AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 211
FINAL EXAMINATION
SPRING 2006
Closed Book, 2 HOURS

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	15	
2	10	
3	15	
4	10	
5	10	
6	10	
TOTAL	70	

1. (10 points)

   (a) (5 points) Let m be a positive integer, show that for $a, b \in \mathbb{Z}$, $a \equiv b(\ \mathbf{mod}\ m)$ if and only if $a\ \mathbf{mod}\ m = b\ \mathbf{mod}\ m$.

   (b) (5 points) Use mathematical induction to prove that $\frac{1}{2n} \leq \frac{1.3.5....(2n-1)}{2.4...(2n)}$, whenever n is a positive integer.

   (c) (5 points) Show that if $2^n - 1$ is prime then n is prime.

2. (10 points) Let S be the set of **all bit strings of length at least equal to 3**. Consider the relation R on S, such that :

$$R = \{(x, y) | \, x \text{ and } y \text{ agree in their first 3 bits.}\}$$

(a) (3 points) Show that R is an equivalence relation on S.

(b) (7 points) Is S finitely or infinitely partitioned by R? Specify the classes of equivalences of R.

3. (15 points) Consider the recurrence relation :

(1) $$a_n = a_{n-1} + a_{n-2}$$

(a) (2 points) Find the general solution of (1).

(b) (3 points) For which initial conditions a_0 and a_1, do we get the **Fibonacci sequence** $\{f_n\}$. Give the general expression of f_n.

(c) (3 points) Find a such that $f_n = \Theta(a^n)$.

(d) (5 points) Solve the recurrence relation :

$$a_n = a_{n-1} + a_{n-2} + 2^n,\ a_0 = 0,\ a_1 = 1.$$

(e) (2 points) Find b such that $a_n = \Theta(b^n)$.

4. (10 points) Given a sequence of $n+1$, $(n \geq 1)$ real numbers $\{a(0), a(1), ..., a(n)\}$ and a real number c, consider the following algorithm :

```
p :=a(n)
for i from n-1 downto 0 {,i.e. i=n-1,...,1,0}
    p=c*p+a(i)
end
```

(a) (3 points) Check which function `p(c)` is computed for $n = 3, 2, 1$?

(b) (2 points) Give $f(n)$ the number of artithmetic operations needed to execute this algorithm.

(c) (2 points) Give α in $f(n) = \Theta(n^\alpha)$.

(d) (3 points) (independent from (a) and (b)) Assume that $f(n) = \Theta(n^3 \ln(n))$ and $g(n) = \Theta(n^{0.5})$. Give $\Theta(f(n) + g(n))$ and $\Theta(\frac{f(n)}{g(n)})$.

5. (10 points) Let S be a set with n elements.

(a) (3 points) How many subsets of S have 3 elements?

(b) (7 points) How many subsets of S have more than 2 elements?

6. (10 points) A street named `QuizStreet` has addresses numbered from 100 to 199.

   (a) (5 points) Find the number of **disjoint distinct subsets** of the form $\{x, x+1\}$.

   (b) (5 points) Show that among 51 distinct addresses in `QuizStreet`, 2 must be consecutive.