms of n.

10. a) Study the variation of the function $f(x) = x^{-1/2} \ln x$ for $x \geq 1$ and find a specific constant $A \in \mathbf{R}$ such that $\forall x \geq 1,\ 0 \leq f(x) \leq A$.

b) Find **explicit constants** $c, C, k > 0$ such that $\forall x \geq k,\ cx \leq x + \ln x \leq Cx$. (This shows that $x + \ln x$ is $\Theta(x)$. Note that you may express c, C, k in terms of A even if you were not able to solve part (a) above.)

11. Answer the following exercise in GAP or in pseudocode. You are **not** allowed to use high-level builtin GAP functions such as `ForAny()` or `Filtered()`. However, you **are** allowed to use `IsEmpty()`, `addfirst()`, `addlast()`, `removefirst()`, and `removelast()` just as in the example file `example3.txt` on recursion accompanying the last computer assignment.

a) Write a **recursive** function `myforany(ll,P)` which returns `true` if and only if there exists some element `x` in the list `ll` for which `P(x)` is true.

b) Write a **recursive** function `myfiltered(ll,P)` which returns the sublist of the list `ll` consisting of all the elements `x` (in order) in `ll` for which `P(x)` is true.

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