Math 211 — Discrete Structures, Fall 2006–07
Course website: `http://people.aub.edu.lb/~kmakdisi/`
Solutions to Quiz 2

1. Note: the different parts are UNRELATED.

a) (4 pts) In how many ways can one rearrange the letters of `ALIBABA`?

b) (5 pts) A bakery sells three types of manaqish: zaatar, cheese, and kishk. In how many ways can one buy **no more than 12** manaqish? (It is possible to buy any number between zero and twelve manaqish.)

c) (5 pts) How many bitstrings of length 9 are there that have 000 at the beginning, the end, OR in the middle three positions? (In other words, the bitstring should have the form $000xxxxxx$ –**or**– $xxx000xxx$ –**or**– $xxxxxx000$.)

Answer. a) `ALIBABA` contains 3 `A`s, 2 `B`s, 1 `I`, and 1 `L` for a total of 7 letters. The number of permutations where some elements are indistinguishable is therefore $\frac{7!}{3!2!1!1!}$. (This is incidentally equal to 420.)

b) Let x_1, x_2, and x_3 be the numbers of zaatar, cheese, and kishk manaqish respectively. We want to count the cardinality $\left|\{(x_1,x_2,x_3)\in\mathbf{N}^3 \mid x_1+x_2+x_3\le 12\}\right|$. It is easier to introduce a "slack" variable $x_4 = 12-x_1-x_2-x_3\in\mathbf{N}$, and to count instead

$$\left|\{(x_1,x_2,x_3,x_4)\in\mathbf{N}^4 \mid x_1+x_2+x_3+x_4=12\}\right|.$$

This last quantity is $\binom{12+4-1}{12}$. (If you wish, you can simplify $\binom{15}{12}=\binom{15}{3}$, which is equal to 455.)

c) Let A, B, and C be the respectively sets of bitstrings of the form $000xxxxxx$, $xxx000xxx$, and $xxxxxx000$. We want to count $|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$. Note that $|A|=|B|=|C|=2^6$, since in each case we can choose six xs from $\{0,1\}$. Similarly $|A\cap B|=|A\cap C|=|B\cap C|=2^3$ because we have three xs to choose: indeed, $A\cap B$ is the set of bitstrings of the form $000000xxx$, $A\cap C$ is the set of bitstrings of the form $000xxx000$, and $B\cap C$ is the set of bitstrings of the form $xxx000000$. Finally, $|A\cap B\cap C|=1$ because $A\cap B\cap C=\{000000000\}$. Putting this all together, we see our answer is $2^6+2^6+2^6-2^3-2^3-2^3+1$ (incidentally equal to 169).

2. a) (6 pts) Show algebraically that $\binom{n}{k}\binom{n-k}{\ell}=\binom{n}{\ell}\binom{n-\ell}{k}$.

b) (8 pts) Show the above identity by a combinatorial argument.

Answer. a) $\binom{n}{k}\binom{n-k}{\ell}=\frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{\ell!(n-k-\ell)!}=\frac{n!}{k!\ell!(n-k-\ell)!}$. Exchanging the roles of k and ℓ, we again obtain $\binom{n}{\ell}\binom{n-\ell}{k}=\frac{n!}{\ell!k!(n-\ell-k)!}$, which is the same value as above. So the two values are equal, as claimed.

b) Let X be a set with $|X|=n$ elements. We shall count the number of pairs (A,B) of two **disjoint** subsets $A,B\subseteq X$ such that $|A|=k$ and $|B|=\ell$. Such a pair (A,B) can be counted either by first choosing the subset $A\subseteq X$ in $\binom{n}{k}$ ways, and then choosing the subset $B\subseteq X-A$ (to ensure that $B\cap A=\emptyset$) in $\binom{|X-A|}{\ell}=\binom{n-k}{\ell}$. Thus the total number of pairs (A,B) is $\binom{n}{k}\binom{n-k}{\ell}$. Similarly, if we choose the set B first and **then** choose the set $A\subseteq X-B$, this can be done in $\binom{n}{\ell}\binom{n-\ell}{k}$ ways. Thus the two numbers are equal.

A more concrete way to express the above solution is that we have a room with n people in it, and we wish to count in how many ways we can put k red hats and ℓ green hats on the heads of people in the room. (Either place the red hats first, then the green hats, or vice-versa.)

3. a) (3 pts) Fill in the blanks for the definition of the function `implies`. Its input are boolean (i.e., logical) values `p, q` $\in$ {`true, false`} and its output is `true` or `false` depending on whether `p`$\to$`q` is true or not.

b) (6 pts) Write a function function `sumofpositives` (in GAP or pseudocode) which takes as input a list `thelist`, and which outputs the sum of all the positive elements of `thelist`. For example, `sumofpositives([2,3,-4,5,-1]) = 10` because $2+3+5=10$.

c) (6 pts) Write a function `listofsquares` (in GAP or pseudocode) that takes as input a list `thelist`, and that outputs a list of the squares of the elements in `thelist`. For example, `listofsquares([-1,2,3,-1,1]) = [1,4,9,1,1]`. You may, if you wish, use the command `Add(ll,y)` which replaces the list `ll` by putting `y` in after the end of `ll`.

Answer. a) First, if `p = true`, then the value of `p` $\to$ `q` is the same as the value of `q`. (This is immediate from the truth table for $\to$.) On the other hand, if `p = false`, then `p` $\to$ `q` is always `true`. Thus the answer is:

```
implies := function(p,q)
        if p then return(___  q  ___);
             else return(___  true  ___);
        fi;
end;
```

b) We keep track of the sum so far in a variable `answer`. We start with `answer = 0`, and we keep adding to it every positive entry we find in `thelist`:

```
sumofpositives := function(thelist)
        local x, answer;
        answer := 0;
        for x in thelist do
             if (x > 0) then answer := answer + x; fi;
        od;
        return(answer);
end;
```

Note that we can also write the loop in terms of a variable `i` that goes from `1` to the length `Length(thelist)` of our input. In that case we examine the element `thelist[i]` instead of the element `x` above, and add it to `answer` whenever it is positive.

c) This time our variable `answer` is a list that keeps track of the squares that we have seen so far. Note that the algorithm is structurally very similar to the algorithm of part (b). This time, however, we use `Add` to put a new element at the end of the list `answer` instead of adding numbers into a sum.

```
listofsquares := function(thelist)
        local x, answer;
        answer := [];
        for x in thelist do
             Add(answer,x^2);
        od;
        return(answer);
end;
```

4. a) (7 pts) Show that

$$\forall x \geq 0, \quad x^4 \leq 5e^x.$$

Hint: study the function $h(x) = x^4 e^{-x}$, and use the fact that $4^4 e^{-4} \approx 4.689 < 5$.

b) (7 pts) Show from the definition of Θ that $e^x + 2x^4 + x^3$ is $\Theta(e^x)$. You may use the result of part (a) even if you were not able to solve it.

Answer. a) The derivative of $h(x)$ is $h'(x) = 4x^3 e^{-x} - x^4 e^{-x} = (4-x)x^3 e^{-x}$. Thus, for $0 < x < 4$ we have $h'(x) > 0$, so $h(x)$ is increasing; on the other hand, for $x > 4$. we have $h'(x) < 0$, so $h(x)$ is decreasing. Thus the maximum value of $h(x)$ for $x \in [0, +\infty)$ is the value $h(4) = 4^4 e^{-4} \approx 4.689$. In other words:

$$\forall x \geq 0, \quad h(x) \leq h(4) < 5$$

and hence $\forall x \geq 0, \quad x^4 e^{-x} < 5$. Multiplying both sides by e^x (which is positive!) then gives us the result that we want to prove.

b) For $x > 0$, the function $f(x) = e^x + 2x^4 + x^3$ is positive (all the terms in the sum are positive), and we will not bother to distinguish between $f(x)$ and the absolute value $|f(x)|$. Moreover, for $x \geq 1$ we have $x^3 \leq x^4$. Thus, using part (a) for the last $\leq$ below, we obtain:

$$\forall x \geq 1, \quad e^x \leq e^x + 2x^4 + x^3 = f(x) \leq e^x + 3x^4 \leq e^x + 15e^x = 16e^x.$$

Thus we have found constants $c, C > 0$ (namely, $c = 1$ and $C = 16$) and a constant $k = 1$ such that $\forall x \geq k, \quad ce^x \leq f(x) \leq Ce^x$. This shows that $f(x)$ is $\Theta(e^x)$.

5. In this problem, we count functions $f : \{1,2,3,4,5,6\} \to \{10,20,30,40,50\}$ satisfying certain properties.

a) (3 pts) How many functions f are injective?

b) (6 pts) How many functions f satisfy $f^{-1}(\{10,20\}) = \{1,2,3,4\}$?

c) (6 pts) How many functions f satisfy $f(\{1,2,3,4\}) = \{10,20\}$? Hint: first count the number of surjective functions $g : \{1,2,3,4\} \to \{10,20\}$.

Answer. a) The answer is 0 by the pigeonhole principle: you cannot have an injective function from a set with 6 elements to a set with 5 elements.

b) The condition $f^{-1}(\{10,20\}) = \{1,2,3,4\}$ means that

$$f(x) \in \{10,20\} \quad \leftrightarrow \quad x \in \{1,2,3,4\}.$$

In other words, $f(1), f(2), f(3), f(4) \in \{10,20\}$ and $f(5), f(6) \notin \{10,20\}$. Thus we have 2 choices for each of the values $f(1), f(2), f(3), f(4)$ and 3 choices for each of $f(5), f(6)$ (which must belong to $\{30,40,50\}$). The total number of functions f is therefore $2^4 \cdot 3^2 = 144$.

c) The condition $f(\{1,2,3,4\}) = \{10,20\}$ means that $f(1), f(2), f(3), f(4) \in \{10,20\}$ **and** that there exist $x, y \in \{1,2,3,4\}$ for which $f(x) = 10$ and $f(y) = 20$. However, we have **no restrictions** on the values of $f(5)$ and $f(6)$. As a result, to count the functions f, we start with a surjective function $g : \{1,2,3,4\} \to \{10,20\}$, which gives us the values $f(1) = g(1), \ldots, f(4) = g(4)$. This can be done in $2^4 - 2 \cdot 1^4$ ways by the formula for surjective functions. We then choose arbitrary values $f(5), f(6) \in \{10,20,30,40,50\}$, which can be done in 5^2 ways. Thus the total number of functions f in this exercise is $(2^4 - 2 \cdot 1^4)(5^2) = 350$.

Note: you can count the number of surjective functions g without the formula. Just note that among all 2^4 choices of functions from $\{1,2,3,4\}$ to $\{10,20\}$, there are exactly 2 functions that are not surjective: (1) the function whose value on $1,2,3,4$ is always 10, and (2) the function whose value on $1,2,3,4$ is always 20.