AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 211
FINAL EXAM
Spring 2007-2008
Wednesday June 4, 11:30 am
Closed Book, 2H

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

1. (20 points) Let $L(a_n) = a_n + 5a_{n-1} + 6a_{n-2}$. Solve the following recurrence relations.

   (a) (10 points) Find the general solution of:

$$L(a_n) = a_n + 5a_{n-1} + 6a_{n-2} = 4^n,\ n \geq 2.$$

(b) (10 points) Find the solution to:

$$L(a_n) = a_n + 5a_{n-1} + 6a_{n-2} = (-2)^n,\ n \geq 2 \quad a_0 = 0, \quad a_1 = 1.$$

2. (20 points) Let a and b be two positive integers greater than 1 and let P be the set of all prime numbers

$$P = \{p_1, p_2, p_3, \cdots p_n \cdots\}$$

where $p_1 = 2$, $p_2 = 3$, $\cdots$.

(a) (3 points) Write, in terms of P, the prime number factorization of any integer a, $(a \geq 2)$.

(b) (4 points) Prove that the least common multiplier of a and b, $lcm(a, b)$, and their greatest common divisor $gcd(a, b)$ verify:

$$lcm(a, b) \times gcd(a, b) = a \times b$$

(c) (3 points) Let $a = 882$ and $b = 308$. Find the prime factorization of a and b.

(d) (2 points) Find $lcm(a, b)$ and $gcd(a, b)$.

(e) (2 points) Are a and b relatively prime (justify your answer).

(f) (6 points) Let n and d be positive integers. How many positive integers not exceeding n are divisible by d? (justify your answer)

3. (20 points) Solve the following counting problems.

   (a) (6 points) 8 women and 12 men want to form a committee of 5 persons in which there are at least 2 men and 2 women.

      i. (3 points) In how many ways can they form the committee?

      ii. (3 points) If Mr. X and Mrs. Y refuse to work together, how many committees can be formed?

   (b) (9 points) Four married couples have bought 8 seats in the same row for a concert. In how many different ways can they be seated:

      i. (3 points) With no restrictions.

ii. (3 points) If each couple is to sit together.

iii. (3 points) If all the men sit together next to the right of all the women.

(c) (5 points) Using pigeonhole principle find the minimum number of students required in Math 211 to be sure that at least six will receive the same grade assuming that there are exactly five possible grades, A, B, C, D and F?

4. (20 points) Let A be a finite set of objects, $|A| = n$ and let $f : A \rightarrow \mathbb{N}$ be a function which domain is the power set of A (Powerset(A) - all subsets of A) and its codomain the set of natural numbers $\{m \in \mathbb{N} | 0 \leq m \leq 2^n\}$, whereby,

$$\text{for } s \subseteq A,\ f(s) \text{ is the cardinality of set } s$$

(a) (5 points) Let $A = \{1, 2, 3\}$. Complete the following table thats assigns the value of f for any subset of A:

Subset s of A	$f(s)$

(b) (5 points) In general, is f one-to-one? Onto?

(c) Let R be a relation on the powerset of A defined by:

$$s\,R\,t \text{ if } f(s) = f(t)$$

i. (6 points) Is R an equivalence relation?

ii. (4 points) Find the equivalence class of $\{a, b\}$ when $A = \{a, b, c, d\}$.

5. (20 points) For $n \in \mathbb{N}$, let S_n be the set of bits strings (i.e. made of o's and 1's).

(a) (5 points) Give a recursive definition of S_n, by completing the following pseudocode:
function $S(n)$
`% Return a set of bit strings of length n`

- if $n = 0$ then $S(n) = \{\Lambda\}$ (Λ is the empty string)
- else $S(n) = \{tx | t \in \qquad\qquad , x \in \qquad\qquad \}$
- end

`% Note that x is a bit, whereas t is a string`

Let $s_n = |S_n|$, the cardinality of S_n. Find a recurrence relation on $\{s_n, n \geq 0\}$ with initial condition then solve this relation.

(b) (3 points) Let T_n be the subset of S_n, whereby T_n is the set of strings that have at least one occurrence of the 2 bits pattern 11. Complete the following table:

n	The set T_n
0	{} (empty set)
1	
2	{ }
3	{ }

(c) (6 points) Give a recursive definition of T_n, by completing the following pseudocode (that should use the previous one on $S(n)$):
function $T(n)$
`% Return a set of bit strings of length n`

- if $n = 0$ or $n = 1$ then $T(n) = \{\}$

- else if $n \geq 2$, $T(n) = \{t0|t \in \qquad\qquad\}\cup\{t11|t \in \qquad\qquad\}\cup\{\qquad\qquad\qquad\}$

- end

(d) (6 points) Let $t_n = |T_n|$, the cardinality of T_n.

- Using (c) <u>OR</u> independently, find the recurrence relation and the initial conditions verified by $\{t_n\}$.

- Solve the resulting recurrence relation.