Math 211 — Fall 2007–08
Discrete Structures
Quiz 1, October 26 — Duration: 1 hour

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME: SOLUTIONS

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation W 4	Recitation W 11	Recitation Th 12:30	Recitation Th 3:30
Professor Makdisi	Ms. Karam	Ms. Karam	Ms. Karam

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your reasoning precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.
3. You may use the back of each page for scratchwork OR for solutions. There are two extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Open book and notes. ~~NO~~ CALCULATORS ARE ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

1. a) (6 pts) Show the logical equivalence $p \wedge (q \to r) \equiv (\neg p \vee q) \to (p \wedge r)$ USING a truth table:

p	q	r	$q \to r$	$p \wedge (q \to r)$	$\neg p \vee q$	$p \wedge r$	$(\neg p \vee q) \to (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	T	T	F	T	T
T	F	F	T	T	F	F	T
F	T	T	T	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	T	F	T	F	F
F	F	F	T	F	T	F	F

the circled columns are identical, so $p \wedge (q \to r) \equiv (\neg p \vee q) \to (p \wedge r)$.

b) (6 pts) Give a direct proof (WITHOUT using a truth table or other logical equivalence) that if $p \wedge (q \to r)$, then $(\neg p \vee q) \to (p \wedge r)$.

$\boxed{\text{Given}}$ $p \wedge (q \to r)$ ①
we can deduce p ②
and we can deduce $q \to r$ ③.

Hypothesis: assume $\neg p \vee q$ ④
Combining ④ & ②, deduce q ⑤
from ⑤ & ③, $\therefore r$ ⑥
from ① & ⑥, get $\therefore p \wedge r$ ⑦

the reasoning in ④–⑦ shows us that
$(\neg p \vee q) \to (p \wedge r)$. (the desired conclusion) ⑧

2. Let $f : \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f(n) = \left\lfloor \frac{n^2}{10} \right\rfloor$.

a) (3 pts) Find the image set $f(\{-3, 3, 6, 8, 10\})$.

This is $\{f(-3), f(3), f(6), f(8), f(10)\}$

$= \{\lfloor \frac{9}{10} \rfloor, \lfloor \frac{9}{10} \rfloor, \lfloor \frac{36}{10} \rfloor, \lfloor \frac{64}{10} \rfloor, \lfloor \frac{100}{10} \rfloor\}$

$= \{0, 0, 3, 6, 10\}$

$= \{0, 3, 6, 10\}$

b) (5 pts) Find the inverse image set $f^{-1}(\{0, 5\})$.

This is $\{n \in \mathbb{Z} \mid f(n) \in \{0, 5\}\}$

$= \{n \in \mathbb{Z} \mid \lfloor \frac{n^2}{10} \rfloor = 0 \;\vee\; \lfloor \frac{n^2}{10} \rfloor = 5\}$

$= \{n \in \mathbb{Z} \mid (0 \leq \frac{n^2}{10} < 1) \;\vee\; (5 \leq \frac{n^2}{10} < 6)\}$

$= \{n \in \mathbb{Z} \mid (0 \leq n^2 < 10) \;\vee\; (50 \leq n^2 < 60)\}$

$= \{n \in \mathbb{Z} \mid \underbrace{|n| < \underset{\text{between } 3 \,\&\, 4}{\sqrt{10}}}_{\text{this means } |n| \in \{0,1,2,3\}} \;\vee\; \underbrace{\underset{\text{between } 7 \,\&\, 8}{\sqrt{50}} \leq |n| < \underset{\text{between } 7 \,\&\, 8}{\sqrt{60}}}_{\text{there are no such integers } |n|.}\}$

$= \{-3, -2, -1, 0, 1, 2, 3\}$

c) (4 pts) Explain why f is neither an injection nor a surjection.

f is not an injection because $-3 \neq 3$ but $f(-3) = f(3) = 0$

(many other examples of $x_1, x_2 \in \mathbb{Z}$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$ are possible).

f is not a surjection because the argument in part (b) above shows that there does not exist any $n \in \mathbb{Z}$ for which $\lfloor \frac{n^2}{10} \rfloor = 5$.

So 5 has no preimage.

(again, other counterexamples are possible.)

3. Given the sets $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{2, 3, 5\}$.

a) (3 pts) Find $A \cup (B - C)$.

$B - C = \{3, 4, 5, 6\} - \{2, 3, 5\} = \{4, 6\}$ $\quad (B - C = \{x \mid x \in B \wedge x \notin C\})$

$A \cup (B - C) = \{1, 2, 3, 4\} \cup \{4, 6\} = \{1, 2, 3, 4, 6\}$.

b) (3 pts) Is the following statement true? Explain why or why not.

$$\forall x \in A, \; \underbrace{\exists y \in C \text{ s.t. } (x = y + 1 \;\vee\; x = y - 1)}.$$

Check for all $x \in A$ whether this predicate $P(x)$ is true.

$x = 1$: does there exist $y \in C$ s.t. $(1 = y + 1) \vee (1 = y - 1)$? yes, take $y = 2$ so $x = y - 1$

$x = 2$: yes, take $y = 3$ so $x = y - 1$

$x = 3$: yes, take $y = 2$ so $x = y + 1$

$x = 4$: yes, take $y = 5$ so $x = y - 1$ [we can also take $y = 3$]

Since $P(x)$ is true in each of the cases $x = 1$, $x = 2$, $x = 3$, $x = 4$, for all elts of A, we conclude that <u>YES</u>, the above statement is true.

c) (3 pts) Is the following statement true? Explain why or why not.

$$\forall x \in B, \; \exists y \in C \text{ s.t. } (x = y + 1 \;\vee\; x = y - 1).$$

Check the same thing as in part (b) above, for $x \in \underline{B}$:

for $x = 3$, $P(x)$ is true since we can take $y = 2$

for $x = 4$, $P(x)$ is true since we can take $y = 3$

(you don't have to mention these in your solution)

<u>But</u> for $x = 5$, $P(x)$ is <u>FALSE</u> since $\begin{cases} x = y + 1 \text{ can only happen for } y = 4 \\ x = y - 1 \text{ can only happen for } y = 6 \end{cases}$ <u>and</u> neither 4 nor 6 belongs to C

So the answer is NO, $P(x)$ is not always true for $x \in B$, since $P(5)$ is false!

of course, you must mention this case ($x = 5$) in your solution.

d) (3 pts) Briefly explain why the following statement is FALSE:

$$\exists x \in \mathbf{Z} \text{ s.t. } \forall y \in C, \; (\overset{(y = x - 1)}{x = y + 1} \;\vee\; \overset{(y = x + 1)}{x = y - 1}).$$

This asks: can there exist some $x \in \mathbb{Z}$ that satisfies $(y \in C \rightarrow y = x - 1 \vee y = x + 1)$?

this would be saying that $C \subseteq \{x - 1, x + 1\}$

But C has 3 elements while $\{x - 1, x + 1\}$ is a set with 2 elements only (no matter what x is chosen).

<u>Hence</u> <u>No such x can exist</u> & the above statement is FALSE.

4. Let A, B, C be sets.

a) (4 pts) Show that if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Given: $A \subseteq B$, meaning: $\forall x, (x \in A \rightarrow x \in B)$ ①

Given: $A \subseteq C$, meaning: $\forall x, (x \in A \rightarrow x \in C)$ ②

Let x be arbitrary and <u>assume</u> $x \in A$ ③

Combining ③ and ①, we obtain that $x \in B$ ④

Combining ③ and ②, we obtain that $x \in C$ ⑤

④ and ⑤ say that $x \in B \cap C$ ⑥

The reasoning in ③–⑥ shows us that

$\forall x, (x \in A \rightarrow x \in B \cap C)$. ⑦

This means that $A \subseteq B \cap C$, which is what we wanted to prove.

b) (8 pts) UNRELATED: Show that if $A \cap B = \emptyset$ and $A \subseteq B \cup C$, then $A \subseteq C$.

Given: $A \cap B = \emptyset$ ①

Given: $A \subseteq B \cup C$, which means

$\forall x, [x \in A \longrightarrow (x \in B \vee x \in C)]$ ②

Let x be arbitrary & assume that $x \in A$ ③

By ② and ③, we obtain that $x \in B \vee x \in C$ ④

Subassumption: suppose $x \in B$. ⑤

Combining ③ and ⑤, we obtain $x \in A \cap B$ ⑥

But ⑥ <u>contradicts</u> ①, since we know $A \cap B$ cannot contain any elements. ⑦

The proof by contradiction in ⑤–⑦ tells us that $x \notin B$. ⑧

Combining ④ and ⑧, we deduce that <u>$x \in C$</u>. ⑨

The reasoning in steps ③–⑨ shows us that

$\forall x, (x \in A \longrightarrow x \in C)$. ⑩

This is the same as saying that $A \subseteq C$. Our proof is now complete.

5. (12 pts) Let x be an irrational number. Prove the statement

$$\forall a \in \mathbb{Q}, \quad (a \neq 0 \leftrightarrow ax \text{ is } \underline{\text{irrational}}).$$

Note that this is an <u>if and only if</u> statement — means $ax \notin \mathbb{Q}$

Work directly from the definition of rational numbers in terms of integers. Do NOT assume any other properties of rational or irrational numbers.

We are given that $x \notin \mathbb{Q}$. ⓪

Recall that $\mathbb{Q} = \{r \in \mathbb{R} \mid \exists m, n \in \mathbb{Z} \text{ s.t. } (n \neq 0 \wedge r = \frac{m}{n})\}$.

We must show two statements, for arbitrary $a \in \mathbb{Q}$:

$\boxed{I}$ Show that $a \neq 0 \longrightarrow ax \notin \mathbb{Q}$.

Proof by contradiction:

We are given $a \neq 0$, and $a \in \mathbb{Q}$ ①

Suppose $ax \in \mathbb{Q}$. ②

From ①, we know $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$ and $a = m/n$. ③

Furthermore, since $a \neq 0$, we have $m \neq 0$ ④

Since $ax \in \mathbb{Q}$, we know $\exists s, t \in \mathbb{Z}$ s.t. $t \neq 0$ and $ax = \frac{s}{t}$ ⑤

We conclude that

$$x = \frac{ax}{a} = \frac{(s/t)}{(m/n)} = \frac{sn}{tm} \quad ⑥$$

(note how we used $a \neq 0$ to allow division by a and to conclude $tm \neq 0$.)

Here in ⑥, we know $sn, tm \in \mathbb{Z}$ and $tm \neq 0$ (since $s, t, m, n \in \mathbb{Z}$ and t, m are both nonzero).

So ⑥ says that $x \in \mathbb{Q}$, contradicting the original given ⓪. ⑦

This argument (②–⑦) shows us that $ax \notin \mathbb{Q}$ by contradiction. ⑧

The reasoning ①–⑧ proves $\boxed{I}$.

$\boxed{II}$ Show that $ax \notin \mathbb{Q} \longrightarrow a \neq 0$. (you <u>MUST</u> prove the converse also!!)

This can also be done by contradiction (if a were $= 0$, we would have $ax = 0 \cdot x = 0 \in \mathbb{Q}$, contradiction.)

<u>OR</u>: Suppose $ax \notin \mathbb{Q}$.

Since $0 \in \mathbb{Q}$, we conclude $ax \neq 0$.

this forces $a \neq 0$.

by either of the above proofs, we obtain $\boxed{II}$

& combining $\boxed{I}$ and $\boxed{II}$ gives us

6. Let $f : X \to Y$ be a function.

a) (6 pts) Given subsets $A, B, C \subseteq X$. Show that if $A \subseteq B \cup C$, then $f(A) \subseteq$ ~~$f(B \cup C)$~~

CORRECTION: $f(B) \cup f(C)$

Our given states that $A \subseteq B \cup C$, in other words

$\forall x, (x \in A \to x \in B \vee x \in C)$. ①

Let y be an arbitrary element of $f(A)$. ②

Then by the definition $f(A) = \{y_{\in Y} \mid \exists x \in A \text{ s.t. } y = f(x)\}$

we conclude $\exists x \in A$ s.t. $y = f(x)$. ③

For this x, we have $x \in A$, so combining with ① we obtain $x \in B \vee x \in C$. ④

Case 1
if $x \in B$:
then $y = f(x)$ but $x \in B$
so $\exists x \in B$ s.t. $y = f(x)$
so $y \in f(B)$, hence $y \in f(B) \cup f(C)$

Case 2
if $x \in C$:
Similarly $\exists x \in C$ s.t. $y = f(x)$,
so $y \in f(C)$, so $y \in f(B) \cup f(C)$.

In both cases, $y \in f(B) \cup f(C)$. ⑤

NOTE you MUST start with this. starting with $x \in A$ and considering $f(x)$ (as many of you did) is a BAD IDEA...

the reasoning in ②–⑤ shows that

$\boxed{\forall y,\ y \in f(A) \to y \in f(B) \cup f(C)}$ ⑥

which means that

$f(A) \subseteq f(B) \cup f(C)$

as desired.

b) (6 pts) Given a subset $T \subseteq Y$. Show that if f is surjective, then $f(f^{-1}(T)) = T$.

Given f is surjective, I.E. $\forall y \in Y, \exists x \in X$ s.t. $y = f(x)$ ①.

Part I Show that $f(f^{-1}(T)) \underset{??}{\subseteq} T$

Let y be an arbitrary element of $f(f^{-1}(T))$. ②

By def. of f(a set), this means

$\exists x \in f^{-1}(T)$ s.t. $y = f(x)$. ③

Since $x \in f^{-1}(T)$, this means $f(x) \in T$ ④

by def. of f^{-1}(a set).

by ③ and ④, we obtain $y \in T$ ⑤

$\therefore \forall y,\ y \in f(f^{-1}(T)) \to y \in T$ ⑥ this proves part I.

Part II Show that $T \underset{??}{\subseteq} f(f^{-1}(T))$

(continued on p. 7)

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Ex 6b, continued

we want to show that $T \underset{??}{\subseteq} f(f^{-1}(T))$ (part II).

Let y be an arbitrary element of T. ⑦

Since f is surjective, $\exists x \in X$ s.t. $y = f(x)$. ⑧

We have $f(x) = y \in T$; hence $x \in f^{-1}(T)$ ⑨

because of the definition of $f^{-1}(T)$

Thus the x we have constructed belongs to $f^{-1}(T)$ and $y = f(x)$,

So we have shown

$\exists x \in f^{-1}(T)$ st. $y = f(x)$ ⑩

this means, by definition, that $y \in f(f^{-1}(T))$. ⑪

note that surjection only tells you that you find $x \in X$

you need this step to observe that the x you found actually belongs to $f^{-1}(T)$.

The reasoning in ⑦–⑪ shows that

$\forall y,\ y \in T \to y \in f(f^{-1}(T))$. ⑫ this proves part II.

Now, combine statements ⑥ (which says that $f(f^{-1}(T)) \subseteq T$) and ⑫ (which says that $T \subseteq f(f^{-1}(T))$)

to deduce that $\boxed{f(f^{-1}(T)) = T}$, as desired.