Math 211 — Discrete Structures, Fall 2006–07
Course website: `http://people.aub.edu.lb/~kmakdisi/`
Solutions to Quiz 1

1. (total 12 pts) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 5, 10\}$.
a) (2 pts) Find $A \cup B$.
b) (2 pts) Find $A \cap \overline{B}$. (The answer does not depend on the choice of universal set.)
c) (4 pts) Write the following sentence using symbols, and determine whether it is true or false (briefly say why): "Every element of A is a divisor of some element of B"
d) (4 pts) Write the following sentence using symbols, and determine whether it is true or false (briefly say why): "Some element of A is a divisor of every element of B"

Answer. (a) $A \cup B = \{x \mid (x \in A) \vee (x \in B)\} = \{1, 2, 3, 5, 10\}$.
(b) $A \cap \overline{B} = \{x \mid (x \in A) \wedge (x \notin B)\} = \{3\}$.
(c) $\forall x \in A, \quad \exists y \in B$ s.t. $x|y$. (Alternatively: $\forall x \in A, \quad \exists y \in B, \quad \exists n \in \mathbf{Z}$ s.t. $y = nx$.
This statement is false since $3 \in A$ and $\neg(3|1) \wedge \neg(3|2) \wedge \neg(3|5) \wedge \neg(3|10)$.
(d) $\exists x \in A$ s.t. $\forall y \in B, \quad x|y$.
This statement is true since $1 \in A$ and $(1|1) \wedge (1|2) \wedge (1|5) \wedge (1|10)$.

2. (total 12 pts) Let $f : \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f(n) = \left\lceil \frac{n}{3} \right\rceil$.
a) (2 pts) Find the image set $f(\{1, 3, 4, 100\})$.
b) (2 pts) Find the inverse image set $f^{-1}(\{-1, 10\})$.
c) (4 pts) Is f an injection? Why or why not (give a brief explanation)?
d) (4 pts) Is f a surjection? Why or why not (give a brief explanation)?

Answer. (a) $f(1) = 1$, $f(3) = 1$, $f(4) = 2$ and $f(100) = 34$. Hence $f(\{1, 3, 4, 100\}) = \{1, 2, 34\}$.
(b) $f^{-1}(\{-1, 10\}) = \left\{n \in \mathbf{Z} \mid \left\lceil \frac{n}{3} \right\rceil \in \{-1, 10\}\right\}$. So for $n \in \mathbf{Z}$, we have

$$n \in f^{-1}(\{-1, 10\}) \leftrightarrow \left(\left\lceil \frac{n}{3} \right\rceil = -1\right) \vee \left(\left\lceil \frac{n}{3} \right\rceil = 10\right)$$
$$\leftrightarrow \left(-2 < \frac{n}{3} \leq -1\right) \vee \left(9 < \frac{n}{3} \leq 10\right) \leftrightarrow (-6 < n \leq -3) \vee (27 < n \leq 30).$$

This means that $f^{-1}(\{-1, 10\}) = \{-5, -4, -3, 28, 29, 30\}$.
(c) f is not an injection since $f(1) = f(2) = 1$ even though $1 \neq 2$. In other words, there are distinct elements of $\mathbf{Z}$ having the same image by f.
(d) f is a surjection since for all $n \in \mathbf{Z}$, we can find a preimage $k = 3n$ for which $f(3n) = n$. Hence every integer n has at least one preimage. (Note that the other preimages of n are $3n - 1$ and $3n - 2$.)

3. (total 12 pts) Show that $p \to (q \to r)$ is logically equivalent to $(p \wedge q) \to r$ in two different ways, (a) and (b+c):
a) (4 pts) Use a truth table.
b) (4 pts) Give a direct proof that if $p \to (q \to r)$, then $(p \wedge q) \to r$.
c) (4 pts) Give a direct proof of the converse of the statement in (b).

Answer. (a) Since the fifth and seventh columns below are identical, we obtain the desired logical equivalence.

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T

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F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

(b) We shall give a fairly formal proof, to emphasize the structure of the argument (e.g., where we make an assumption, where we make a subassumption, and so forth). Note how we use indentation to show where an internal argument begins and ends.

[1] Assumption: $p \to (q \to r)$
[2] Sub-assumption: $p \wedge q$
[3] therefore p (from [2])
[4] and q (also from [2])
[5] therefore $q \to r$ (combining [3] with [1])
[6] therefore r (combining [4] with [5])
[7] The internal argument in steps [2–6] shows that $(p \wedge q) \to r$.
The above reasoning in [1–7] proves that $[p \to (q \to r)] \to [(p \wedge q) \to r]$.

(c) Same comment as for part (b). Note that here we have an internal argument and an "internal internal argument".

[1] Assumption: $(p \wedge q) \to r$
[2] Sub-assumption: p
[3] Sub-sub-assumption q
[4] therefore $p \wedge q$ (combining [2] with [3])
[5] therefore r (combining [4] with [1])
[6] The internal internal argument in steps [3–5] shows that $q \to r$
[7] The internal argument in steps [2–6] shows that $p \to (q \to r)$.
The above reasoning in [1–7] proves that $[(p \wedge q) \to r] \to [p \to (q \to r)]$.

(Wrapping up b and c) We have shown implications in both directions:
if $[p \to (q \to r)]$ then $[(p \wedge q) \to r]$, if $[(p \wedge q) \to r]$ then $[p \to (q \to r)]$.
Therefore $[p \to (q \to r)] \equiv [(p \wedge q) \to r]$.

Remark: There is another way to do (b) and (c) simultaneously, without doing everything we did above. Use the tautology $x \to y \equiv \neg x \vee y$ and De Morgan's rules to show that $(p \wedge q) \to r \equiv \neg p \vee \neg q \vee r \equiv p \to (q \to r)$. The details are left to you.

4. (total 12 pts) Given the sequence $a_1, a_2, a_3, \ldots$ of real numbers, defined (recursively) by:

$$a_1 = \sqrt{2}, \qquad \forall n \geq 1, \quad a_{n+1} = 1 + \frac{1}{a_n}.$$

a) (6 pts) Show that $\forall n \geq 1, \quad 1 < a_n < 2$.

b) (6 pts) Show that $\forall n \geq 1, \quad a_n$ is irrational. (You already know that $\sqrt{2}$ is irrational, so just prove the inductive step.)

Answer. (a) Let $P(n)$ be the statement $1 < a_n < 2$. We use induction to prove that $\forall n \geq 1, \quad P(n)$. First we prove the base case: $P(1)$ states that $1 < \sqrt{2} < 2$, which is true because we can deduce it by taking square roots of the inequalities $1 < 2 < 4$. Second, we prove the inductive step: let $n \geq 1$ be arbitrary, and assume $P(n)$. Thus $1 < a_n < 2$. Taking reciprocals shows that $1/2 < 1/a_n < 1$, and then adding 1 gives $3/2 < a_{n+1} = 1 + 1/a_n < 2$. This implies that $1 < a_{n+1} < 2$, so we obtain $P(n+1)$. The preceding discussion shows that $\forall n \geq 1, \quad P(n) \to P(n+1)$. Thus we have proved the inductive step. Now we combine the base case and the inductive step to conclude $\forall n \geq 1, \quad P(n)$ by induction. This completes the desired proof.

(b) Let $Q(n)$ be the statement that a_n is irrational. We already know $Q(1)$ from class, so we obtain the base case. The best way to prove the inductive step is by proving the **contrapositive** as follows. Let n be arbitrary and assume that a_{n+1} is **rational**. Then there exist $p, q \in \mathbf{Z}$ such that $a_{n+1} = p/q$. But this means that $1 + 1/a_n = p/q$, in other words that $1/a_n = (p-q)/p$, and hence that $a_n = p/(p-q)$, with $p, p-q \in \mathbf{Z}$. This shows that a_n is rational. [Note that $p \neq q$ because we have already proved that the terms in the sequence are never equal to 1.] The above discussion shows that $\forall n \geq 1, \quad (\neg Q(n+1)) \to (\neg Q(n))$. Replacing with the contrapositive shows that $\forall n \geq 1, \quad Q(n) \to Q(n+1)$. This proves the inductive step. Combining the base case and the inductive step, we obtain by induction that $\forall n \geq 1, \quad Q(n)$, which is what we wanted to prove.

5. (total 12 pts) Given sets X, Y, and a function $f : X \to Y$. Prove the following statements **FROM THE DEFINITION** of the image set and inverse image sets. I.e., you are **NOT ALLOWED** to use identities such as $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, unless you include a proof of the identities you use. (It is possible to give a short solution that does not involve proving these identities first.)

a) (6 pts) Assume that $A, B, C \subseteq X$ satisfy $A \subseteq C$ and $B \subseteq C$. Show that $f(A) \cup f(B) \subseteq f(C)$.

b) (6 pts) Assume that $S, T \subseteq Y$ satisfy $S \cap T = \emptyset$. Show that $f^{-1}(S) \cap f^{-1}(T) = \emptyset$.

Answer. (a) We need to show an inclusion of sets. So we must start with an arbitrary element $y \in f(A) \cup f(B)$, and somehow conclude that y must belong to $f(C)$. This will show that $\forall y, \quad y \in f(A) \cup f(B) \to y \in f(C)$, which will prove what we want. This is our plan for the proof.

We now carry out the plan: let $y \in f(A) \cup f(B)$ be arbitrary. We have **two cases**, (i) $y \in f(A)$ and (ii) $y \in f(B)$. If we are in case (i), then $\exists x \in A$ s.t. $y = f(x)$, by the definition of the set $f(A)$. But since $A \subseteq C$ and $x \in A$, we obtain that $x \in C$. Thus $y = f(x)$ for this specific $x \in C$, and this shows that $y \in f(C)$. On the other hand, if we are in case (ii), then $\exists x \in B$ s.t. $y = f(x)$. Once again, since $B \subseteq C$, we obtain $x \in C$ and $y = f(x) \in f(C)$. Thus in **both cases** we can conclude that $y \in f(C)$. Hence we have shown that $\forall y, \quad y \in f(A) \cup f(B) \to y \in f(C)$, and this means that $f(A) \cup f(B) \subseteq f(C)$.

(b) This is easiest to prove by **contradiction**. Assume that $f^{-1}(S) \cap f^{-1}(T) \neq \emptyset$. Then $\exists x \in f^{-1}(S) \cap f^{-1}(T)$. In particular, $x \in f^{-1}(S)$, and so by the definition of $f^{-1}(S)$ we conclude that $f(x) \in S$. Similarly, $x \in f^{-1}(T)$ and this implies that $f(x) \in T$. Thus we have produced an element $f(x) \in S \cap T$, contradicting the given fact that $S \cap T = \emptyset$. The above reasoning by contradiction shows that $f^{-1}(S) \cap f^{-1}(T) = \emptyset$ after all.

6. (12 pts) Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. How many functions $f : A \to B$ are there such that EITHER $\Big[f(a) = 1\Big]$ OR $\Big[f(a) = 2 \text{ AND } f \text{ is injective}\Big]$?

Answer. Define two sets $\mathcal{F} = \{\text{functions } f : A \to B \mid f(a) = 1\}$ and $\mathcal{G} = \{\text{functions } f : A \to B \mid (f(a) = 2) \wedge (f \text{ injective})\}$. Our task is to compute $|\mathcal{F} \cup \mathcal{G}|$. Note that $\mathcal{F} \cap \mathcal{G} = \emptyset$, since a given function cannot simultaneously satisfy $f(a) = 1$ and $f(a) = 2$ (whether or not f is injective). Thus $|\mathcal{F} \cup \mathcal{G}| = |\mathcal{F}| + |\mathcal{G}|$ by the addition principle.

We now compute $|\mathcal{F}|$ as follows: to choose $f \in \mathcal{F}$, we already know $f(a) = 1$, and we need to choose $f(b) \in \{1, 2, 3, 4\}$, followed by $f(c) \in \{1, 2, 3, 4\}$, followed by $f(d) \in \{1, 2, 3, 4\}$. By the multiplication principle, our total choices are $4 \cdot 4 \cdot 4 = 64$, and so $|\mathcal{F}| = 64$. [Note: this result can also be obtained by making a bijection between $\mathcal{F}$ and the set of functions from $\{b, c, d\}$ to $\{1, 2, 3, 4\}$.]

We next compute $|\mathcal{G}|$. Once again, for $f \in \mathcal{G}$, we already know $f(a) = 2$. We must now choose $f(b) \in B - \{f(a)\} = \{1, 3, 4\}$ because $f(b)$ must be different from $f(a)$, so there are 3 choices for $f(b)$. We then choose $f(c) \in B - \{f(a), f(b)\}$, which gives us 2 choices for $f(c)$. Lastly we choose $f(d) \in B - \{f(a), f(b), f(c)\}$, which gives us 1 choice for $f(d)$. All in all we obtain $3 \cdot 2 \cdot 1 = 6$ choices for f, and so $|\mathcal{G}| = 6$. [Note: this result can also be obtained by making a bijection between $\mathcal{G}$ and the set of injections from $\{b, c, d\}$ to $\{1, 3, 4\}$.]

Putting all of the above together, we obtain that $|\mathcal{F} \cup \mathcal{G}| = 64 + 6 = 70$.