AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 211
QUIZ 1
Spring 2006-2007
Monday March 26, 12 :00 pm
Closed Book, 1H15MIN

Solution

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	15	
2	10	
3	15	
4	10	
5	10	
TOTAL	60	

1. (15 points)

(a) (5 points) Prove using truth tables that :

$$(p \to q) \land (p \to r) \equiv (p \to (q \land r)).$$

p	q	r	$p \to q$	$p \to r$	$(p \to q) \land (p \to r)$	$q \land r$	$p \to (q \land r)$	$[(p \to q) \land (p \to r)] \leftrightarrow [p \to (q \land r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	F	F	F	T
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

Since the biconditional $[(p \to q) \land (p \to r)] \leftrightarrow [p \to (q \land r)]$ is a tautology, it follows that the two propositions are logically equivalent: $(p \to q) \land (p \to r) \equiv p \to (q \land r)$

(b) (4 points) Prove **without using truth tables** that :

$$(\neg p \to (p \to q)) \text{ is a Tautology.}$$

Method 1 : To prove that an implication is a tautology, it is enough to show that the conclusion is true whenever the premise is true.

Assumption : $\neg p$ is true

Therefore p is false

So $p \to q$ is true, whatever is the truth value of q

Thus : $\neg p \Rightarrow (p \to q)$

(or $\neg p \to (p \to q)$ is a tautology)

Method 2 : (Using logical equivalences)

$$\begin{aligned}\neg p \to (p \to q) &\equiv \neg(\neg p) \lor (p \to q)\\ &\equiv p \lor (\neg p \lor q)\\ &\equiv (p \lor \neg p) \lor q\\ &\equiv T \lor q\\ &\equiv T\end{aligned}$$

(c) (6 points) Prove **without using truth tables** that, given a fixed proposition Q, and a predicate $P(x)$ on a universe of discourse U that consists of a finite number of elements $U := \{x_1, x_2, ..., x_n\}$, that :

$$\forall x (P(x) \to Q) \equiv (\exists x P(x)) \to Q$$

$$\begin{aligned}
\forall x \, [P(x) \to Q] &\equiv [P(x_1) \to Q] \wedge [P(x_2) \to Q] \wedge \cdots \wedge [P(x_n) \to Q] \\
&\equiv [\neg P(x_1) \vee Q] \wedge [\neg P(x_2) \vee Q] \wedge \cdots \wedge [\neg P(x_n) \vee Q] \\
&\equiv [\neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n)] \vee Q \\
&\equiv [\forall x \, \neg P(x)] \vee Q \\
&\equiv \neg [(\exists x \, P(x)] \vee Q \\
&\equiv [\exists x \, P(x)] \to Q
\end{aligned}$$

2. (10 points) Let $C(x, y)$ be the statement 'x and y have chatted over the Internet', where the domain for the variables x and y consists of all students in your class.

(a) (5 points) Use quantifiers to express the following statement :

"There are at least two students in your class who have not chatted with the same person in your class."

$\exists x \exists y \left[(x \neq y) \wedge \forall z \neg (C(x,z) \wedge C(y,z)) \right]$

(b) (5 points) Write the following in English

$$\exists x \exists y (C(x, y) \wedge (\forall z (z \neq x \longrightarrow \neg C(z, y))))$$

There is at least one student of your class who has chatted with one and only one other student in the class.

3. (15 points) Consider the function $f(x) = tan(x)$ from $(-\pi/2, \pi/2)$ to $\mathbb{R}$ and $g(x) = \exp(x)$ from $\mathbb{R}$ to $\mathbb{R}_+ = (0, \infty)$.

(a) (3 points) Study the function $h := g \circ f$. If properly defined, find the domain, the co-domain and the expression of h.

$f : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$, $x \longmapsto f(x) = \tan x$ $\qquad$ $g : \mathbb{R} \longrightarrow (0, \infty)$, $x \longmapsto g(x) = \exp(x)$

- $h = g \circ f$ is properly defined since the codomain of f is a subset of the domain of g.
- $$\begin{tikzcd} (-\pi/2, \pi/2) \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R} \arrow[r, "g"] & (0, \infty) \end{tikzcd}$$

$$g \circ f(x) = g[f(x)] = g[\tan x] = \exp(\tan x)$$

Domain of $g \circ f$: $(-\pi/2, \pi/2)$

Codomain of $g \circ f$: $(0, \infty)$

$$\boxed{h(x) = g \circ f(x) = e^{\tan x}}$$

(b) (5 points) Find the inverse image set $h^{-1}(S)$ where $S := [1/e, e]$, with $e = \exp(1)$.

$$h^{-1}(S) = h^{-1}\left(\left[\tfrac{1}{e}, e\right]\right)$$
$$= \left\{ x \in (-\pi/2, \pi/2) \mid \tfrac{1}{e} \leq h(x) \leq e \right\}$$

But $\frac{1}{e} \leq h(x) \leq e$ is equivalent to

$$\frac{1}{e} \leq e^{\tan x} \leq e$$

$\ln \frac{1}{e} \leq \tan x \leq \ln e$ $\quad$ by applying the increasing function $\ln$

$$-\ln e \leq \tan x \leq \ln e$$

$$-1 \leq \tan x \leq 1$$

$\tan^{-1}(-1) \leq x \leq \tan^{-1}(1)$ $\quad$ by applying the increasing function $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$$

Therefore: $h^{-1}(S) = \left\{ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mid -\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \right\}$

$$\boxed{h^{-1}(S) = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]}$$

(c) (4 points) Is h invertible? If yes, say why and find the expression of the inverse function h^{-1}.

* $h(x) = e^{\tan x}$

We know that the function $\tan x$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and defines a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$.

The function $\exp x$ is also strictly increasing on $\mathbb{R}$ and defines a bijection from $\mathbb{R}$ to $(0,\infty)$.

It follows that the composition h of these 2 functio[ns] is a bijection from $(-\pi/2, \pi/2)$ to $(0,\infty)$.

Therefore h is invertible.

* $h^{-1}(x) = y \iff x = h(y)$

$$x = e^{\tan y}$$
$$\tan y = \ln x$$
$$y = \tan^{-1}(\ln x)$$

or:

$$h^{-1}(x) = (g \circ f)^{-1}(x) = f^{-1} \circ g^{-1}(x) = \tan^{-1}(\ln(x))$$

Therefore: $\boxed{h^{-1}(x) = \tan^{-1}(\ln x)}$

(d) (3 points) Is $f \circ g$ defined? If no, say why?

$$g : \mathbb{R} \longrightarrow (0,\infty), \quad x \longmapsto \exp(x)$$

$$f : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}, \quad x \longmapsto \tan x$$

The range of g is not a subset of the domain of f.

Therefore $f \circ g$ is not defined.

4. (10 points) Solve the following :

(a) (4 points) Given two sets A, B, subsets of a universal set U,

$$\chi_{A-B} = \chi_A(1-\chi_B).$$

$\forall x \in (A-B)$, $(x \in A) \wedge (x \notin B)$

So: $[\chi_A(x)=1] \wedge [\chi_B(x)=0]$

$\chi_A(x)[1-\chi_B(x)] = 1.(1-0) = 1$

But, $\forall x \in (A-B)$, $\chi_{A-B}(x) = 1$

So: $\forall x \in (A-B)$ $\chi_{A-B}(x) = \chi_A(x)[1-\chi_B(x)]$

That is: $\boxed{\chi_{A-B} = \chi_A(1-\chi_B)}$

(Here, 1 denotes the constant function that assigns 1 to any x)

(b) (6 points) Let $n \in \mathbb{Z}$. Consider the sets $\{A_n := (n,\infty) | n \in \mathbb{Z}\}$. For $x \in (0,3]$, graph the function :

$$\boxed{f(x) = \chi_{A_0}(x) + \chi_{A_1}(x) + \chi_{A_2}(x).}$$

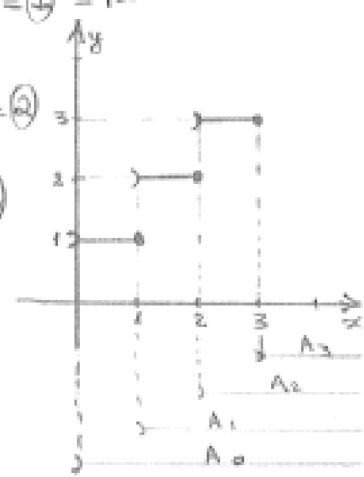
Is this function equal to $\lfloor x \rfloor$ or $\lceil x \rceil$ for $x \in (0,3]$? Justify your answer.

$\begin{cases} A_0 = (0,\infty) \\ A_1 = (1,\infty) \\ A_2 = (2,\infty) \\ A_3 = (3,\infty) \end{cases}$ therefore: $\begin{cases} A_0 - A_1 = (0,1] \\ A_1 - A_2 = (1,2] \\ A_2 - A_3 = (2,3] \end{cases}$ and: $A_3 \subset A_2 \subset A_1 \subset A_0$

$\forall x \in A_0 - A_1$, $\underbrace{0 < x \leq 1}$ and $f(x) = \chi_{A_0}(x) + \chi_{A_1}(x) + \chi_{A_2}(x) = 1+0+0 = 1 = \lceil x \rceil$

$\forall x \in A_1 - A_2$, $\underbrace{1 < x \leq 2}$ and $f(x) = \chi_{A_0}(x) + \chi_{A_1}(x) + \chi_{A_2}(x) = 1+1+0 = 2 = \lceil x \rceil$

$\forall x \in A_2 - A_3$, $\underbrace{2 < x \leq 3}$ $f(x) = \chi_{A_0}(x) + \chi_{A_1}(x) + \chi_{A_2}(x) = 1+1+1 = 3 = \lceil x \rceil$



Therefore: $\underline{\forall x \in (0,3]}$, $\boxed{f(x) = \lceil x \rceil}$

5. (10 points) Consider the relation $R \subseteq A \times A$, where $A := \{1, 2, 3, 4\}$, where

$$R := \{(x, y) | x \leq y\}.$$

(a) (4 points) Find the elements of R and, as well, its table representation.

$$R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4) \}$$

A \ A	1	2	3	4
1	x	x	x	x
2		x	x	x
3			x	x
4				x

(b) (6 points) Is R reflexive ? Symmetric ? Anti-symmetric ? Transitive ? Justify your answers.

* R is reflexive since $\forall x \in A \quad x \leq x \Leftrightarrow (x,x) \in R$ (which is true for all real number)

* R is not symmetric since for example: $1 \leq 2$ but $2 \not\leq 1$

  So: $(1,2) \in R$ and $(2,1) \notin R$

* R is antisymmetric since:

  $$\forall x \in A, \forall y \in A, \ \underbrace{[(x,y) \in R \wedge (y,x) \in R]}_{} \Rightarrow x = y$$

  (the premise is false whenever $x \neq y$)

* R is transitive since:

  $\forall x \forall y \forall z$, if $(x,y) \in R$ and $(y,z) \in R$

  then $x \leq y$ and $y \leq z$

  so $x \leq z$

  that is: $(x,z) \in R$