Math 211 — Fall 2007–08
Discrete Structures
Quiz 2, November 29 — Duration: 1 hour 15 minutes

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME: Questions and Solutions

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation W 4	Recitation W 11	Recitation Th 12:30	Recitation Th 3:30
Professor Makdisi	Ms. Karam	Ms. Karam	Ms. Karam

INSTRUCTIONS:

1. **Write your NAME and AUB ID number, and circle your SECTION above.**
2. **Solve the problems inside the booklet. Explain your reasoning precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.**
3. **You may use the back of each page for scratchwork OR for solutions. There are two extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.**
4. **Open book and notes. CALCULATORS ARE ALLOWED. Turn OFF and put away any cell phones.**

NEW! 5. You do not need to simplify your answers to counting problems — for example, if the answer is $\binom{211}{30} + (12!)7 \cdot 6 \cdot 5$, just leave it that way.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points. Each part of a problem is weighted equally, EXCEPT FOR problems 3 and 4. Take a minute to look at all the questions, BEFORE you start solving.

1. Let $X = \left\{ n \in \mathbf{N} \;\middle|\; (7 \leq n \leq 77) \wedge (n \text{ is a multiple of } 3) \right\}$.
 a) Find the cardinality $N = |X|$ of the set X.
 b) Give a bijection $f : \{1, 2, \ldots, N\} \to X$. You do not have to prove that f is a bijection.

2. There are 12 passengers on the AUB shuttle bus. Each passenger gets off at one of 5 stops: A, E, J, M, O (meaning Agriculture, Engineering, Jafet, Marquand, Oval).
 a) Show that there exists a stop where at least 3 passengers get off.
 b) How many ways are there for the passengers to get off the bus if the passengers are indistinguishable?
 c) Same question as (b) except that the passengers are distinguishable?
 d) Same question as (b) except that the passengers are distinguishable, **and** 3 passengers get off at each of A, E, J, while 1 passenger gets off at M, and 2 get off at O?

3. a) **(8 pts)** Show by **mathematical induction** that $\forall n \in \mathbf{N}, \binom{2n}{n} \leq 4^n$.
 b) **(4 pts)** Give a different proof of the above by expanding $(1+1)^{2n}$.
 Remark: one can prove by induction the stronger statement that $\binom{2n}{n} \leq 4^n/\sqrt{2n+1}$. Do that at home, **after** the test.

4. a) **(3 pts)** Fill in the blanks on page 4 for a function `oddprod(n)` that computes the product of the odd positive integers `i` $\leq$ `n`. For example, `oddprod(10)` $= 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$ and `oddprod(11)` $= 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$.
 b) **(3 pts)** We are given a list of lists `l`, and we wish to make a new list `l2` consisting of the sublists of `l` whose elements are **all odd**. For example:

```
If    l = [[1,3,5], [4,5,6], [9,11,17], [3,1,1,7], [2]]
then  l2 = [[1,3,5], [9,11,17], [3,1,1,7]].
```

Use the `Filtered()` and `ForAll()` functions in GAP to do this in one line (see page 4).

```
Reminder: Filtered([1..10], x->IsOdd(x)) = [1,3,5,7,9]
          ForAll([1..10], x->IsOdd(x)) = false.
```

 c) **(6 pts)** Write a function `totallyoddsublists()` that solves part b) using **nested loops**. So you can write `l2 := totallyoddsublists(l)` to achieve the same result as in part (b). **Do not use** `Filtered()`, `ForAll()`, or even `ForAny()` in this part.
 Hint: the outer loop goes over elements of `l`, which are sublists. The inner loop goes through the elements of a sublist and uses a helping variable `allodd` $\in$ {`true`, `false`} to keep track of whether the elements that we have examined so far in this sublist are all odd.

5. A problem about the number of permutations $P(n, r)$.
 a) Give an **algebraic proof** that $P(n, r) = P(n-1, r) + rP(n-1, r-1)$.
 b) Give a **combinatorial proof** of the above identity.

6. Consider the function $f : \{a, b, c\} \to \{1, 2, 3, 4, 5, 6, 7\}$ that is defined by $f(a) = 5$, $f(b) = 6$, $f(c) = 7$.
 a) How many left inverses g does f have? [Recall that a left inverse g is a function $g : \{1, 2, 3, 4, 5, 6, 7\} \to \{a, b, c\}$ such that $g \circ f = \mathrm{id}_{\{a,b,c\}}$.]
 b) How many left inverses g satisfy $[g(1) = a$ **or** $g(2) = b$ **or** $g(3) = c]$?

1. Let $X = \left\{ n \in \mathbf{N} \;\middle|\; (7 \le n \le 77) \wedge (n \text{ is a multiple of } 3) \right\}$.

a) Find the cardinality $N = |X|$ of the set X.

$X = \{9, 12, 15, \dots, 75\}$. Elements of X can be written in the form $n = 3k$, with $7 \le 3k \le 77$ and $k \in \mathbb{N}$. (Here we could also have said $k \in \mathbb{Z}$.)

Equivalently, $\frac{7}{3} \le k \le \frac{77}{3}$ and $k \in \mathbb{N}$, which is the same as

$$3 = \left\lceil \frac{7}{3} \right\rceil \le k \le \left\lfloor \frac{77}{3} \right\rfloor = 25 \quad \text{and } k \in \mathbb{N},$$ because k is an integer.

The number of possible k is $25 - 3 + 1 = \boxed{23}$, each k corresponding to precisely one value of $n = 3k$.

[Note that this proof has set up a bijection between X and the set of integers $Y = \{k \in \mathbb{N} \mid 3 \le k \le 25\} = \{3, 4, 5, \dots, 25\}$. Namely, we have $g: Y \to X$ defined by $g(k) = 3k$ and its inverse $g^{-1}: X \to Y$ defined by $g^{-1}(n) = n/3$.]

b) Give a bijection $f : \{1, 2, \dots, N\} \to X$. You do not have to prove that f is a bijection.

We want a bijection $f: \{1, 2, \dots, 23\} \longrightarrow X$, where $X = \{9, 12, 15, \dots, 75\}$.

<u>1st way</u>. put $Z = \{1, 2, \dots, 23\}$ and make a bijection with $Y = \{3, 4, \dots, 25\}$. Here $j \in Z$ corresponds to $k = j + 2 \in Y$. But $k = j + 2 \in Y$ corresponds to $n = 3k = 3j + 6 \in X$. So we can take

$$\boxed{f(j) = 3j + 6}.$$

<u>2nd way</u> (less abstract): if $Z = \{1, 2, 3, \dots, 23\}$ and $X = \{9, 12, 15, \dots, 75\}$ try to find a nice formula for f such that $f(1) = 9$, $f(2) = 12$, $f(3) = 15$, …, $f(23) = 75$.

We have $f(j+1) = f(j) + 3$ so try f of the form $f(j) = 3j + b$ for some suitable b. It turns out easily that $b = 6$ works, so again we obtain $\boxed{f(j) = 3j + 6}$.

<u>N.B.</u> There are many other possible bijections, but the one given above is the simplest. The next simplest is $\tilde{f}(j) = 78 - 3j$.

2. There are 12 passengers on the AUB shuttle bus. Each passenger gets off at one of 5 stops: A, E, J, M, O (meaning Agriculture, Engineering, Jafet, Marquand, Oval).

a) Show that there exists a stop where at least 3 passengers get off.

Pigeonhole principle: we must distribute 12 passengers among 5 stops. At least one stop must receive $\left\lceil \frac{12}{5} \right\rceil = 3$ passengers or more.

<u>Note</u> that we can think of this in terms of a function $f: \underbrace{\{p_1, \dots, p_{12}\}}_{\text{set of passengers}} \longrightarrow \{A, E, J, M, O\}$.

b) How many ways are there for the passengers to get off the bus if the passengers are indistinguishable?

Let x_1 be the number of passengers who get off at A
" x_2 " " " " " " " " " " E
" x_3 " " " " " " " " " " J
" x_4 " " " " " " " " " " M
" x_5 " " " " " " " " " " O

only the <u>number</u> of passengers getting off at each stop matters.

We have $x_1, \dots, x_5 \in \mathbb{N}$ and $\sum_{i=1}^{5} x_i = 12$.

So we wish to count $\{(x_1, \dots, x_5) \in \mathbb{N}^5 \mid \sum_{i=1}^{5} x_i = 12\}$.

This set has cardinality $\binom{12+5-1}{5-1} = \boxed{\binom{16}{4}}$ (which is equal to $\binom{16}{12}$)

c) Same question as (b) except that the passengers are distinguishable?

Here each passenger has a choice of 5 stops. The total number of choices by the product rule is

$$\underbrace{5}_{\text{choices of passenger 1}} \cdot \underbrace{5}_{\text{choices of passenger 2}} \cdot \underbrace{5}_{\text{of passenger 3}} \cdots \underbrace{5}_{\text{of passenger 12}} = \boxed{5^{12}}.$$

<u>Note</u> this is the same as counting the number of functions $f: \{p_1, \dots, p_{12}\} \longrightarrow \{A, E, J, M, O\}$.

d) Same question as (b) except that the passengers are distinguishable, and 3 passengers get off at each of A, E, J, while 1 passenger gets off at M, and 2 get off at O?

This is like the problem of putting 12 distinguishable balls into 5 distinguishable boxes with n_i balls in box i. Here the "balls" are the passengers $p_1 \dots p_{12}$ & the "boxes" are $\{A, E, J, M, O\}$, and $n_1 = n_2 = n_3 = 3$, $n_4 = 1$, $n_5 = 2$. The answer is hence

$$\boxed{\frac{12!}{3!\,3!\,3!\,1!\,2!}}.$$

[this is also the number of functions $f: \{p_1 \dots p_{12}\} \to \{A, \dots, O\}$ with $|f^{-1}(\{A\})| = |f^{-1}(\{E\})| = |f^{-1}(\{J\})| = 3 \wedge |f^{-1}(\{M\})| = 1 \wedge |f^{-1}(\{O\})| = 2$]

<u>Note</u> this can also be calculated as $\binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{1} \cdot \binom{2}{2} = \frac{12!}{3! \cdot \cancel{9!}} \cdot \frac{\cancel{9!}}{3! \cdot \cancel{6!}} \cdot \frac{\cancel{6!}}{3!\,\cancel{3!}} \cdot \frac{\cancel{3!}}{1!\,\cancel{2!}} \cdot \frac{2!}{2!\,0!}$

1. first choose the 3 passengers who get off at A

2. from the rest, choose 3 who get off at E

3. a) (8 pts) Show by mathematical induction that $\forall n \in \mathbf{N}, \binom{2n}{n} \leq 4^n$.

Let $P(n)$ be the statement $\binom{2n}{n} \leq 4^n$.

Base case: check $P(0)$. $\binom{2\cdot 0}{0} = \binom{0}{0} = 1$ is indeed $\leq 4^0 = 1$, so $P(0)$ is true.

Inductive step: show that $\forall n\ (P(n) \xrightarrow{?} P(n+1))$.

Let n be an arbitrary element of $\mathbb{N}$, and assume that $\binom{2n}{n} \leq 4^n$.

Then
$$\binom{2(n+1)}{n+1} = \binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$$
$$= \frac{(2n+2)(2n+1)(2n)!}{(n+1)\,n!\,(n+1)\,n!}$$
$$= \frac{\cancel{(2n+2)}^{\,2}(2n+1)}{\cancel{(n+1)}\,(n+1)}\ \frac{2n!}{n!\,n!} = \left(\frac{4n+2}{n+1}\right)\cdot\binom{2n}{n}$$

hence by our hypothesis that $\binom{2n}{n} \leq 4^n$, we get
$$\binom{2(n+1)}{n+1} \leqslant \frac{4n+2}{n+1}\cdot 4^n \leq 4\cdot 4^n = 4^{n+1}$$
because $\frac{4n+2}{n+1} \leqslant \frac{4n+4}{n+1} = 4$

The above shows that $\forall n,\ (P(n) \rightarrow P(n+1))$.

Hence by the base case & inductive step, we can conclude that $\forall n\ P(n)$ by mathematical induction.

b) (4 pts) Give a different proof of the above by expanding $(1+1)^{2n}$.

$$(1+1)^{2n} = \binom{2n}{0}1^{2n}1^0 + \binom{2n}{1}1^{2n-1}1^1 + \cdots + \binom{2n}{n}1^n1^n + \cdots + \binom{2n}{2n}1^0 1^{2n}$$
$$\therefore\ 2^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n}.$$

But the terms $\binom{2n}{k}$ are all $\geqslant 0$

so $\binom{2n}{n} \leq \binom{2n}{0} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n} = 2^{2n} = 4^n$, as desired.

Remark: one can prove by induction the stronger statement that $\binom{2n}{n} \leq 4^n/\sqrt{2n+1}$. Do that at home, after the test.

Key step: show that $\frac{4n+2}{n+1}\cdot\frac{1}{\sqrt{2n+1}} \overset{?}{\leqslant} \frac{4}{\sqrt{2n+3}}$. Write $4n+2 = 2(2n+1)$ & simplify to get the equivalent statement $\frac{\sqrt{2n+1}}{n+1} \overset{?}{\leqslant} \frac{2}{\sqrt{2n+3}}$, which is equivalent to $(2n+1)(2n+3) \overset{?}{\leq} [2(n+1)]^2$. left to you.

4. a) (3 pts) Fill in the blanks below for a function oddprod(n) that computes the product of the odd positive integers $i \leq n$. For example, oddprod(10) $= 1\cdot 3\cdot 5\cdot 7\cdot 9$ and oddprod(11) $= 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11$.

```
oddprod := function(n)
            local answer, i;
            answer := 1;          # start the product with 1 before doing
                                    anything (just like an "empty sum" is 0,
                                    an "empty product" is 1)
            for i in [1..n] do
                if IsOdd(i)
                  then answer := answer * i;
                fi;
            od;
            return(answer);
end;
```

Note 1 we can also test if i is odd by if (i mod 2 = 1) then ...

Note 2 I wrote IsOdd(i) to see if i is odd. In GAP it should be IsOddInt(i).

b) (3 pts) We are given a list of lists l, and we wish to make a new list l2 consisting of the sublists of l whose elements are all odd. For example:

If   l = [[1,3,5], [4,5,6], [9,11,17], [3,1,1,7], [2]]

then   l2 = [[1,3,5], [9,11,17], [3,1,1,7]].

Use the Filtered() and ForAll() functions in GAP to do this in one line below.

Reminder: Filtered([1..10], x->IsOdd(x)) = [1,3,5,7,9]

ForAll([1..10], x->IsOdd(x)) = false.

Answer: (see Note 2 above)

l2 := Filtered(l, x-> ForAll(x, y-> IsOdd(y)));

Note: we can also write Filtered(l, x-> ForAll(x, IsOdd)).

c) (6 pts) Write a function totallyoddsublists() that solves part b) using nested loops. So you can write l2 := totallyoddsublists(l) to achieve the same result as in part (b). Do not use Filtered(), ForAll(), or even ForAny() in this part.

Hint: the outer loop goes over elements of l, which are sublists. The inner loop goes through the elements of a sublist and uses a helping variable allodd $\in$ {true, false} to keep track of whether the elements that we have examined so far in this sublist are all odd.

Answer (will take several lines): Here is one way. There are several others.

```
totallyoddsublists := function(l)
   local answer, sublist, allodd, elt;
   answer := [];
   for sublist in l do
       allodd := true;   # IMPORTANT: reinitialize allodd before examining each new sublist.
       for elt in sublist do
           if not IsOdd(elt) then allodd := false; fi;   # ALTERNATIVELY allodd := allodd and IsOdd(elt)
       od;
       if allodd then Add(answer, sublist);
   od;
   return(answer);
end;
```

5. A problem about the number of permutations $P(n,r)$.

a) Give an algebraic proof that $P(n,r) = P(n-1,r) + rP(n-1,r-1)$.

Recall that $P(n,r) = \dfrac{n!}{(n-r)!}$.

We have
$$P(n-1,r) + r\,P(n-1,r-1) = \frac{(n-1)!}{(n-1-r)!} + \frac{r\,(n-1)!}{(n-1-r+1)!}$$
$$= \frac{(n-1)!}{(n-1-r)!} + \frac{r(n-1)!}{(n-r)!} \qquad \text{now multiply the 1st term by } \frac{n-r}{n-r}$$
$$= \frac{(n-1)!\,(n-r)}{(n-r)!} + \frac{r(n-1)!}{(n-r)!}$$
$$= \frac{(n-1)!}{(n-r)!}[n-r+r] = \frac{(n-1)!\,n}{(n-r)!} = \frac{n!}{(n-r)!} = P(n,r)$$
as desired.

b) Give a combinatorial proof of the above identity.

Let $A = \{1,2,\dots,n\}$. $P(n,r)$ is the number of tuples $(a_1,\dots,a_r)$ with $a_i \in A$, all a_i distinct.

We count these tuples in a different way. There are two disjoint possibilities for a tuple:

① either the tuple does not contain n (so $\forall i,\ a_i \neq n$)

② or (xor!) the tuple does contain n (so $\exists i$ st. $a_i = n$).

Case ① then $(a_1,\dots a_r)$ is an r-permutation of the set $A-\{n\} = \{1,2,\dots,n-1\}$ and this can be done in $\underline{P(n-1,r)}$ ways.

Case ② Because the a_i are distinct, $\exists!\ i$ s.t. $a_i = n$. (so n occurs in exactly one place in the tuple $(a_1,\dots,a_r)$)

To count these tuples: First choose where "n" goes. There are r possibilities. Second choose the remaining $(r-1)$ entries of the tuple from the set $A-\{n\}$. Since order matters, there are $\underline{P(n-1,r-1)}$ ways.

so case ② can be done in $r \cdot P(n-1,r-1)$ ways.

Adding up cases ① & ②, we get $P(n-1,r) + r\cdot P(n-1,r-1)$ ways to get an r-permutation of $\{1,\dots,n\}$. This is $\therefore$ equal to $P(n,r)$.

Alternative, more concrete way: We have a group with n people (one man and $n-1$ women, say) and we want to make r of them stand in line. This can be done in $P(n,r)$ ways. Now count this differently: either ① we stand r women in a line (out of the $n-1$ women), in $P(n-1,r)$ ways, or (xor) ② we choose one of the r positions & stand the man there, then stand $r-1$ of the women in the other places, doable in $r\cdot P(n-1,r-1)$ ways. Add up ① & ② to get $P(n,r)$.

6. Consider the function $f: \{a,b,c\} \to \{1,2,3,4,5,6,7\}$ that is defined by $f(a) = 5$, $f(b) = 6$, $f(c) = 7$.

a) How many left inverses g does f have? [Recall that a left inverse g is a function $g: \{1,2,3,4,5,6,7\} \to \{a,b,c\}$ such that $g \circ f = \mathrm{id}_{\{a,b,c\}}$.]

We want $g(f(x)) = x$ for all $x \in \{a,b,c\}$.

Thus $\left.\begin{array}{l} g(f(a)) = a \\ g(f(b)) = b \\ g(f(c)) = c \end{array}\right\}$ required

so we require $\left.\begin{array}{l} g(5) = a \\ g(6) = b \\ g(7) = c \end{array}\right\}$ and there are no further restrictions on g

I.e. we can choose
$g(1) \in \{a,b,c\}$ to be anything : 3 choices
$g(2) \in \{a,b,c\}$ " " " : 3 choices
$g(3) \in \{a,b,c\}$ " " " : 3 choices
$g(4) \in \{a,b,c\}$ " " " : 3 choices

for a total of $\boxed{3^4 \text{ choices of } g}$.

b) How many left inverses g satisfy $[g(1) = a$ or $g(2) = b$ or $g(3) = c]$?

1st way: inclusion-exclusion. Define
$A_1 = \{$left inverses g of f satisfying $g(1) = a\}$
$A_2 = \{$ " " " " " $g(2) = b\}$
$A_3 = \{$ " " " " " $g(3) = c\}$.

We want $|A_1 \cup A_2 \cup A_3|$. So calculate:

- $|A_1| = 3^3$ because for $g \in A_1$, we know $\left[\begin{array}{l} g(1) = a \\ g(5) = a \\ g(6) = b \\ g(7) = c \end{array}\right.$ so we must only choose $g(2), g(3), g(4) \in \{a,b,c\}$.

from A_1

from before

similarly $|A_2| = 3^3$ (choose $g(1), g(3), g(4)$ at will) and $|A_3| = 3^3$.

- $|A_{12}| = 3^2$ since we know $\left[\begin{array}{l} g(1) = a \\ g(2) = b \end{array}\right.$ $\left[\begin{array}{l} g(5) = a \\ g(6) = b \\ g(7) = c \end{array}\right.$ & we can choose just $g(3), g(4)$ at will.

similarly $|A_{23}| = |A_{13}| = 3^2$.

- $|A_{123}| = 3$ since we only get to choose $g(4)$.

So Answer $= |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_{12}| - |A_{13}| - |A_{23}| + |A_{123}|$
$= 3^3 + 3^3 + 3^3 - 3^2 - 3^2 - 3^2 + 3 \qquad (=57)$.

2nd way: take the complements: count those left inverses that do not satisfy the condition. So our answer will be $3^4 - |\{g$ not satisfying the condition$\}|$. For g not to satisfy the condition, we must have $(g(1) \neq a) \wedge (g(2) \neq b) \wedge (g(3) \neq c)$. This leaves 2 choices for $g(1) \in \{b,c\}$, 2 choices for $g(2) \in \{a,c\}$, 2 choices for $g(3) \in \{a,b\}$, 3 choices for $g(4) \in \{a,b,c\}$.

So the answer is $3^4 - 2\cdot2\cdot2\cdot3 = 57$ again.