AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 211
FINAL EXAMINATION
SPRING 2007
Closed Book, 2 HOURS

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	5	
2	10	
3	10	
4	10	
5	15	
6	10	
7	10	
TOTAL	70	

1. (5 points) Express this statement using quantifiers on specific universe of discourses :

   "There is exactly one student in this class who has taken exactly one mathematics course at this school".

2. (10 points) Let $X_n = \{x_1, x_2, ..., x_n\}$ be a set of n objects forming the universe of discourse for the propositional functions (predicates) $P(x)$ and $Q(x)$. Prove by (soft) mathematical induction that :

$$\exists x \in X_n, (P(x) \rightarrow Q(x)) \text{ and } (\forall x \in X_n, P(x)) \rightarrow (\exists x \in X_n, Q(x))$$

are logically equivalent.

Specify the base and inductive steps. (Hint : Use the equivalence $p \rightarrow q \equiv \neg p \vee q$.)

3. (10 points) Prove that for all $x \in \mathbb{R}$:

$$\lfloor \lfloor x/2) \rfloor /2 \rfloor = \lfloor x/4 \rfloor$$

Hint : use $x/4 = \lfloor x/4 \rfloor + \alpha$.

4. (10 points) In what follows, a, b, c and m are integers with $m > 1$.

   (a) (5 points) Prove or disprove the following statement : If $ac \equiv bc (mod\, m)$, then $a \equiv b (mod\, m)$.

   (b) (5 points) Show that if $ac \equiv bc (mod\, m)$, then $a \equiv b (mod\, (m/d))$, where $d = gcd(m, c)$.

5. (15 points) Consider the recurrence relation :

$$a_n = a_{n-1}^3 a_{n-2}^2, \text{ with } a_0 = 2 \text{ and } a_1 = 2 \tag{1}$$

(a) (3 points) Use logarithms to change (1) into a linear recurrence relation for ($b_n = \log_2(a_n)$). Find such recurrence relation for $\{b_n\}$:

(2) ....

(b) (8 points) Solve (2) and find the expressions of b_n and a_n.

(c) (4 points) Find functions $f(n)$ and $g(n)$ such that $b_n = \Theta(g(n))$ and $a_n = \Theta(f(n))$.

6. (10 points)

   (a) (5 points) How many strings of length 10 over the alphabet $\{a, b, c\}$ have exactly 3 a's or exactly 4 b's ?

   (b) (5 points) There are 12 signs in the zodiac. How many people are needed to guarantee that at least six of these people have the same sign ?

7. (10 points) Given a sequence of $n+1$, $(n \geq 1)$ real numbers $\{a(0), a(1), ..., a(n)\}$ and a real number c, consider the following algorithm :

```
p :=a(0) t :=c
for i from 1 to n {,i.e. i=1,2,...,n-1,n}
    p=p+a(i)*t
    t=c*t
end
```

(a) (3 points) Check which function `p(c)` is computed for $n = 3, 2, 1$?

(b) (5 points) Give $f(n)$ the number of artithmetic operations needed to execute this algorithm.

(c) (2 points) Give a and α in $f(n) = \Theta(a \times n^{\alpha})$.