# Chapter 8: Reflection, Transmission, and Waveguides

## Lessons #50 and 51

**Chapter — Section:** 8-1

**Topics:** Normal incidence

**Highlights:**

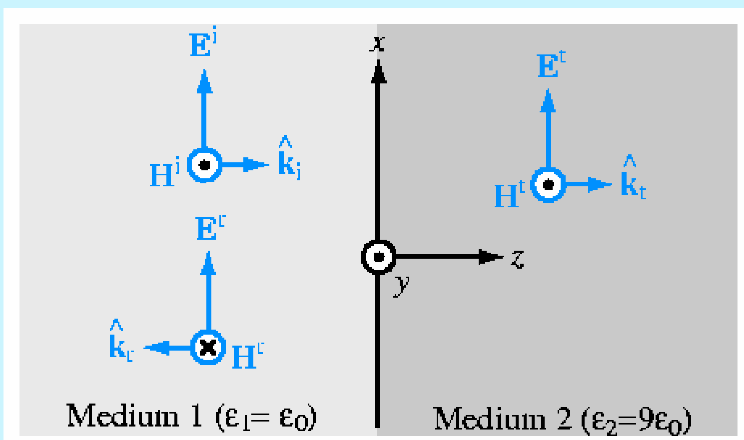
- Analogy to transmission line
- Reflection and transmission coefficient

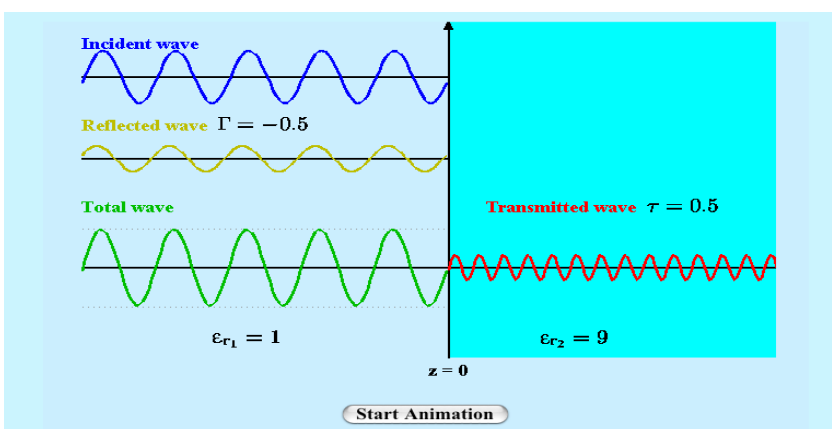
**Special Illustrations:**

- Example 8-1
- CD-ROM Modules 8.1-8.5
- CD-ROM Demos 8.2

Demo 8.2: Medium-contrast Interface

Consider a 6-GHz plane wave in air incident upon the planar surface of a lossless dielectric medium with $\varepsilon_r = 9$.



# Lesson #52

**Chapter — Section:** 8-2

**Topics:** Snell's laws

**Highlights:**

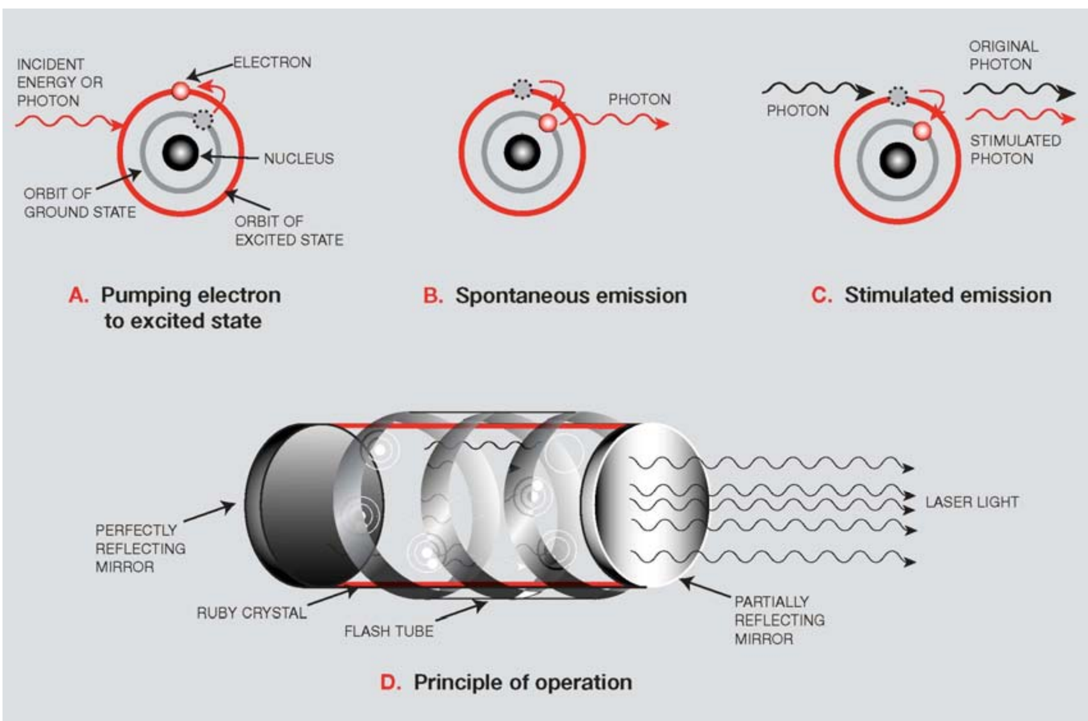
- Reflection and refraction
- Index of refraction

**Special Illustrations:**

- Example 8-4
- Technology Brief on "Lasers" (CD-ROM)

**Lasers**

Lasers are used in CD and DVD players, bar-code readers, eye surgery and multitudes of other systems and applications. A laser—acronym for light amplification by stimulated emission of radiation—is a source of monochromatic (single wavelength), coherent (uniform wavefront), narrow-beam light, in contrast with other sources of light (such as the sun or a light bulb) which usually encompass waves of many different wavelengths with random phase (incoherent). A laser source generating microwaves is called a maser. The first maser was built in 1953 by Charles Townes and the first laser was constructed in 1960 by Theodore Maiman.



# Lesson #53

**Chapter — Section:** 8-3

**Topics:** Fiber optics

**Highlights:**

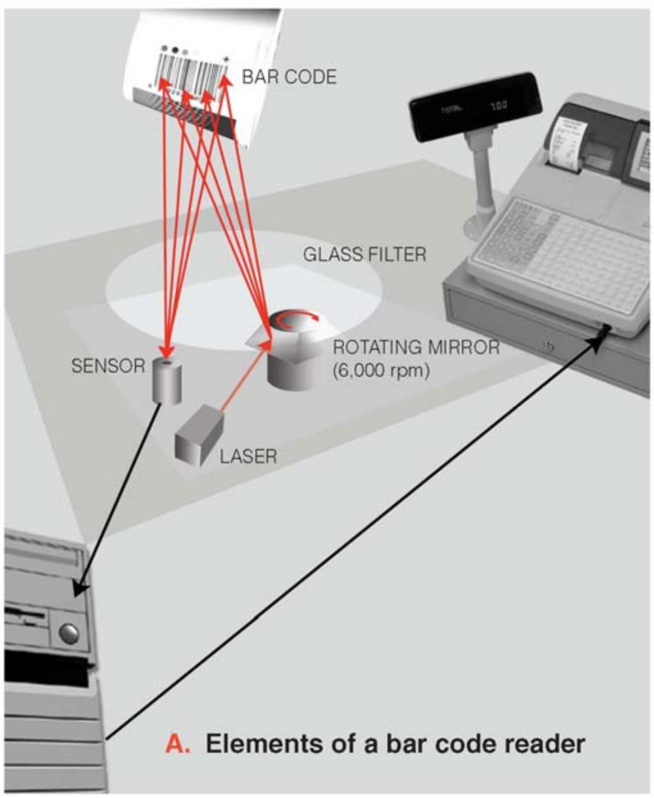
- Structure of an optical fiber
- Dispersion

**Special Illustrations:**

- Example 8-5
- Technology Brief on "Bar-Code Reader" (CD-ROM)

### Bar Code Readers

A bar code consists of a sequence of parallel bars of certain widths, usually printed in black against a white background, configured to represent a particular binary code of information about a product and its manufacturer. Laser scanners can read the code and transfer the information to a computer, a cash register, or a display screen. For both stationary scanners built into checkout counters at grocery stores and handheld units that can be pointed at the bar-coded object like a gun, the basic operation of a bar-code reader is the same.




A. Elements of a bar code reader

## Lessons #54 and 55

**Chapter — Section:** 8-4

**Topics:** Oblique incidence

**Highlights:**

- Parallel and perpendicular polarizations
- Brewster angle
- Total internal reflection

**Special Illustrations:**

- Example 8-6 and 8-7
- CD-ROM Demos 8.4-8.6

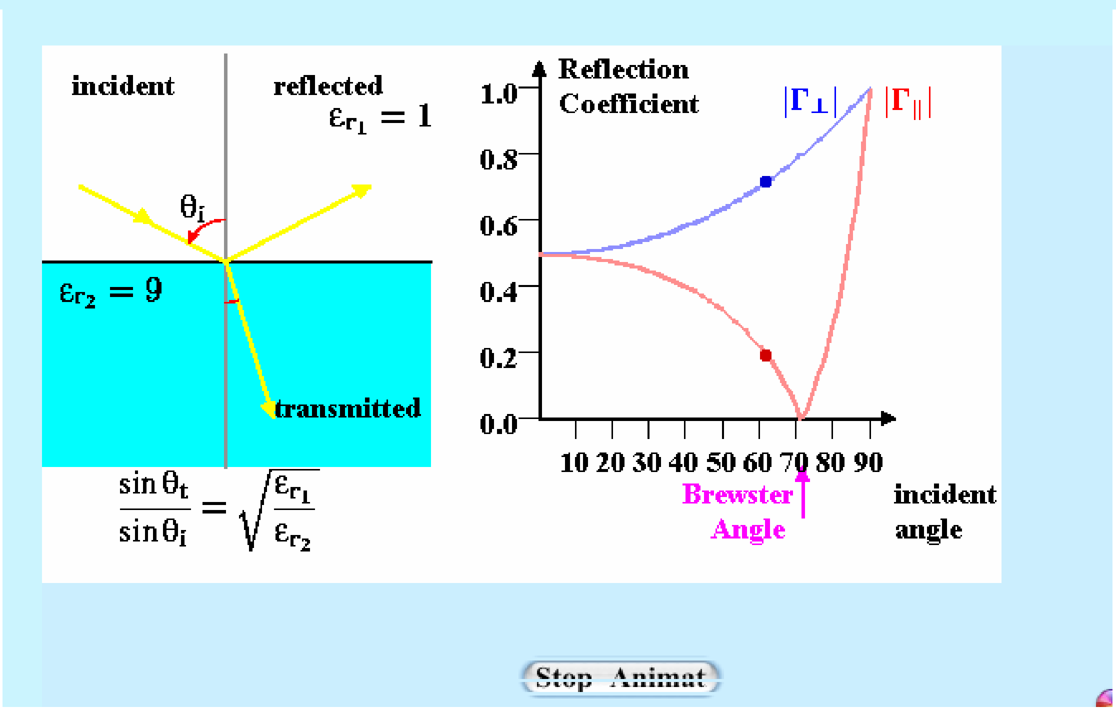
Demo 8.5: Moderate-contrast Interface

Consider a plane wave in air incident upon the planar surface of a lossless dielectric medium with $\varepsilon_r = 9$.

Press to display the following:

(1) The directions of the incident, reflected and transmitted rays as a function of the incidence angle.

(2) The magnitude of the reflection coefficient for both parallel and perpendicular polarizations as a function of the incidence angle.



# Lesson #56

**Chapter — Section:** 8-5

**Topics:** Reflectivity and transmissivity

**Highlights:**

- Power relations

**Special Illustrations:**

- Example 8-7

## Lessons #57–59

**Chapter — Section:** 8-6 to 8-10

**Topics:** Waveguides

**Highlights:**

- TE and TM modes
- Cutoff frequency
- Phase and group velocities

**Special Illustrations:**

- Examples 8-8, 8-9, and 8-10

## Lesson #60

**Chapter — Section:** 8-11

**Topics:** Cavity Resonators

**Highlights:**

- Resonant frequency
- Q factor
- Applications

# Chapter 8

## Section 8-1: Reflection and Transmission at Normal Incidence

**Problem 8.1** A plane wave in air with an electric field amplitude of 20 V/m is incident normally upon the surface of a lossless, nonmagnetic medium with $\varepsilon_r = 25$. Determine:

**(a)** the reflection and transmission coefficients,
**(b)** the standing-wave ratio in the air medium, and
**(c)** the average power densities of the incident, reflected, and transmitted waves.

**Solution:**

**(a)**

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{120\pi}{5} = 24\pi \quad (\Omega).$$

From Eqs. (8.8a) and (8.9),

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{24\pi - 120\pi}{24\pi + 120\pi} = \frac{-96}{144} = -0.67,$$
$$\tau = 1 + \Gamma = 1 - 0.67 = 0.33.$$

**(b)**

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.67}{1 - 0.67} = 5.$$

**(c)** According to Eqs. (8.19) and (8.20),

$$S^{\rm i}_{\rm av} = \frac{|E^{\rm i}_0|^2}{2\eta_0} = \frac{400}{2 \times 120\pi} = 0.52 \text{ W/m}^2,$$
$$S^{\rm r}_{\rm av} = |\Gamma|^2 S^{\rm i}_{\rm av} = (0.67)^2 \times 0.52 = 0.24 \text{ W/m}^2,$$
$$S^{\rm t}_{\rm av} = |\tau|^2 \frac{|E^{\rm i}_0|^2}{2\eta_2} = |\tau|^2 \frac{\eta_1}{\eta_2} S^{\rm i}_{\rm av} = (0.33)^2 \times \frac{120\pi}{24\pi} \times 0.52 = 0.28 \text{ W/m}^2.$$

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**Problem 8.2** A plane wave traveling in medium 1 with $\varepsilon_{r1} = 2.25$ is normally incident upon medium 2 with $\varepsilon_{r2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the electric field of the incident wave is given by

$$\mathbf{E}^{\rm i} = \hat{\mathbf{y}} 8 \cos(6\pi \times 10^9 t - 30\pi x) \quad \text{(V/m)},$$

**(a)** obtain time-domain expressions for the electric and magnetic fields in each of the two media, and

**(b)** determine the average power densities of the incident, reflected and transmitted waves.

**Solution:**

**(a)**

$$\mathbf{E}^{\mathrm{i}} = \hat{\mathbf{y}}\, 8\cos(6\pi\times 10^9 t - 30\pi x) \quad \text{(V/m)},$$
$$\eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_1}}} = \frac{\eta_0}{\sqrt{2.25}} = \frac{\eta_0}{1.5} = \frac{377}{1.5} = 251.33\ \Omega,$$
$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{\eta_0}{\sqrt{4}} = \frac{377}{2} = 188.5\ \Omega,$$
$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1/2 - 1/1.5}{1/2 + 1/1.5} = -0.143,$$
$$\tau = 1 + \Gamma = 1 - 0.143 = 0.857,$$
$$\mathbf{E}^{\mathrm{r}} = \Gamma \mathbf{E}^{\mathrm{i}} = -1.14\, \hat{\mathbf{y}} \cos(6\pi\times 10^9 t + 30\pi x) \quad \text{(V/m)}.$$

Note that the coefficient of $x$ is positive, denoting the fact that $\mathbf{E}^{\mathrm{r}}$ belongs to a wave traveling in $-x$-direction.

$$\mathbf{E}_1 = \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{r}} = \hat{\mathbf{y}}\,[8\cos(6\pi\times 10^9 t - 30\pi x) - 1.14\cos(6\pi\times 10^9 t + 30\pi x)] \quad \text{(A/m)},$$
$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}\frac{8}{\eta_1}\cos(6\pi\times 10^9 t - 30\pi x) = \hat{\mathbf{z}}\, 31.83\cos(6\pi\times 10^9 t - 30\pi x) \quad \text{(mA/m)},$$
$$\mathbf{H}^{\mathrm{r}} = \hat{\mathbf{z}}\frac{1.14}{\eta_1}\cos(6\pi\times 10^9 t + 30\pi x) = \hat{\mathbf{z}}\, 4.54\cos(6\pi\times 10^9 t + 30\pi x) \quad \text{(mA/m)},$$
$$\begin{aligned}\mathbf{H}_1 &= \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{r}} \\ &= \hat{\mathbf{z}}\,[31.83\cos(6\pi\times 10^9 t - 30\pi x) + 4.54\cos(6\pi\times 10^9 t + 30\pi x)] \quad \text{(mA/m)}.\end{aligned}$$

Since $k_1 = \omega\sqrt{\mu\varepsilon_1}$ and $k_2 = \omega\sqrt{\mu\varepsilon_2}$,

$$k_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, k_1 = \sqrt{\frac{4}{2.25}}\, 30\pi = 40\pi \quad \text{(rad/m)},$$
$$\mathbf{E}_2 = \mathbf{E}^{\mathrm{t}} = \hat{\mathbf{y}}\, 8\tau\cos(6\pi\times 10^9 t - 40\pi x) = \hat{\mathbf{y}}\, 6.86\cos(6\pi\times 10^9 t - 40\pi x) \quad \text{(V/m)},$$
$$\mathbf{H}_2 = \mathbf{H}^{\mathrm{t}} = \hat{\mathbf{z}}\,\frac{8\tau}{\eta_2}\cos(6\pi\times 10^9 t - 40\pi x) = \hat{\mathbf{z}}\, 36.38\cos(6\pi\times 10^9 t - 40\pi x) \quad \text{(mA/m)}.$$

**(b)**

$$\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = \hat{\mathbf{x}}\,\frac{8^2}{2\eta_1} = \frac{64}{2\times 251.33} = \hat{\mathbf{x}}\, 127.3 \quad (\mathrm{mW/m^2}),$$
$$\mathbf{S}^{\mathrm{r}}_{\mathrm{av}} = -|\Gamma|^2 \mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = -\hat{\mathbf{x}}\,(0.143)^2 \times 0.127 = -\hat{\mathbf{x}}\, 2.6 \quad (\mathrm{mW/m^2}),$$

$$\begin{aligned}\mathbf{S}_{\mathrm{av}}^{\mathrm{t}} &= \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} \\ &= \hat{\mathbf{x}}\,\tau^2\frac{(8)^2}{2\eta_2} = \hat{\mathbf{x}}\,\frac{(0.86)^2 64}{2\times 188.5} = \hat{\mathbf{x}}\,124.7 \quad (\mathrm{mW/m^2}).\end{aligned}$$

Within calculation error, $\mathbf{S}_{\mathrm{av}}^{\mathrm{i}} + \mathbf{S}_{\mathrm{av}}^{\mathrm{r}} = \mathbf{S}_{\mathrm{av}}^{\mathrm{t}}$.

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**Problem 8.3** A plane wave traveling in a medium with $\varepsilon_{\mathrm{r}_1} = 9$ is normally incident upon a second medium with $\varepsilon_{\mathrm{r}_2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the magnetic field of the incident plane wave is given by

$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}\,2\cos(2\pi\times 10^9 t - ky) \quad (\mathrm{A/m}),$$

**(a)** obtain time domain expressions for the electric and magnetic fields in each of the two media, and

**(b)** determine the average power densities of the incident, reflected and transmitted waves.

**Solution:**

**(a)** In medium 1,

$$\begin{aligned}
u_{\mathrm{p}} &= \frac{c}{\sqrt{\varepsilon_{\mathrm{r}_1}}} = \frac{3\times 10^8}{\sqrt{9}} = 1\times 10^8 \quad (\mathrm{m/s}),\\
k_1 &= \frac{\omega}{u_{\mathrm{p}}} = \frac{2\pi\times 10^9}{1\times 10^8} = 20\pi \quad (\mathrm{rad/m}),\\
\mathbf{H}^{\mathrm{i}} &= \hat{\mathbf{z}}\,2\cos(2\pi\times 10^9 t - 20\pi y) \quad (\mathrm{A/m}),\\
\eta_1 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_1}}} = \frac{377}{3} = 125.67\ \Omega,\\
\eta_2 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{377}{2} = 188.5\ \Omega,\\
\mathbf{E}^{\mathrm{i}} &= -\hat{\mathbf{x}}\,2\eta_1\cos(2\pi\times 10^9 t - 20\pi y)\\
&= -\hat{\mathbf{x}}\,251.34\cos(2\pi\times 10^9 t - 20\pi y) \quad (\mathrm{V/m}),
\end{aligned}$$

$$\begin{aligned}
\Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{188.5 - 125.67}{188.5 + 125.67} = 0.2,\\
\tau &= 1 + \Gamma = 1.2,\\
\mathbf{E}^{\mathrm{r}} &= -\hat{\mathbf{x}}\,251.34\times 0.2\cos(2\pi\times 10^9 t + 20\pi y)\\
&= -\hat{\mathbf{x}}\,50.27\cos(2\pi\times 10^9 t + 20\pi y) \quad (\mathrm{V/m}),
\end{aligned}$$

$$\begin{aligned}\mathbf{H}^{\mathrm{r}} &= -\hat{\mathbf{z}}\,\frac{50.27}{\eta_1}\cos(2\pi\times 10^9 t + 20\pi y)\\ &= -\hat{\mathbf{z}}\,0.4\cos(2\pi\times 10^9 t + 20\pi y) \quad \text{(A/m)},\end{aligned}$$

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{r}}\\ &= -\hat{\mathbf{x}}\,[25.134\cos(2\pi\times 10^9 t - 20\pi y) + 50.27\cos(2\pi\times 10^9 t + 20\pi y)] \quad \text{(V/m)},\\ \mathbf{H}_1 &= \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{r}} = \hat{\mathbf{z}}\,[2\cos(2\pi\times 10^9 t - 20\pi y) - 0.4\cos(2\pi\times 10^9 t + 20\pi y)] \quad \text{(A/m)}.\end{aligned}$$

In medium 2,

$$\begin{aligned}k_2 &= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,k_1 = \sqrt{\frac{4}{9}}\times 20\pi = \frac{40\pi}{3} \quad \text{(rad/m)},\\ \mathbf{E}_2 = \mathbf{E}^{\mathrm{t}} &= -\hat{\mathbf{x}}\,251.34\,\tau\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right)\\ &= -\hat{\mathbf{x}}\,301.61\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right) \quad \text{(V/m)},\\ \mathbf{H}_2 = \mathbf{H}^{\mathrm{t}} &= \hat{\mathbf{z}}\,\frac{301.61}{\eta_2}\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right)\\ &= \hat{\mathbf{z}}\,1.6\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right) \quad \text{(A/m)}.\end{aligned}$$

**(b)**

$$\begin{aligned}\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} &= \hat{\mathbf{y}}\,\frac{|E_0|^2}{2\eta_1} = \hat{\mathbf{y}}\,\frac{(251.34)^2}{2\times 125.67} = \hat{\mathbf{y}}\,251.34 \quad \text{(W/m}^2\text{)},\\ \mathbf{S}^{\mathrm{r}}_{\mathrm{av}} &= -\hat{\mathbf{y}}\,|\Gamma|^2(251.34) = \hat{\mathbf{y}}\,10.05 \quad \text{(W/m}^2\text{)},\\ \mathbf{S}^{\mathrm{t}}_{\mathrm{av}} &= \hat{\mathbf{y}}\,(251.34 - 10.05) = \hat{\mathbf{y}}\,241.29 \quad \text{(W/m}^2\text{)}.\end{aligned}$$

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**Problem 8.4** A 200-MHz left-hand circularly polarized plane wave with an electric field modulus of 5 V/m is normally incident in air upon a dielectric medium with $\varepsilon_r = 4$ and occupying the region defined by $z \ge 0$.

**(a)** Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.

**(b)** Calculate the reflection and transmission coefficients.

**(c)** Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \le 0$.

**(d)** Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

**Solution:**

**(a)**

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \text{ rad/m},$$
$$k_2 = \frac{\omega}{u_{p_2}} = \frac{\omega}{c}\sqrt{\varepsilon_{r_2}} = \frac{4\pi}{3}\sqrt{4} = \frac{8\pi}{3} \text{ rad/m}.$$

LHC wave:

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = a_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}e^{j\pi/2})e^{-jkz} = a_0(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jkz},$$
$$\mathbf{E}^{\mathrm{i}}(z,t) = \hat{\mathbf{x}}a_0\cos(\omega t - kz) - \hat{\mathbf{y}}a_0\sin(\omega t - kz),$$
$$|\mathbf{E}^{\mathrm{i}}| = [a_0^2\cos^2(\omega t - kz) + a_0^2\sin^2(\omega t - kz)]^{1/2} = a_0 = 5 \quad \text{(V/m)}.$$

Hence,

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j4\pi z/3} \quad \text{(V/m)}.$$

**(b)**

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{2} = 60\pi \quad (\Omega).$$

Equations (8.8a) and (8.9) give

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = \frac{-60}{180} = -\frac{1}{3}, \qquad \tau = 1 + \Gamma = \frac{2}{3}.$$

**(c)**

$$\widetilde{\mathbf{E}}^{\mathrm{r}} = 5\Gamma(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{jk_1z} = -\frac{5}{3}(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{j4\pi z/3} \quad \text{(V/m)},$$
$$\widetilde{\mathbf{E}}^{\mathrm{t}} = 5\tau(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jk_2z} = \frac{10}{3}(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j8\pi z/3} \quad \text{(V/m)},$$
$$\widetilde{\mathbf{E}}_1 = \widetilde{\mathbf{E}}^{\mathrm{i}} + \widetilde{\mathbf{E}}^{\mathrm{r}} = 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\left[e^{-j4\pi z/3} - \frac{1}{3}e^{j4\pi z/3}\right] \quad \text{(V/m)}.$$

**(d)**

$$\%\text{ of reflected power} = 100 \times |\Gamma|^2 = \frac{100}{9} = 11.11\%,$$
$$\%\text{ of transmitted power} = 100 \times |\tau|^2\frac{\eta_1}{\eta_2} = 100 \times \left(\frac{2}{3}\right)^2 \times \frac{120\pi}{60\pi} = 88.89\%.$$

**Problem 8.5** Repeat Problem 8.4 after replacing the dielectric medium with a poor conductor characterized by $\varepsilon_r = 2.25$, $\mu_r = 1$, and $\sigma = 10^{-4}$ S/m.

**Solution:**

**(a)** Medium 1:

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \quad \text{(rad/m)}.$$

Medium 2:

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{10^{-4} \times 36\pi}{2\pi \times 2 \times 10^8 \times 2.25 \times 10^{-9}} = 4 \times 10^{-3}.$$

Hence, medium 2 is a low-loss dielectric. From Table 7-1,

$$\begin{aligned}
\alpha_2 &= \frac{\sigma_2}{2}\sqrt{\frac{\mu_2}{\varepsilon_2}} \\
&= \frac{\sigma_2}{2}\frac{120\pi}{\sqrt{\varepsilon_{r_2}}} = \frac{\sigma_2}{2} \times \frac{120\pi}{\sqrt{2.25}} = \frac{10^{-4}}{2} \times \frac{120\pi}{1.5} = 1.26 \times 10^{-2} \quad \text{(NP/m)}, \\
\beta_2 &= \omega\sqrt{\mu_2\varepsilon_2} = \frac{\omega\sqrt{\varepsilon_{r_2}}}{c} = 2\pi \quad \text{(rad/m)}, \\
\eta_2 &= \sqrt{\frac{\mu_2}{\varepsilon_2}}\left(1 + \frac{j\sigma_2}{2\omega\varepsilon_2}\right) = \frac{120\pi}{\sqrt{\varepsilon_{r_2}}}\left(1 + j2 \times 10^{-3}\right) \simeq \frac{120\pi}{1.5} = 80\pi \quad (\Omega).
\end{aligned}$$

LHC wave:

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{i}} &= a_0(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jk_1 z}, \\
|\widetilde{\mathbf{E}}^{\mathrm{i}}| &= a_0 = 5 \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}^{\mathrm{i}} &= 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j4\pi z/3} \quad \text{(V/m)}.
\end{aligned}$$

**(b)** According to Eqs. (8.8a) and (8.9),

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{80\pi - 120\pi}{80\pi + 120\pi} = -0.2, \qquad \tau = 1 + \Gamma = 1 - 0.2 = 0.8.$$

**(c)**

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{r}} &= 5\Gamma(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{jk_1 z} = -(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{j4\pi z/3} \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}^{\mathrm{t}} &= 5\tau(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-\alpha_2 z}e^{-j\beta_z z} = 4(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-1.26 \times 10^{-2} z}e^{-j2\pi z} \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}_1 &= \widetilde{\mathbf{E}}^{\mathrm{i}} + \widetilde{\mathbf{E}}^{\mathrm{r}} = 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})[e^{-j4\pi z/3} - 0.2e^{j4\pi z/3}] \quad \text{(V/m)}.
\end{aligned}$$

**(d)**

$$\begin{aligned}\text{\% of reflected power} &= 100|\Gamma|^2 = 100(0.2)^2 = 4\%,\\ \text{\% of transmitted power} &= 100|\tau|^2\frac{\eta_1}{\eta_2} = 100(0.8)^2 \times \frac{120\pi}{80\pi} = 96\%.\end{aligned}$$

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**Problem 8.6** A 50-MHz plane wave with electric field amplitude of 50 V/m is normally incident in air onto a semi-infinite, perfect dielectric medium with $\varepsilon_r = 36$. Determine (a) $\Gamma$, (b) the average power densities of the incident and reflected waves, and (c) the distance in the air medium from the boundary to the nearest minimum of the electric field intensity, $|\mathbf{E}|$.

**Solution:**

**(a)**

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{\varepsilon_{r_2}}} = \frac{120\pi}{6} = 20\pi \quad (\Omega),$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{20\pi - 120\pi}{20\pi + 120\pi} = -0.71.$$

Hence, $|\Gamma| = 0.71$ and $\theta_\eta = 180°$.

**(b)**

$$\begin{aligned}S^{\mathrm{i}}_{\mathrm{av}} &= \frac{|E^{\mathrm{i}}_0|^2}{2\eta_1} = \frac{(50)^2}{2 \times 120\pi} = 3.32 \quad (\mathrm{W/m^2}),\\ S^{\mathrm{r}}_{\mathrm{av}} &= |\Gamma|^2 S^{\mathrm{i}}_{\mathrm{av}} = (0.71)^2 \times 3.32 = 1.67 \quad (\mathrm{W/m^2}).\end{aligned}$$

**(c)** In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^7} = 6 \text{ m}.$$

From Eqs. (8.16) and (8.17),

$$\begin{aligned}l_{\max} &= \frac{\theta_r \lambda_1}{4\pi} = \frac{\pi \times 6}{4\pi} = 1.5 \text{ m},\\ l_{\min} &= l_{\max} - \frac{\lambda_1}{4} = 1.5 - 1.5 = 0 \text{ m (at the boundary)}.\end{aligned}$$

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**Problem 8.7** What is the maximum amplitude of the total electric field in the air medium of Problem 8.6, and at what nearest distance from the boundary does it occur?

**Solution:** From Problem 8.6, $\Gamma = -0.71$ and $\lambda = 6$ m.

$$|\widetilde{\mathbf{E}}_1|_{\max} = (1+|\Gamma|)E_0^i = (1+0.71)\times 50 = 85.5 \text{ V/m},$$
$$l_{\max} = \frac{\theta_r\lambda_1}{4\pi} = \frac{\pi\times 6}{4\pi} = 1.5 \text{ m}.$$

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**Problem 8.8** Repeat Problem 8.6 after replacing the dielectric medium with a conductor with $\varepsilon_r = 1$, $\mu_r = 1$, and $\sigma = 2.78\times 10^{-3}$ S/m.

**Solution:**

**(a)** Medium 1:

$$\eta_1 = \eta_0 = 120\pi = 377 \quad (\Omega), \qquad \lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m},$$

Medium 2:

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{2.78\times 10^{-3}\times 36\pi}{2\pi\times 5\times 10^7\times 10^{-9}} = 1.$$

Hence, Medium 2 is a quasi-conductor. From Eq. (7.70),

$$\begin{aligned}\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}\left(1-j\frac{\varepsilon_2''}{\varepsilon_2'}\right)^{-1/2} &= 120\pi\left(1-j\frac{\sigma_2}{\omega\varepsilon_2}\right)^{-1/2}\\ &= 120\pi(1-j1)^{-1/2}\\ &= 120\pi(\sqrt{2})^{-1/2}e^{j22.5^\circ} = (292.88+j121.31) \quad (\Omega).\end{aligned}$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{(292.88+j121.31)-377}{(292.88+j121.31)+377} = -0.09+j0.12 = 0.22\angle 114.5^\circ.$$

**(b)**

$$S_{\text{av}}^{\text{i}} = \frac{|E_0^{\text{i}}|^2}{2\eta_1} = \frac{50^2}{2\times 120\pi} = 3.32 \quad (\text{W/m}^2),$$
$$|S_{\text{av}}^{\text{r}}| = |\Gamma|^2 S_{\text{av}}^{\text{i}} = (0.22)^2(3.32) = 0.16 \quad (\text{W/m}^2).$$

**(c)** In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m}.$$

For $\theta_r = 114.5^\circ = 2$ rad, Eqs. (8.16) and (8.17) give

$$l_{\max} = \frac{\theta_r\lambda_1}{4\pi} + \frac{(0)\lambda_1}{2} = \frac{2(6)}{4} + 0 = 3 \text{ m},$$

$$l_{\min} = l_{\max} - \frac{\lambda_1}{4} = 3 - \frac{6}{4} = 3 - 1.5 = 1.5 \text{ m}.$$

---

**Problem 8.9** The three regions shown in Fig. 8-32 (P8.9) contain perfect dielectrics. For a wave in medium 1 incident normally upon the boundary at $z = -d$, what combination of $\varepsilon_{r_2}$ and $d$ produce no reflection? Express your answers in terms of $\varepsilon_{r_1}$, $\varepsilon_{r_3}$ and the oscillation frequency of the wave, $f$.

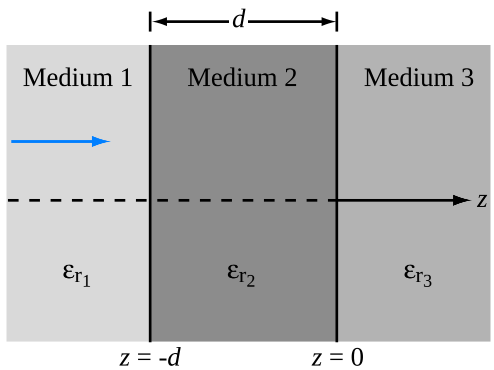



Figure P8.9: Three dielectric regions.

**Solution:** By analogy with the transmission-line case, there will be no reflection at $z = -d$ if medium 2 acts as a quarter-wave transformer, which requires that

$$d = \frac{\lambda_2}{4}$$

and

$$\eta_2 = \sqrt{\eta_1 \eta_3}\,.$$

The second condition may be rewritten as

$$\frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \left[\frac{\eta_0}{\sqrt{\varepsilon_{r_1}}}\,\frac{\eta_0}{\sqrt{\varepsilon_{r_3}}}\right]^{1/2}, \qquad \text{or} \quad \varepsilon_{r_2} = \sqrt{\varepsilon_{r_1}\varepsilon_{r_3}}\,,$$

$$\lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{c}{f(\varepsilon_{r_1}\varepsilon_{r_3})^{1/4}}\,,$$

and

$$d = \frac{c}{4f(\varepsilon_{r_1}\varepsilon_{r_3})^{1/4}}\,.$$

---

**Problem 8.10** For the configuration shown in Fig. 8-32 (P8.9), use transmission-line equations (or the Smith chart) to calculate the input impedance at $z = -d$ for $\varepsilon_{r_1} = 1$, $\varepsilon_{r_2} = 9$, $\varepsilon_{r_3} = 4$, $d = 1.2$ m, and $f = 50$ MHz. Also determine the fraction of the incident average power density reflected by the structure. Assume all media are lossless and nonmagnetic.

**Solution:** In medium 2,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{3\times 10^8}{5\times 10^7 \times 3} = 2 \text{ m}.$$

Hence,

$$\beta_2 = \frac{2\pi}{\lambda_2} = \pi \text{ rad/m}, \qquad \beta_2 d = 1.2\pi \text{ rad}.$$

At $z = -d$, the input impedance of a transmission line with load impedance $Z_L$ is given by Eq. (2.63) as

$$Z_{in}(-d) = Z_0 \left( \frac{Z_L + jZ_0 \tan\beta_2 d}{Z_0 + jZ_L \tan\beta_2 d} \right).$$

In the present case, $Z_0 = \eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} = \eta_0/3$ and $Z_L = \eta_3 = \eta_0/\sqrt{\varepsilon_{r_3}} = \eta_0/2$, where $\eta_0 = 120\pi$ ($\Omega$). Hence,

$$Z_{in}(-d) = \eta_2 \left( \frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d} \right) = \frac{\eta_0}{3} \left( \frac{\frac{1}{2} + j\left(\frac{1}{3}\right)\tan 1.2\pi}{\frac{1}{3} + j\left(\frac{1}{2}\right)\tan 1.2\pi} \right) = \eta_0(0.35 - j0.14).$$

At $z = -d$,

$$\Gamma = \frac{Z_{in} - Z_1}{Z_{in} + Z_1} = \frac{\eta_0(0.35 - j0.14) - \eta_0}{\eta_0(0.35 - j0.14) + \eta_0} = 0.49e^{-j162.14^\circ}.$$

Fraction of incident power reflected by the structure is $|\Gamma|^2 = |0.49|^2 = 0.24$.

---

**Problem 8.11** Repeat Problem 8.10 after interchanging $\varepsilon_{r_1}$ and $\varepsilon_{r_3}$.

**Solution:** In medium 2,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{3\times 10^8}{5\times 10^7 \times 3} = 2 \text{ m}.$$

Hence,

$$\beta_2 = \frac{2\pi}{\lambda_2} = \pi \text{ rad/m}, \qquad \beta_2 d = 1.2\pi \text{ rad}.$$

At $z = -d$, the input impedance of a transmission line with impedance $Z_L$ is given as Eq. (2.63),

$$Z_{in}(-d) = Z_0 \left( \frac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta 2d} \right).$$

In the present case, $Z_0 = \eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} = \eta_0/3$, $Z_L = \eta_3 = \eta_0/\sqrt{\varepsilon_{r_1}} = \eta_0$, where $\eta_0 = 120\pi$ ($\Omega$). Hence,

$$\begin{aligned} Z_{in}(-d) &= \eta_2 \left( \frac{\eta_3 + j\eta_2 \tan 1.2\pi}{\eta_2 + j\eta_3 \tan 1.2\pi} \right) \\ &= \frac{\eta_0}{3} \left( \frac{1 + (j/3)\tan 1.2\pi}{(1/3) + j\tan 1.2\pi} \right) \\ &= \eta_0 \left( \frac{1 + (j/3)\tan 1.2\pi}{1 + j3\tan 1.2\pi} \right) = (0.266 - j0.337)\eta_0 = 0.43\eta_0 \angle{-51.7^\circ}. \end{aligned}$$

At $z = -d$,

$$\Gamma = \frac{Z_{in} - Z_1}{Z_{in} + Z_1} = \frac{0.43\angle{-51.7^\circ} - \frac{1}{2}}{0.43\angle{-51.7^\circ} + \frac{1}{2}} = 0.49\angle{-101.1^\circ}.$$

Fraction of incident power reflected by structure is $|\Gamma|^2 = 0.24$.

---

**Problem 8.12** Orange light of wavelength 0.61 $\mu$m in air enters a block of glass with $\varepsilon_r = 1.44$. What color would it appear to a sensor embedded in the glass? The wavelength ranges of colors are violet (0.39 to 0.45 $\mu$m), blue (0.45 to 0.49 $\mu$m), green (0.49 to 0.58 $\mu$m), yellow (0.58 to 0.60 $\mu$m), orange (0.60 to 0.62 $\mu$m), and red (0.62 to 0.78 $\mu$m).

**Solution:** In the glass,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{0.61}{\sqrt{1.44}} = 0.508 \ \mu\text{m}.$$

The light would appear green.

---

**Problem 8.13** A plane wave of unknown frequency is normally incident in air upon the surface of a perfect conductor. Using an electric-field meter, it was determined that the total electric field in the air medium is always zero when measured at a distance of 2 m from the conductor surface. Moreover, no such nulls were observed at distances closer to the conductor. What is the frequency of the incident wave?

**Solution:** The electric field of the standing wave is zero at the conductor surface, and the standing wave pattern repeats itself every $\lambda/2$. Hence,

$$\frac{\lambda}{2} = 2 \text{ m}, \qquad \text{or } \lambda = 4 \text{ m},$$

in which case

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{4} = 7.5 \times 10^7 = 75 \text{ MHz}.$$

---

**Problem 8.14** Consider a thin film of soap in air under illumination by yellow light with $\lambda = 0.6$ $\mu$m in vacuum. If the film is treated as a planar dielectric slab with $\varepsilon_r = 1.72$, surrounded on both sides by air, what film thickness would produce strong reflection of the yellow light at normal incidence?

**Solution:** The transmission line analogue of the soap-bubble wave problem is shown in Fig. P8.14(b) where the load $Z_L$ is equal to $\eta_0$, the impedance of the air medium on the other side of the bubble. That is,

$$\eta_0 = 377 \ \Omega, \qquad \eta_1 = \frac{377}{\sqrt{1.72}} = 287.5 \ \Omega.$$

The normalized load impedance is

$$z_L = \frac{\eta_0}{\eta_1} = 1.31.$$

For the reflection by the soap bubble to be the largest, $Z_{in}$ needs to be the most different from $\eta_0$. This happens when $z_L$ is transformed through a length $\lambda/4$. Hence,

$$L = \frac{\lambda}{4} = \frac{\lambda_0}{4\sqrt{\varepsilon_r}} = \frac{0.6 \ \mu\text{m}}{4\sqrt{1.72}} = 0.115 \ \mu\text{m},$$

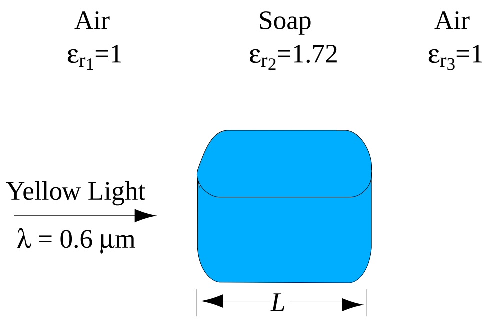
where $\lambda$ is the wavelength of the soap bubble material. Strong reflections will also occur if the thickness is greater than $L$ by integer multiples of $n\lambda/2 = (0.23n)$ $\mu$m.

Hence, in general

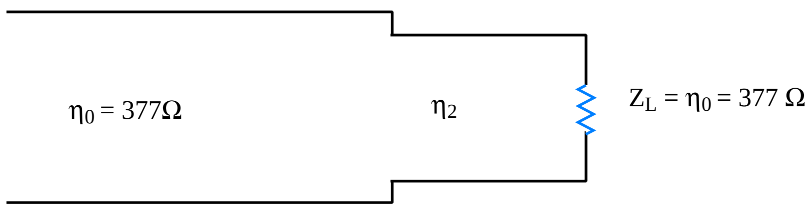
$$L = (0.115 + 0.23n) \ \mu\text{m}, \qquad n = 0, 1, 2, \dots .$$

According to Section 2-7.5, transforming a load $Z_L = 377$ $\Omega$ through a $\lambda/4$ section of $Z_0 = 287.5$ $\Omega$ ends up presenting an input impedance of

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{(287.5)^2}{377} = 219.25 \ \Omega.$$




(a) Yellow light incident on soap bubble.




(b) Transmission-line equivalent circuit

Figure P8.14: Diagrams for Problem 8.14.

This $Z_{\rm in}$ is at the input side of the soap bubble. The reflection coefficient at that interface is

$$\Gamma = \frac{Z_{\rm in} - \eta_0}{Z_{\rm in} + \eta_0} = \frac{219.25 - 377}{219.25 + 377} = -0.27.$$

Any other thickness would produce a reflection coeffficient with a smaller magnitude.

---

**Problem 8.15** A 5-MHz plane wave with electric field amplitude of 10 (V/m) is normally incident in air onto the plane surface of a semi-infinite conducting material with $\varepsilon_r = 4$, $\mu_r = 1$, and $\sigma = 100$ (S/m). Determine the average power dissipated (lost) per unit cross-sectional area in a 2-mm penetration of the conducting medium.

**Solution:** For convenience, let us choose $\mathbf{E}^i$ to be along $\hat{\mathbf{x}}$ and the incident direction to be $+\hat{\mathbf{z}}$. With

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 5 \times 10^6}{3 \times 10^8} = \frac{\pi}{30} \quad \text{(rad/m)},$$

we have

$$\mathbf{E}^{\rm i} = \hat{\mathbf{x}}\,10\cos\left(\pi\times 10^7 t - \frac{\pi}{30}z\right) \quad \text{(V/m)},$$
$$\eta_1 = \eta_0 = 377\ \Omega.$$

From Table 7-1,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon_{\rm r}\varepsilon_0} = \frac{100\times 36\pi}{\pi\times 10^7\times 4\times 10^{-9}} = 9\times 10^4,$$

which makes the material a good conductor, for which

$$\alpha_2 = \sqrt{\pi f\mu\sigma} = \sqrt{\pi\times 5\times 10^6\times 4\pi\times 10^{-7}\times 100} = 44.43 \quad \text{(Np/m)},$$
$$\beta_2 = 44.43 \quad \text{(rad/m)},$$
$$\eta_{\rm c_2} = (1+j)\,\frac{\alpha_2}{\sigma} = (1+j)\,\frac{44.43}{100} = 0.44\,(1+j)\ \Omega.$$

According to the expression for $\mathbf{S}_{\rm av_2}$ given in the answer to Exercise 8.3,

$$\mathbf{S}_{\rm av_2} = \hat{\mathbf{z}}\,|\tau|^2\,\frac{|E_0^{\rm i}|^2}{2}\,e^{-2\alpha_2 z}\,\mathfrak{Re}\left(\frac{1}{\eta^*_{\rm c_2}}\right).$$

The power lost is equal to the difference between $\mathbf{S}_{\rm av_2}$ at $z=0$ and $\mathbf{S}_{\rm av_2}$ at $z=2$ mm. Thus,

$$\begin{aligned} P' &= \text{power lost per unit cross-sectional area} \\ &= S_{\rm av_2}(0) - S_{\rm av_2}(z=2\text{ mm}) \\ &= |\tau|^2\,\frac{|E_0^{\rm i}|^2}{2}\,\mathfrak{Re}\left(\frac{1}{\eta^*_{\rm c_2}}\right)\,[1-e^{-2\alpha_2 z_1}] \end{aligned}$$

where $z_1 = 2$ mm.

$$\begin{aligned} \tau &= 1+\Gamma \\ &= 1+\frac{\eta_2-\eta_1}{\eta_2+\eta_1} = 1+\frac{0.44\,(1+j)-377}{0.44\,(1+j)+377} \approx 0.0023\,(1+j) = 3.3\times 10^{-3}e^{j45^\circ}. \end{aligned}$$

$$\begin{aligned} \mathfrak{Re}\left(\frac{1}{\eta^*_{\rm c_2}}\right) &= \mathfrak{Re}\left(\frac{1}{0.44\,(1+j)^*}\right) \\ &= \mathfrak{Re}\left(\frac{1}{0.44\,(1-j)}\right) = \mathfrak{Re}\left(\frac{1+j}{0.44\times 2}\right) = \frac{1}{0.88} = 1.14, \end{aligned}$$

$$P' = (3.3\times 10^{-3})^2\,\frac{10^2}{2}\times 1.14\,[1-e^{-2\times 44.43\times 2\times 10^{-3}}] = 1.01\times 10^{-4} \quad (\text{W/m}^2).$$

**Problem 8.16** A 0.5-MHz antenna carried by an airplane flying over the ocean surface generates a wave that approaches the water surface in the form of a normally incident plane wave with an electric-field amplitude of 3,000 (V/m). Sea water is characterized by $\varepsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m). The plane is trying to communicate a message to a submarine submerged at a depth $d$ below the water surface. If the submarine's receiver requires a minimum signal amplitude of 0.01 ($\mu$V/m), what is the maximum depth $d$ to which successful communication is still possible?

**Solution:** For sea water at 0.5 MHz,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{4 \times 36\pi}{2\pi \times 0.5 \times 10^6 \times 72 \times 10^{-9}} = 2000.$$

Hence, sea water is a good conductor, in which case we use the following expressions from Table 7-1:

$$\begin{aligned}
\alpha_2 &= \sqrt{\pi f \mu\sigma} = \sqrt{\pi \times 0.5 \times 10^6 \times 4\pi \times 10^{-7} \times 4} = 2.81 \quad \text{(Np/m)},\\
\beta_2 &= 2.81 \quad \text{(rad/m)},\\
\eta_{c_2} &= (1+j)\,\frac{\alpha_2}{\sigma} = (1+j)\,\frac{2.81}{4} = 0.7\,(1+j)\ \Omega,\\
\Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{0.7\,(1+j) - 377}{0.7\,(1+j) + 377} = (-0.9963 + j3.7 \times 10^{-3}),\\
\tau &= 1 + \Gamma = 5.24 \times 10^{-3} e^{j44.89^\circ},\\
|E^{\mathrm{t}}| &= |\tau E_0^{\mathrm{i}} e^{-\alpha_2 d}|.
\end{aligned}$$

We need to find the depth $z$ at which $|E^{\mathrm{t}}| = 0.01\ \mu\text{V/m} = 10^{-8}$ V/m.

$$\begin{aligned}
10^{-8} &= 5.24 \times 10^{-3} \times 3 \times 10^3 e^{-2.81\,d},\\
e^{-2.81\,d} &= 6.36 \times 10^{-10},\\
-2.81\,d &= \ln(6.36 \times 10^{-10}) = -21.18,
\end{aligned}$$

or

$$d = 7.54 \quad \text{(m)}.$$

## Sections 8-2 and 8-3: Snell's Laws and Fiber Optics

**Problem 8.17** A light ray is incident on a prism at an angle $\theta$ as shown in Fig. 8-33 (P8.17). The ray is refracted at the first surface and again at the second surface. In terms of the apex angle $\phi$ of the prism and its index of refraction $n$, determine the smallest value of $\theta$ for which the ray will emerge from the other side. Find this minimum $\theta$ for $n = 4$ and $\phi = 60°$.

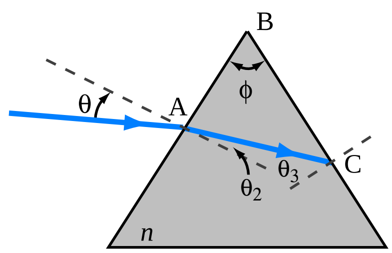



Figure P8.17: Prism of Problem 8.17.

**Solution:** For the beam to emerge at the second boundary, it is necessary that

$$\theta_3 < \theta_c,$$

where $\sin\theta_c = 1/n$. From the geometry of triangle *ABC*,

$$180° = \phi + (90° - \theta_2) + (90° - \theta_3),$$

or $\theta_2 = \phi - \theta_3$. At the first boundary, $\sin\theta = n\sin\theta_2$. Hence,

$$\sin\theta_{\text{min}} = n\sin(\phi - \theta_3) = n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right),$$

or

$$\theta_{\text{min}} = \sin^{-1}\left[n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right)\right].$$

For $n = 4$ and $\phi = 60°$,

$$\theta_{\text{min}} = \sin^{-1}\left[4\sin(60° - \sin^{-1}\left(\frac{1}{4}\right)\right] = 20.4°.$$

**Problem 8.18** For some types of glass, the index of refraction varies with wavelength. A prism made of a material with

$$n = 1.71 - \frac{4}{30}\lambda_0, \quad (\lambda_0 \text{ in } \mu\text{m}),$$

where $\lambda_0$ is the wavelength in vacuum, was used to disperse white light as shown in Fig. 8-34 (P8.18). The white light is incident at an angle of 50°, the wavelength $\lambda_0$ of red light is 0.7 $\mu$m and that of violet light is 0.4 $\mu$m. Determine the angular dispersion in degrees.

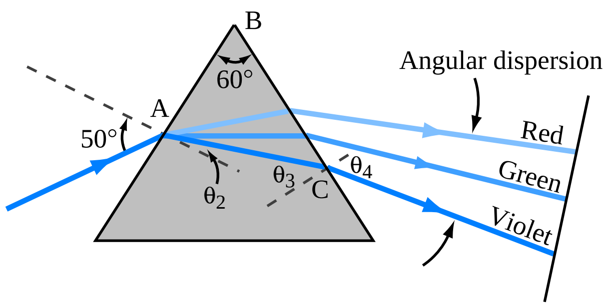



Figure P8.18: Prism of Problem 8.18.

**Solution:** For violet,

$$n_{\text{v}} = 1.71 - \frac{4}{30} \times 0.4 = 1.66, \qquad \sin\theta_2 = \frac{\sin\theta}{n_{\text{v}}} = \frac{\sin 50^\circ}{1.66},$$

or

$$\theta_2 = 27.48^\circ.$$

From the geometry of triangle *ABC*,

$$180^\circ = 60^\circ + (90^\circ - \theta_2) + (90^\circ - \theta_3),$$

or

$$\theta_3 = 60^\circ - \theta_2 = 60 - 27.48^\circ = 32.52^\circ,$$

and

$$\sin\theta_4 = n_{\text{v}} \sin\theta_3 = 1.66 \sin 32.52^\circ = 0.89,$$

or

$$\theta_4 = 63.18^\circ.$$

For red,

$$n_r = 1.71 - \frac{4}{30} \times 0.7 = 1.62,$$
$$\theta_2 = \sin^{-1}\left[\frac{\sin 50^\circ}{1.62}\right] = 28.22^\circ,$$
$$\theta_3 = 60^\circ - 28.22^\circ = 31.78^\circ,$$
$$\theta_4 = \sin^{-1}[1.62 \sin 31.78^\circ] = 58.56^\circ.$$

Hence, angular dispersion $= 63.18^\circ - 58.56^\circ = 4.62^\circ$.

---

**Problem 8.19** The two prisms in Fig. 8-35 (P8.19) are made of glass with $n = 1.5$. What fraction of the power density carried by the ray incident upon the top prism emerges from bottom prism? Neglect multiple internal reflections.

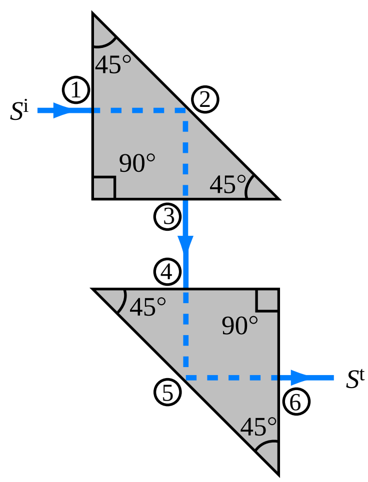



Figure P8.19: Periscope problem.

**Solution:** Using $\eta = \eta_0/n$, at interfaces 1 and 4,

$$\Gamma_a = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2.$$

At interfaces 3 and 6,

$$\Gamma_b = -\Gamma_a = 0.2.$$

At interfaces 2 and 5,

$$\theta_c = \sin^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.81^\circ.$$

Hence, total internal reflection takes place at those interfaces. At interfaces 1, 3, 4 and 6, the ratio of power density transmitted to that incident is $(1-\Gamma^2)$. Hence,

$$\frac{S^t}{S^i} = (1-\Gamma^2)^4 = (1-(0.2)^2)^4 = 0.85.$$

---

**Problem 8.20** A light ray incident at 45° passes through two dielectric materials with the indices of refraction and thicknesses given in Fig. 8-36 (P8.20). If the ray strikes the surface of the first dielectric at a height of 2 cm, at what height will it strike the screen?

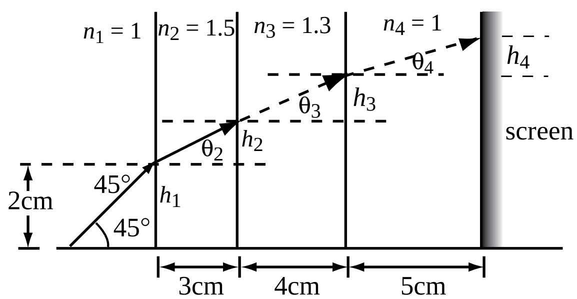



Figure P8.20: Light incident on a screen through a multi-layered dielectric (Problem 8.20).

**Solution:**

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{1}{1.5}\sin 45^\circ = 0.47.$$

Hence,

$$\theta_2 = 28.13^\circ,$$
$$h_2 = 3\text{ cm} \times \tan\theta_2 = 3\text{ cm} \times 0.53 = 1.6\text{ cm},$$
$$\sin\theta_3 = \frac{n_2}{n_3}\sin\theta_2 = \frac{1.5}{1.3}\sin 28.13^\circ = 0.54.$$

Hence,

$$\theta_3 = 32.96^\circ,$$

$$h_3 = 4 \text{ cm} \times \tan 32.96^\circ = 2.6 \text{ cm},$$
$$\sin\theta_4 = \frac{n_3}{n_4}\sin\theta_3 = 0.707.$$

Hence,

$$\theta_4 = 45^\circ,$$
$$h_4 = 5 \text{ cm} \times \tan 45^\circ = 5 \text{ cm}.$$

Total height $= h_1 + h_2 + h_3 + h_4 = (2 + 1.6 + 2.6 + 5) = 11.2$ cm.

---

**Problem 8.21** Figure P8.21 depicts a beaker containing a block of glass on the bottom and water over it. The glass block contains a small air bubble at an unknown depth below the water surface. When viewed from above at an angle of 60°, the air bubble appears at a depth of 6.81 cm. What is the true depth of the air bubble?

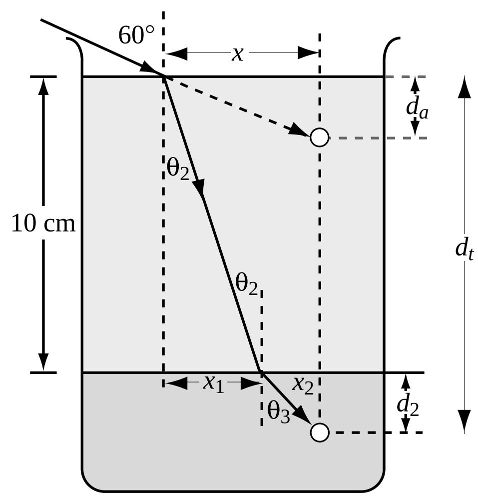



Figure P8.21: Apparent position of the air bubble in Problem 8.21.

**Solution:** Let

$$d_a = 6.81 \text{ cm} = \text{apparent depth},$$
$$d_t = \text{true depth}.$$

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_i\right] = \sin^{-1}\left[\frac{1}{1.33}\sin 60^\circ\right] = 40.6^\circ,$$

$$\theta_3 = \sin^{-1}\left[\frac{n_1}{n_3}\sin\theta_i\right] = \sin^{-1}\left[\frac{1}{1.6}\sin 60^\circ\right] = 32.77^\circ,$$
$$x_1 = (10 \text{ cm}) \times \tan 40.6^\circ = 8.58 \text{ cm},$$
$$x = d_a \cot 30^\circ = 6.81 \cot 30^\circ = 11.8 \text{ cm}.$$

Hence,

$$x_2 = x - x_1 = 11.8 - 8.58 = 3.22 \text{ cm},$$

and

$$d_2 = x_2 \cot 32.77^\circ = (3.22 \text{ cm}) \times \cot 32.77^\circ = 5 \text{ cm}.$$

Hence, $d_t = (10+5) = 15$ cm.

---

**Problem 8.22** A glass semicylinder with $n = 1.5$ is positioned such that its flat face is horizontal, as shown in Fig. 8-38 (P8.22). Its horizontal surface supports a drop of oil, as shown. When light is directed radially toward the oil, total internal reflection occurs if $\theta$ exceeds 53°. What is the index of refraction of the oil?

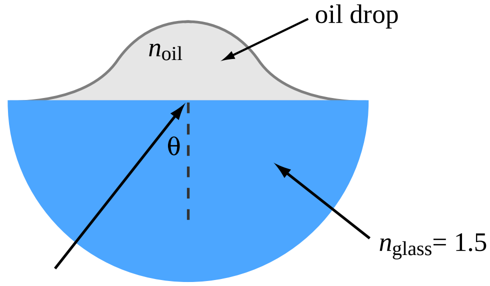



Figure P8.22: Oil drop on the flat surface of a glass semicylinder (Problem 8.22).

**Solution:**

$$\sin\theta_c = \frac{n_2}{n_1} = \frac{n_{oil}}{1.5},$$
$$n_{oil} = 1.5 \sin 53^\circ = 1.2.$$

---

**Problem 8.23** A penny lies at the bottom of a water fountain at a depth of 30 cm. Determine the diameter of a piece of paper which, if placed to float on the surface of

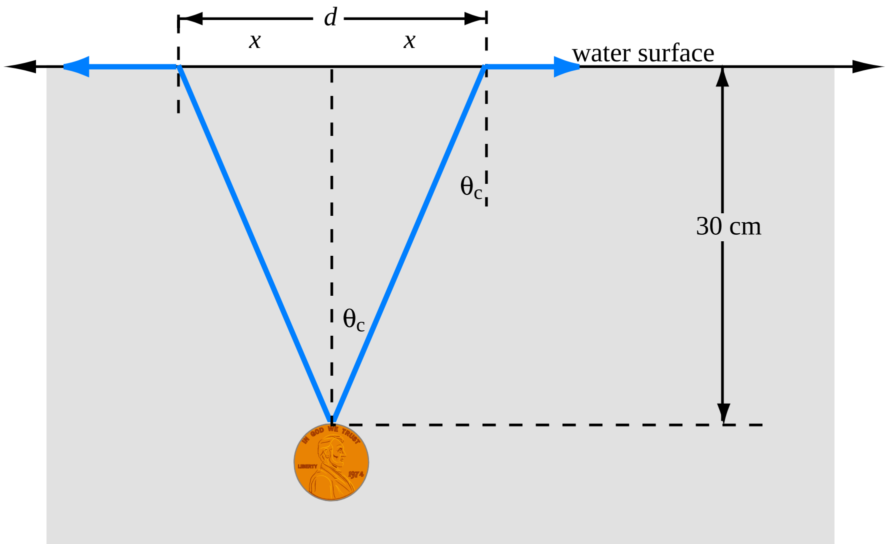



Figure P8.23: Light cone bounded by total internal reflection.

the water directly above the penny, would totally obscure the penny from view. Treat the penny as a point and assume that $n = 1.33$ for water.

**Solution:**

$$\theta_c = \sin^{-1}\left[\frac{1}{1.33}\right] = 48.75^\circ,$$
$$d = 2x = 2[(30 \text{ cm})\tan\theta_c] = (60 \text{ cm}) \times \tan 48.75^\circ = 68.42 \text{ cm}.$$

---

**Problem 8.24** Suppose the optical fiber of Example 8-5 is submerged in water (with $n = 1.33$) instead of air. Determine $\theta_a$ and $f_p$ in that case.

**Solution:** With $n_0 = 1.33$, $n_f = 1.52$ and $n_c = 1.49$, Eq. (8.40) gives

$$\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2} = \frac{1}{1.33}\left[(1.52)^2 - (1.49)^2\right]^{1/2} = 0.23,$$

or

$$\theta_a = 13.1^\circ.$$

The data rate $f_p$ given by Eq. (8.45) is not a function of $n_0$, and therefore it remains unchanged at 4.9 (Mb/s).

**Problem 8.25** Equation (8.45) was derived for the case where the light incident upon the sending end of the optical fiber extends over the entire acceptance cone shown in Fig. 8-12(b). Suppose the incident light is constrained to a narrower range extending between normal incidence and $\theta'$, where $\theta' < \theta_a$.

**(a)** Obtain an expression for the maximum data rate $f_p$ in terms of $\theta'$.

**(b)** Evaluate $f_p$ for the fiber of Example 8-5 when $\theta' = 5°$.

**Solution:**

**(a)** For $\theta_i = \theta'$,

$$\sin\theta_2 = \frac{1}{n_f}\sin\theta',$$

$$l_{max} = \frac{l}{\cos\theta_2} = \frac{l}{\sqrt{1-\sin^2\theta_2}} = \frac{l}{\sqrt{1-\left(\frac{\sin\theta'}{n_f}\right)^2}} = \frac{ln_f}{\sqrt{n_f^2-(\sin\theta')^2}},$$

$$t_{max} = \frac{l_{max}}{u_p} = \frac{l_{max}n_f}{c} = \frac{ln_f^2}{c\sqrt{n_f^2-(\sin\theta')^2}},$$

$$t_{min} = \frac{l}{u_p} = l\,\frac{n_f}{c},$$

$$\tau = \Delta t = t_{max} - t_{min} = l\,\frac{n_f}{c}\left[\frac{n_f}{\sqrt{n_f^2-(\sin\theta')^2}} - 1\right],$$

$$f_p = \frac{1}{2\tau} = \frac{c}{2ln_f}\left[\frac{n_f}{\sqrt{n_f^2-(\sin\theta')^2}} - 1\right]^{-1} \quad \text{(bits/s)}.$$

**(b)** For:

$$n_f = 1.52,$$
$$\theta' = 5°,$$
$$l = 1 \text{ km},$$
$$c = 3\times10^8 \text{ m/s},$$
$$f_p = 59.88 \quad \text{(Mb/s)}.$$

## Sections 8-4 and 8-5: Reflection and Transmission at Oblique Incidence

**Problem 8.26** A plane wave in air with

$$\widetilde{\mathbf{E}}^{\text{i}} = \hat{\mathbf{y}}\,20e^{-j(3x+4z)} \quad \text{(V/m)},$$

is incident upon the planar surface of a dielectric material, with $\varepsilon_{\text{r}} = 4$, occupying the half space $z \geq 0$. Determine:

**(a)** the polarization of the incident wave,
**(b)** the angle of incidence,
**(c)** the time-domain expressions for the reflected electric and magnetic fields,
**(d)** the time-domain expressions for the transmitted electric and magnetic fields, and
**(e)** the average power density carried by the wave in the dielectric medium.

**Solution:**
**(a)** $\widetilde{\mathbf{E}}^{\text{i}} = \hat{\mathbf{y}}\,20e^{-j(3x+4z)}$ V/m.
Since $\mathbf{E}^{\text{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is perpendicularly polarized.
**(b)** From Eq. (8.48a), the argument of the exponential is

$$-jk_1(x\sin\theta_{\text{i}} + z\cos\theta_{\text{i}}) = -j(3x+4z).$$

Hence,

$$k_1\sin\theta_{\text{i}} = 3, \qquad k_1\cos\theta_{\text{i}} = 4,$$

from which we determine that

$$\tan\theta_{\text{i}} = \frac{3}{4} \qquad \text{or} \qquad \theta_{\text{i}} = 36.87^\circ,$$

and

$$k_1 = \sqrt{3^2+4^2} = 5 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\text{p}}k = ck = 3\times10^8\times5 = 1.5\times10^9 \quad \text{(rad/s)}.$$

**(c)**

$$\eta_1 = \eta_0 = 377\ \Omega,$$
$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\text{r}_2}}} = \frac{\eta_0}{2} = 188.5\ \Omega,$$
$$\theta_{\text{t}} = \sin^{-1}\left[\frac{\sin\theta_{\text{i}}}{\sqrt{\varepsilon_{\text{r}_2}}}\right] = \sin^{-1}\left[\frac{\sin 36.87^\circ}{\sqrt{4}}\right] = 17.46^\circ,$$

$$\Gamma_\perp = \frac{\eta_2\cos\theta_\mathrm{i} - \eta_1\cos\theta_\mathrm{t}}{\eta_2\cos\theta_\mathrm{i} + \eta_1\cos\theta_\mathrm{t}} = -0.41,$$
$$\tau_\perp = 1 + \Gamma_\perp = 0.59.$$

In accordance with Eq. (8.49a), and using the relation $E_0^\mathrm{r} = \Gamma_\perp E_0^\mathrm{i}$,

$$\widetilde{\mathbf{E}}^\mathrm{r} = -\hat{\mathbf{y}}\, 8.2\, e^{-j(3x-4z)},$$
$$\widetilde{\mathbf{H}}^\mathrm{r} = (\hat{\mathbf{x}}\cos\theta_\mathrm{i} + \hat{\mathbf{z}}\sin\theta_\mathrm{i})\,\frac{8.2}{\eta_0}\, e^{-j(3x-4z)},$$

where we used the fact that $\theta_\mathrm{i} = \theta_\mathrm{r}$ and the $z$-direction has been reversed.

$$\mathbf{E}^\mathrm{r} = \mathfrak{Re}[\widetilde{\mathbf{E}}^\mathrm{r} e^{j\omega t}] = -\hat{\mathbf{y}}\, 8.2\cos(1.5\times 10^9 t - 3x + 4z) \quad \text{(V/m)},$$
$$\mathbf{H}^\mathrm{r} = (\hat{\mathbf{x}}\, 17.4 + \hat{\mathbf{z}}\, 13.06)\cos(1.5\times 10^9 t - 3x + 4z) \quad \text{(mA/m)}.$$

**(d)** In medium 2,

$$k_2 = k_1\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 5\sqrt{4} = 20 \quad \text{(rad/m)},$$

and

$$\theta_\mathrm{t} = \sin^{-1}\left[\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_\mathrm{i}\right] = \sin^{-1}\left[\frac{1}{2}\sin 36.87^\circ\right] = 17.46^\circ$$

and the exponent of $\mathbf{E}^\mathrm{t}$ and $\mathbf{H}^\mathrm{t}$ is

$$-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t}) = -j10(x\sin 17.46^\circ + z\cos 17.46^\circ) = -j(3x + 9.54z).$$

Hence,

$$\widetilde{\mathbf{E}}^\mathrm{t} = \hat{\mathbf{y}}\, 20\times 0.59\, e^{-j(3x+9.54z)},$$
$$\widetilde{\mathbf{H}}^\mathrm{t} = (-\hat{\mathbf{x}}\cos\theta_\mathrm{t} + \hat{\mathbf{z}}\sin\theta_\mathrm{t})\,\frac{20\times 0.59}{\eta_2}\, e^{-j(3x+9.54z)}.$$
$$\mathbf{E}^\mathrm{t} = \mathfrak{Re}[\widetilde{\mathbf{E}}^\mathrm{t} e^{j\omega t}] = \hat{\mathbf{y}}\, 11.8\cos(1.5\times 10^9 t - 3x - 9.54z) \quad \text{(V/m)},$$
$$\mathbf{H}^\mathrm{t} = (-\hat{\mathbf{x}}\cos 17.46^\circ + \hat{\mathbf{z}}\sin 17.46^\circ)\,\frac{11.8}{188.5}\cos(1.5\times 10^9 t - 3x - 9.54z)$$
$$= (-\hat{\mathbf{x}}\, 59.72 + \hat{\mathbf{z}}\, 18.78)\,\cos(1.5\times 10^9 t - 3x - 9.54z) \quad \text{(mA/m)}.$$

**(e)**

$$S_\mathrm{av}^\mathrm{t} = \frac{|E_0^\mathrm{t}|^2}{2\eta_2} = \frac{(11.8)^2}{2\times 188.5} = 0.36 \quad (\mathrm{W/m^2}).$$

**Problem 8.27** Repeat Problem 8.26 for a wave in air with

$$\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)} \quad \text{(A/m)},$$

incident upon the planar boundary of a dielectric medium ($z \geq 0$) with $\varepsilon_{\mathrm{r}} = 9$.

**Solution:**

**(a)** $\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)}$.

Since $\mathbf{H}^{\mathrm{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is TM polarized, or equivalently, its electric field vector is parallel polarized (parallel to the plane of incidence).

**(b)** From Eq. (8.65b), the argument of the exponential is

$$-jk_1(x \sin\theta_{\mathrm{i}} + z \cos\theta_{\mathrm{i}}) = -j(8x + 6z).$$

Hence,

$$k_1 \sin\theta_{\mathrm{i}} = 8, \qquad k_1 \cos\theta_{\mathrm{i}} = 6,$$

from which we determine

$$\theta_{\mathrm{i}} = \tan^{-1}\left(\frac{8}{6}\right) = 53.13^\circ,$$
$$k_1 = \sqrt{6^2 + 8^2} = 10 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\mathrm{p}} k = ck = 3 \times 10^8 \times 10 = 3 \times 10^9 \quad \text{(rad/s)}.$$

**(c)**

$$\eta_1 = \eta_0 = 377\ \Omega,$$
$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{\eta_0}{3} = 125.67\ \Omega,$$
$$\theta_{\mathrm{t}} = \sin^{-1}\left[\frac{\sin\theta_{\mathrm{i}}}{\sqrt{\varepsilon_{\mathrm{r}_2}}}\right] = \sin^{-1}\left[\frac{\sin 53.13^\circ}{\sqrt{9}}\right] = 15.47^\circ,$$
$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_{\mathrm{t}} - \eta_1 \cos\theta_{\mathrm{i}}}{\eta_2 \cos\theta_{\mathrm{t}} + \eta_1 \cos\theta_{\mathrm{i}}} = -0.30,$$
$$\tau_\parallel = (1 + \Gamma_\parallel)\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}} = 0.44.$$

In accordance with Eqs. (8.65a) to (8.65d), $E_0^{\mathrm{i}} = 2 \times 10^{-2} \eta_1$ and

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = (\hat{\mathbf{x}} \cos\theta_{\mathrm{i}} - \hat{\mathbf{z}} \sin\theta_{\mathrm{i}})\, 2 \times 10^{-2} \eta_1\, e^{-j(8x+6z)} = (\hat{\mathbf{x}}\, 4.52 - \hat{\mathbf{z}}\, 6.03)\, e^{-j(8x+6z)}.$$

$\widetilde{\mathbf{E}}^{\mathrm{r}}$ is similar to $\widetilde{\mathbf{E}}^{\mathrm{i}}$ except for reversal of $z$-components and multiplication of amplitude by $\Gamma_{\parallel}$. Hence, with $\Gamma_{\parallel} = -0.30$,

$$\begin{aligned}
\mathbf{E}^{\mathrm{r}} &= \mathfrak{Re}[\widetilde{\mathbf{E}}^{\mathrm{r}} e^{j\omega t}] = -(\hat{\mathbf{x}}\,1.36 + \hat{\mathbf{z}}\,1.81)\cos(3\times 10^{9}t - 8x + 6z) \text{ V/m},\\
\mathbf{H}^{\mathrm{r}} &= \hat{\mathbf{y}}\,2\times 10^{-2}\Gamma_{\parallel}\cos(3\times 10^{9}t - 8x + 6z)\\
&= -\hat{\mathbf{y}}\,0.6\times 10^{-2}\cos(3\times 10^{9}t - 8x + 6z) \text{ A/m}.
\end{aligned}$$

**(d)** In medium 2,

$$\begin{aligned}
k_2 &= k_1\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 10\sqrt{9} = 30 \text{ rad/m},\\
\theta_{\mathrm{t}} &= \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sin\theta_{\mathrm{i}}\right] = \sin^{-1}\left[\frac{1}{3}\sin 53.13^\circ\right] = 15.47^\circ,
\end{aligned}$$

and the exponent of $\mathbf{E}^{\mathrm{t}}$ and $\mathbf{H}^{\mathrm{t}}$ is

$$-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}}) = -j30(x\sin 15.47^\circ + z\cos 15.47^\circ) = -j(8x + 28.91z).$$

Hence,

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{t}} &= (\hat{\mathbf{x}}\cos\theta_{\mathrm{t}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{t}})E_0^{\mathrm{i}}\tau_{\parallel}e^{-j(8x+28.91z)}\\
&= (\hat{\mathbf{x}}\,0.96 - \hat{\mathbf{z}}\,0.27)\,2\times 10^{-2}\times 377\times 0.44\,e^{-j(8x+28.91z)}\\
&= (\hat{\mathbf{x}}\,3.18 - \hat{\mathbf{z}}\,0.90)\,e^{-j(8x+28.91z)},\\
\widetilde{\mathbf{H}}^{\mathrm{t}} &= \hat{\mathbf{y}}\,\frac{E_0^{\mathrm{i}}\tau_{\parallel}}{\eta_2}\,e^{-j(8x+28.91z)}\\
&= \hat{\mathbf{y}}\,2.64\times 10^{-2}\,e^{-j(8x+28.91z)},\\
\mathbf{E}^{\mathrm{t}} &= \mathfrak{Re}\{\widetilde{\mathbf{E}}^{\mathrm{t}}e^{j\omega t}\}\\
&= (\hat{\mathbf{x}}\,3.18 - \hat{\mathbf{z}}\,0.90)\cos(3\times 10^{9}t - 8x - 28.91z) \text{ V/m},\\
\mathbf{H}^{\mathrm{t}} &= \hat{\mathbf{y}}\,2.64\times 10^{-2}\cos(3\times 10^{9}t - 8x - 28.91z) \text{ A/m}.
\end{aligned}$$

**(e)**

$$S_{\mathrm{av}}^{\mathrm{t}} = \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} = \frac{|H_0^{\mathrm{t}}|^2}{2}\,\eta_2 = \frac{(2.64\times 10^{-2})^2}{2}\times 125.67 = 44 \text{ mW/m}^2.$$

---

**Problem 8.28** Natural light is randomly polarized, which means that, on average, half the light energy is polarized along any given direction (in the plane orthogonal to the direction of propagation) and the other half of the energy is polarized along the direction orthogonal to the first polarization direction. Hence, when treating natural light incident upon a planar boundary, we can consider half of its energy to be in the form of parallel-polarized waves and the other half as perpendicularly polarized waves. Determine the fraction of the incident power reflected by the planar surface of a piece of glass with $n = 1.5$ when illuminated by natural light at $70^\circ$.

**Solution:** Assume the incident power is 1 W. Hence:

Incident power with parallel polarization $= 0.5$ W,
Incident power with perpendicular polarization $= 0.5$ W.

$\varepsilon_2/\varepsilon_1 = (n_2/n_1)^2 = n^2 = 1.5^2 = 2.25$. Equations (8.60) and (8.68) give

$$\Gamma_\perp = \frac{\cos 70^\circ - \sqrt{2.25 - \sin^2 70^\circ}}{\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = -0.55,$$

$$\Gamma_\| = \frac{-2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}}{2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = 0.21.$$

Reflected power with parallel polarization $= 0.5\,(\Gamma_\|)^2$
$= 0.5\,(0.21)^2 = 22$ mW,
Reflected power with perpendicular polarization $= 0.5\,(\Gamma_\perp)^2$
$= 0.5\,(0.55)^2 = 151.3$ mW.

Total reflected power $= 22 + 151.3 = 173.3$ mW, or 17.33%..

---

**Problem 8.29** A parallel polarized plane wave is incident from air onto a dielectric medium with $\varepsilon_r = 9$ at the Brewster angle. What is the refraction angle?

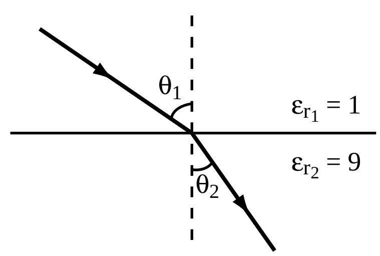



Figure P8.29: Geometry of Problem 8.29.

**Solution:** For nonmagnetic materials, Eq. (8.72) gives

$$\theta_1 = \theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon 1}} = \tan^{-1} 3 = 71.57^\circ.$$

But

$$\sin\theta_2 = \frac{\sin\theta_1}{\sqrt{\varepsilon_{r_2}}} = \frac{\sin\theta_1}{3} = \frac{\sin 71.57^\circ}{3} = 0.32,$$

or $\theta_2 = 18.44^\circ$.

---

**Problem 8.30** A perpendicularly polarized wave in air is obliquely incident upon a planar glass-air interface at an incidence angle of $30^\circ$. The wave frequency is 600 THz (1 THz $= 10^{12}$ Hz), which corresponds to green light, and the index of refraction of the glass is 1.6. If the electric field amplitude of the incident wave is 50 V/m, determine

**(a)** the reflection and transmission coefficients, and

**(b)** the instantaneous expressions for **E** and **H** in the glass medium.

**Solution:**

**(a)** For nonmagnetic materials, $(\varepsilon_2/\varepsilon_1) = (n_2/n_1)^2$. Using this relation in Eq. (8.60) gives

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}} = \frac{\cos 30^\circ - \sqrt{(1.6)^2 - \sin^2 30^\circ}}{\cos 30^\circ + \sqrt{(1.6)^2 - \sin^2 30^\circ}} = -0.27,$$

$$\tau_\perp = 1 + \Gamma_\perp = 1 - 0.27 = 0.73.$$

**(b)** In the glass medium,

$$\sin\theta_t = \frac{\sin\theta_i}{n_2} = \frac{\sin 30^\circ}{1.6} = 0.31,$$

or $\theta_t = 18.21^\circ$.

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{\eta_0}{n_2} = \frac{120\pi}{1.6} = 75\pi = 235.62 \quad (\Omega),$$

$$k_2 = \frac{\omega}{u_p} = \frac{2\pi f}{c/n} = \frac{2\pi f n}{c} = \frac{2\pi \times 600 \times 10^{12} \times 1.6}{3 \times 10^8} = 6.4\pi \times 10^6 \text{ rad/m},$$

$$E_0^t = \tau_\perp E_0^i = 0.73 \times 50 = 36.5 \text{ V/m}.$$

From Eqs. (8.49c) and (8.49d),

$$\widetilde{\mathbf{E}}_\perp^t = \hat{\mathbf{y}} E_0^t e^{-jk_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\widetilde{\mathbf{H}}_\perp^t = (-\hat{\mathbf{x}}\cos\theta_t + \hat{\mathbf{z}}\sin\theta_t)\frac{E_0^t}{\eta_2} e^{-jk_2(x\sin\theta_t + z\cos\theta_t)},$$

and the corresponding instantaneous expressions are:

$$\mathbf{E}^{\rm t}_{\perp}(x,z,t) = \hat{\mathbf{y}}36.5\cos(\omega t - k_2 x\sin\theta_{\rm t} - k_2 z\cos\theta_{\rm t}) \quad \text{(V/m)},$$
$$\mathbf{H}^{\rm t}_{\perp}(x,z,t) = (-\hat{\mathbf{x}}\cos\theta_{\rm t} - \hat{\mathbf{z}}\cos\theta_{\rm t})0.16\cos(\omega t - k_2 x\sin\theta_{\rm t} - k_2 z\cos\theta_{\rm t}) \quad \text{(A/m)},$$

with $\omega = 2\pi \times 10^{15}$ rad/s and $k_2 = 6.4\pi \times 10^6$ rad/m.

---

**Problem 8.31** Show that the reflection coefficient $\Gamma_\perp$ can be written in the form

$$\Gamma_\perp = \frac{\sin(\theta_{\rm t} - \theta_{\rm i})}{\sin(\theta_{\rm t} + \theta_{\rm i})}\,.$$

**Solution:** From Eq. (8.58a),

$$\Gamma_\perp = \frac{\eta_2\cos\theta_{\rm i} - \eta_1\cos\theta_{\rm t}}{\eta_2\cos\theta_{\rm i} + \eta_1\cos\theta_{\rm t}} = \frac{(\eta_2/\eta_1)\cos\theta_{\rm i} - \cos\theta_{\rm t}}{(\eta_2/\eta_1)\cos\theta_{\rm i} + \cos\theta_{\rm t}}\,.$$

Using Snell's law for refraction given by Eq. (8.31), we have

$$\frac{\eta_2}{\eta_1} = \frac{\sin\theta_{\rm t}}{\sin\theta_{\rm i}}\,,$$

we have

$$\Gamma_\perp = \frac{\sin\theta_{\rm t}\cos\theta_{\rm i} - \cos\theta_{\rm t}\sin\theta_{\rm i}}{\sin\theta_{\rm t}\cos\theta_{\rm i} + \cos\theta_{\rm t}\sin\theta_{\rm i}} = \frac{\sin(\theta_{\rm t} - \theta_{\rm i})}{\sin(\theta_{\rm t} + \theta_{\rm i})}\,.$$

---

**Problem 8.32** Show that for nonmagnetic media, the reflection coefficient $\Gamma_\parallel$ can be written in the form

$$\Gamma_\parallel = \frac{\tan(\theta_{\rm t} - \theta_{\rm i})}{\tan(\theta_{\rm t} + \theta_{\rm i})}\,.$$

**Solution:** From Eq. (8.66a), $\Gamma_\parallel$ is given by

$$\Gamma_\parallel = \frac{\eta_2\cos\theta_{\rm t} - \eta_1\cos\theta_{\rm i}}{\eta_2\cos\theta_{\rm t} + \eta_1\cos\theta_{\rm i}} = \frac{(\eta_2/\eta_1)\cos\theta_{\rm t} - \cos\theta_{\rm i}}{(\eta_2/\eta_1)\cos\theta_{\rm t} + \cos\theta_{\rm i}}\,.$$

For nonmagnetic media, $\mu_1 = \mu_2 = \mu_0$ and

$$\frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{n_1}{n_2}\,.$$

Snell's law of refraction is

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}.$$

Hence,

$$\Gamma_\parallel = \frac{\dfrac{\sin\theta_t}{\sin\theta_i}\cos\theta_t - \cos\theta_i}{\dfrac{\sin\theta_t}{\sin\theta_i}\cos\theta_t + \cos\theta_i} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i}.$$

To show that the expression for $\Gamma_\parallel$ is the same as

$$\Gamma_\parallel = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)},$$

we shall proceed with the latter and show that it is equal to the former.

$$\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\sin(\theta_t - \theta_i)\cos(\theta_t + \theta_i)}{\cos(\theta_t - \theta_i)\sin(\theta_t + \theta_i)}.$$

Using the identities (from Appendix C):

$$2\sin x\cos y = \sin(x+y) + \sin(x-y),$$

and if we let $x = \theta_t - \theta_i$ and $y = \theta_t + \theta_i$ in the numerator, while letting $x = \theta_t + \theta_i$ and $y = \theta_t - \theta_i$ in the denominator, then

$$\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\sin(2\theta_t) + \sin(-2\theta_i)}{\sin(2\theta_t) + \sin(2\theta_i)}.$$

But $\sin 2\theta = 2\sin\theta\cos\theta$, and $\sin(-\theta) = -\sin\theta$, hence,

$$\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i},$$

which is the intended result.

---

**Problem 8.33** A parallel polarized beam of light with an electric field amplitude of 10 (V/m) is incident in air on polystyrene with $\mu_r = 1$ and $\varepsilon_r = 2.6$. If the incidence angle at the air–polystyrene planar boundary is 50°, determine

**(a)** the reflectivity and transmissivity, and

**(b)** the power carried by the incident, reflected, and transmitted beams if the spot on the boundary illuminated by the incident beam is 1 m$^2$ in area.

**Solution:**
**(a)** From Eq. (8.68),

$$\begin{aligned}
\Gamma_{\parallel} &= \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}} \\
&= \frac{-2.6\cos 50^\circ + \sqrt{2.6 - \sin^2 50^\circ}}{2.6\cos 50^\circ + \sqrt{2.6 - \sin^2 50^\circ}} = -0.08, \\
R_{\parallel} &= |\Gamma_{\parallel}|^2 = (0.08)^2 = 6.4\times 10^{-3}, \\
T_{\parallel} &= 1 - R_{\parallel} = 0.9936.
\end{aligned}$$

**(b)**

$$\begin{aligned}
P^\mathrm{i}_{\parallel} &= \frac{|E^\mathrm{i}_{\parallel 0}|^2}{2\eta_1} A\cos\theta_\mathrm{i} = \frac{(10)^2}{2\times 120\pi}\times\cos 50^\circ = 85 \text{ mW}, \\
P^\mathrm{r}_{\parallel} &= R_{\parallel} P^\mathrm{i}_{\parallel} = (6.4\times 10^{-3})\times 0.085 = 0.55 \text{ mW}, \\
P^\mathrm{t}_{\parallel} &= T_{\parallel} P^\mathrm{i}_{\parallel} = 0.9936\times 0.085 = 84.45 \text{ mW}.
\end{aligned}$$

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## Sections 8-6 to 8-11

**Problem 8.34** Derive Eq. (8.89b).

**Solution:**
We start with Eqs. (8.88a and e),

$$\frac{\partial\tilde{e}_z}{\partial y} + j\beta\tilde{e}_y = -j\omega\mu\tilde{h}_x,$$

$$-j\beta\tilde{h}_x - \frac{\partial\tilde{h}_z}{\partial x} = j\omega\varepsilon\tilde{e}_y.$$

To eliminate $\tilde{h}_x$, we multiply the top equation by $\beta$ and the bottom equation by $\omega\mu$, and then we add them together. The result is:

$$\beta\frac{\partial\tilde{e}_z}{\partial y} + j\beta^2\tilde{e}_y - \omega\mu\frac{\partial\tilde{h}_z}{\partial x} = j\omega^2\mu\varepsilon\tilde{e}_y.$$

Multiplying all terms by $e^{-j\beta z}$ to convert $\widetilde{e}_y$ to $\widetilde{E}_y$ (and similarly for the other field components), and then solving for $\widetilde{E}_y$ leads to

$$\begin{aligned}\widetilde{E}_y &= \frac{1}{j(\beta^2-\omega^2\mu\varepsilon)}\left(-\beta\frac{\partial\widetilde{E}_z}{\partial y}+\omega\mu\frac{\partial\widetilde{H}_z}{\partial x}\right)\\ &= \frac{j}{k_c^2}\left(-\beta\frac{\partial\widetilde{E}_z}{\partial y}+\omega\mu\frac{\partial\widetilde{H}_z}{\partial x}\right),\end{aligned}$$

where we used the relation

$$k_c^2 = \omega^2\mu\varepsilon-\beta^2.$$

---

**Problem 8.35** A hollow rectangular waveguide is to be used to transmit signals at a carrier frequency of 6 GHz. Choose its dimensions so that the cutoff frequency of the dominant TE mode is lower than the carrier by 25% and that of the next mode is at least 25% higher than the carrier.

**Solution:**

For $m = 1$ and $n = 0$ ($\text{TE}_{10}$ mode) and $u_{p_0} = c$ (hollow guide), Eq. (8.106) reduces to

$$f_{10} = \frac{c}{2a}.$$

Denote the carrier frequency as $f_0 = 6$ GHz. Setting

$$f_{10} = 0.75 f_0 = 0.75\times 6 \text{ GHz} = 4.5 \text{ GHz},$$

we have

$$a = \frac{c}{2f_{10}} = \frac{3\times10^8}{2\times4.5\times10^9} = 3.33 \text{ cm}.$$

If $b$ is chosen such that $a > b > \frac{a}{2}$, the second mode will be $\text{TE}_{01}$, followed by $\text{TE}_{20}$ at $f_{20} = 9$ GHz. For $\text{TE}_{01}$,

$$f_{01} = \frac{c}{2b}.$$

Setting $f_{01} = 1.25 f_0 = 7.5$ GHz, we get

$$b = \frac{c}{2f_{01}} = \frac{3\times10^8}{2\times7.5\times10^9} = 2 \text{ cm}.$$

---

**Problem 8.36** A TE wave propagating in a dielectric-filled waveguide of unknown permittivity has dimensions $a = 5$ cm and $b = 3$ cm. If the $x$-component of its electric field is given by

$$\begin{aligned} E_x = &-36\cos(40\pi x)\sin(100\pi y) \\ &\cdot\sin(2.4\pi\times 10^{10}t - 52.9\pi z), \quad \text{(V/m)} \end{aligned}$$

determine:

**(a)** the mode number,
**(b)** $\varepsilon_r$ of the material in the guide,
**(c)** the cutoff frequency, and
**(d)** the expression for $H_y$.

**Solution:**

**(a)** Comparison of the given expression with Eq. (8.110a) reveals that

$$\frac{m\pi}{a} = 40\pi, \qquad \text{hence } m = 2$$
$$\frac{n\pi}{b} = 100\pi, \qquad \text{hence } n = 3.$$

Mode is $\text{TE}_{23}$.

**(b)** From $\sin(\omega t - \beta z)$, we deduce that

$$\omega = 2.4\pi\times 10^{10} \text{ rad/s}, \qquad \beta = 52.9\pi \text{ rad/m}.$$

Using Eq. (8.105) to solve for $\varepsilon_r$, we have

$$\begin{aligned} \varepsilon_r &= \frac{c^2}{\omega^2}\left[\beta^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \\ &= 2.25. \end{aligned}$$

**(c)**

$$\begin{aligned} u_{p_0} &= \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times 10^8}{\sqrt{2.25}} = 2\times 10^8 \text{ m/s.} \\ f_{23} &= \frac{u_{p_0}}{2}\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{3}{b}\right)^2} \\ &= 10.77 \text{ GHz.} \end{aligned}$$

**(d)**

$$Z_{\text{TE}} = \frac{E_x}{H_y} = \eta \Big/ \sqrt{1-(f_{23}/f)^2}$$
$$= \frac{377}{\sqrt{\varepsilon_{\text{r}}}} \Bigg/ \sqrt{1-\left(\frac{10.77}{12}\right)^2} = 569.9\ \Omega.$$

Hence,

$$H_y = \frac{E_x}{Z_{\text{TE}}}$$
$$= -0.063\cos(40\pi x)\sin(100\pi y)\sin(2.4\pi\times10^{10}t - 52.9\pi z) \quad \text{(A/m)}.$$

---

**Problem 8.37** A waveguide filled with a material whose $\varepsilon_{\text{r}} = 2.25$ has dimensions $a = 2$ cm and $b = 1.4$ cm. If the guide is to transmit 10.5-GHz signals, what possible modes can be used for the transmission?

**Solution:**
Application of Eq. (8.106) with $u_{\text{p}_0} = c/\sqrt{\varepsilon_{\text{r}}} = 3\times10^8/\sqrt{2.25} = 2\times10^8$ m/s, gives:

$$f_{10} = 5 \text{ GHz (TE only)}$$
$$f_{01} = 7.14 \text{ GHz (TE only)}$$
$$f_{11} = 8.72 \text{ GHz (TE or TM)}$$
$$f_{20} = 10 \text{ GHz (TE only)}$$
$$f_{21} = 12.28 \text{ GHz (TE or TM)}$$
$$f_{12} = 15.1 \text{ GHz (TE or TM).}$$

Hence, any one of the first four modes can be used to transmit 10.5-GHz signals.

---

**Problem 8.38** For a rectangular waveguide operating in the $\text{TE}_{10}$ mode, obtain expressions for the surface charge density $\widetilde{\rho}_{\text{s}}$ and surface current density $\widetilde{\mathbf{J}}_{\text{s}}$ on each of the four walls of the guide.

**Solution:**

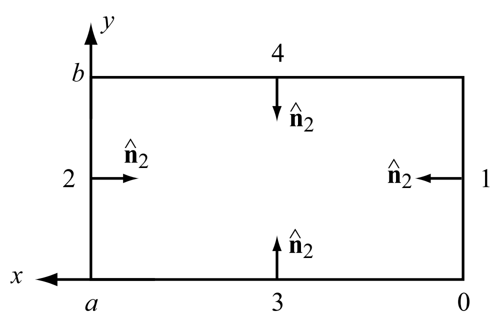
For $\text{TE}_{10}$, the expressions for $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are given by Eq. (8.110) with $m = 1$ and $n = 0$,

$$\widetilde{E}_x = 0,$$
$$\widetilde{E}_y = -j\frac{\omega\mu\pi H_0}{k_c^2 a}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z},$$
$$\widetilde{E}_z = 0,$$
$$\widetilde{H}_x = j\frac{\beta\pi H_0}{k_c^2 a}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z},$$
$$\widetilde{H}_y = 0,$$
$$\widetilde{H}_z = H_0\cos\left(\frac{\pi x}{a}\right)e^{-j\beta z}.$$

The applicable boundary conditions are given in Table 6-2. At the boundary between a dielectric (medium 1) and a conductor (medium 2),

$$\widetilde{\rho}_s = \hat{\mathbf{n}}_2\cdot\widetilde{\mathbf{D}}_1 = \varepsilon_1\,\hat{\mathbf{n}}_2\cdot\widetilde{\mathbf{E}}_1,$$
$$\widetilde{\mathbf{J}}_s = \hat{\mathbf{n}}_2\times\widetilde{\mathbf{H}}_1,$$

where $\widetilde{\mathbf{E}}_1$ and $\widetilde{\mathbf{H}}_1$ are the fields inside the guide, $\varepsilon_1$ is the permittivity of the material filling the guide, and $\hat{\mathbf{n}}_2$ is the normal to the guide wall, pointing away from the wall (inwardly). In view of the coordinate system defined for the guide, $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$ for side wall at $x = 0$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$ for wall at $x = a$, etc.




**(a)** At side wall 1 at $x = 0$, $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$. Hence,

$$\rho_s = \varepsilon_1\,\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}\,E_y|_{x=0} = 0$$
$$\begin{aligned}\mathbf{J}_s &= \hat{\mathbf{x}}\times(\hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{z}}\widetilde{H}_z)|_{x=0}\\ &= -\hat{\mathbf{y}}\widetilde{H}_z|_{x=0}\\ &= -\hat{\mathbf{y}}H_0 e^{-j\beta z}.\end{aligned}$$

**(b)** At side wall 2 at $x = a$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$. Hence,

$$\rho_s = 0$$
$$\mathbf{J}_s = \hat{\mathbf{y}} H_0 e^{-j\beta z}.$$

**(c)** At bottom surface at $y = 0$, $\hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$. Hence,

$$\begin{aligned}
\rho_s &= \varepsilon_1 \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} E_y|_{y=0} \\
&= -j \frac{\omega\varepsilon\mu\pi H_0}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\
\widetilde{\mathbf{J}}_s &= \hat{\mathbf{y}} \times (\hat{\mathbf{x}} \widetilde{H}_x + \hat{\mathbf{z}} \widetilde{H}_z) \\
&= H_0 \left[\hat{\mathbf{x}} \cos\left(\frac{\pi x}{a}\right) - \hat{\mathbf{z}}\, j \frac{\beta\pi}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right)\right] e^{-j\beta z}.
\end{aligned}$$

**(d)** At top surface at $y - b$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{y}}$. Hence,

$$\begin{aligned}
\widetilde{\rho}_s &= j \frac{\omega\varepsilon\mu\pi H_0}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\
\widetilde{\mathbf{J}}_s &= H_0 \left[-\hat{\mathbf{x}} \cos\left(\frac{\pi x}{a}\right) + \hat{\mathbf{z}}\, j \frac{\beta\pi}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right)\right] e^{-j\beta z}.
\end{aligned}$$

---

**Problem 8.39** A waveguide, with dimensions $a = 1$ cm and $b = 0.7$ cm, is to be used at 20 GHz. Determine the wave impedance for the dominant mode when

**(a)** the guide is empty, and

**(b)** the guide is filled with polyethylene (whose $\varepsilon_r = 2.25$).

**Solution:**

For the $\text{TE}_{10}$ mode,

$$f_{10} = \frac{u_{p_0}}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}}.$$

When empty,

$$f_{10} = \frac{3 \times 10^8}{2 \times 10^{-2}} = 15 \text{ GHz}.$$

When filled with polyethylene, $f_{10} = 10$ GHz.

According to Eq. (8.111),

$$Z_{\text{TE}} = \frac{\eta}{\sqrt{1 - (f_{10}/f)^2}} = \frac{\eta_0}{\sqrt{\varepsilon_r}\sqrt{1 - (f_{10}/f)^2}}.$$

When empty,

$$Z_{\text{TE}} = \frac{377}{\sqrt{1-(15/20)^2}} = 570\ \Omega.$$

When filled,

$$Z_{\text{TE}} = \frac{377}{\sqrt{2.25}\sqrt{1-(10/20)^2}} = 290\ \Omega.$$

---

**Problem 8.40** A narrow rectangular pulse superimposed on a carrier with a frequency of 9.5 GHz was used to excite all possible modes in a hollow guide with $a = 3$ cm and $b = 2.0$ cm. If the guide is 100 m in length, how long will it take each of the excited modes to arrive at the receiving end?

**Solution:**

With $a = 3$ cm, $b = 2$ cm, and $u_{p_0} = c = 3 \times 10^8$ m/s, application of Eq. (8.106) leads to:

$$\begin{aligned} f_{10} &= 5 \text{ GHz} \\ f_{01} &= 7.5 \text{ GHz} \\ f_{11} &= 9.01 \text{ GHz} \\ f_{20} &= 10 \text{ GHz} \end{aligned}$$

Hence, the pulse with a 9.5-GHz carrier can excite the top three modes. Their group velocities can be calculated with the help of Eq. (8.114),

$$u_{\text{g}} = c\sqrt{1-(f_{mn}/f)^2},$$

which gives:

$$u_{\text{g}} = \begin{cases} 0.85c = 2.55 \times 10^8 \text{ m/s}, & \text{for TE}_{10} \\ 0.61c = 1.84 \times 10^8 \text{ m/s}, & \text{for TE}_{01} \\ 0.32c = 0.95 \times 10^8 \text{ m/s}, & \text{for TE}_{11} \text{ and TM}_{11} \end{cases}$$

Travel time associated with these modes is:

$$T = \frac{d}{u_{\text{g}}} = \frac{100}{u_{\text{g}}} = \begin{cases} 0.39\ \mu\text{s}, & \text{for TE}_{10} \\ 0.54\ \mu\text{s}, & \text{for TE}_{01} \\ 1.05\ \mu\text{s}, & \text{for TE}_{11} \text{ and TM}_{11}. \end{cases}$$

---

**Problem 8.41** If the zigzag angle $\theta'$ is $42^\circ$ for the $TE_{10}$ mode, what would it be for the $TE_{20}$ mode?

**Solution:**

For $TE_{10}$, the derivation that started with Eq. (8.116) led to

$$\theta'_{10} = \tan^{-1}\left(\frac{\pi}{\beta a}\right), \qquad TE_{10} \text{ mode.}$$

Had the derivation been for $n = 2$ (instead of $n = 1$), the $x$-dependence would have involved a phase factor $(2\pi x/a)$ (instead of $(\pi x/a)$). The sequence of steps would have led to

$$\theta'_{20} = \tan^{-1}\left(\frac{2\pi}{\beta a}\right), \qquad TE_{20} \text{ mode.}$$

Given that $\theta'_{10} = 42^\circ$, it follows that

$$\frac{\pi}{\beta a} = \tan 42^\circ = 0.90$$

Hence,

$$\theta'_{20} = \tan^{-1}(2 \times 0.9) = 60.9^\circ.$$

---

**Problem 8.42** Measurement of the $TE_{101}$ frequency response of an air-filled cubic cavity revealed that its $Q$ is 4802. If its volume is 64 mm$^3$, what material are its sides made of?

**Solution:**

According to Eq. (8.121), the $TE_{101}$ resonant frequency of a cubic cavity is given by

$$f_{101} = \frac{3 \times 10^8}{\sqrt{2}\,a} = \frac{3 \times 10^8}{\sqrt{2} \times 4 \times 10^{-3}} = 53.0 \text{ GHz.}$$

Its $Q$ is given by

$$Q = \frac{a}{3\delta_s} = 4802,$$

which gives $\delta_s = 2.78 \times 10^{-7}$ m. Applying

$$\delta_s = \frac{1}{\sqrt{\pi f_{101} \mu_0 \sigma_c}},$$

and solving for $\sigma_c$ leads to

$$\sigma_c \simeq 6.2 \times 10^7 \text{ S/m.}$$

According to Appendix B, the material is silver.

---

**Problem 8.43** A hollow cavity made of aluminum has dimensions $a = 4$ cm and $d = 3$ cm. Calculate $Q$ of the $\mathrm{TE}_{101}$ mode for

**(a)** $b = 2$ cm, and

**(b)** $b = 3$ cm.

**Solution:**

For the $\mathrm{TE}_{101}$ mode, $f_{101}$ is independent of $b$,

$$\begin{aligned} f_{101} &= \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2} \\ &= \frac{3\times 10^8}{2}\sqrt{\left(\frac{1}{4\times 10^{-2}}\right)^2 + \left(\frac{1}{3\times 10^{-2}}\right)^2} \\ &= 6.25 \text{ GHz}. \end{aligned}$$

For aluminum with $\sigma_\mathrm{c} = 3.5\times 10^7$ S/m (Appendix B),

$$\delta_\mathrm{s} = \frac{1}{\sqrt{\pi f_{101}\mu_0\sigma_\mathrm{c}}} = 1.08\times 10^{-6} \text{ m}.$$

**(a)** For $a = 4$ cm, $b = 2$ cm and $d = 3$ cm,

$$\begin{aligned} Q &= \frac{1}{\delta_\mathrm{s}}\,\frac{abd(a^2+d^2)}{[a^3(d+2b)+d^3(a+2b)]} \\ &= 8367. \end{aligned}$$

**(b)** For $a = 4$ cm, $b = 3$ cm, and $d = 3$ cm,

$$Q = 9850.$$

---

**Problem 8.44** A 50-MHz right-hand circularly polarized plane wave with an electric field modulus of 30 V/m is normally incident in air upon a dielectric medium with $\varepsilon_\mathrm{r} = 9$ and occupying the region defined by $z \geq 0$.

**(a)** Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.

**(b)** Calculate the reflection and transmission coefficients.

**(c)** Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \leq 0$.

**(d)** Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

**Solution:**

**(a)**

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 50 \times 10^6}{3 \times 10^8} = \frac{\pi}{3} \text{ rad/m},$$
$$k_2 = \frac{\omega}{c}\sqrt{\varepsilon_{r_2}} = \frac{\pi}{3}\sqrt{9} = \pi \text{ rad/m}.$$

From (7.57), RHC wave traveling in $+z$ direction:

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{i}} &= a_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}\, e^{-j\pi/2})e^{-jk_1 z} = a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-jk_1 z} \\
\mathbf{E}^{\mathrm{i}}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}^{\mathrm{i}} e^{j\omega t}\right] \\
&= \mathfrak{Re}\left[a_0(\hat{\mathbf{x}}\, e^{j(\omega t - k_1 z)} + \hat{\mathbf{y}}\, e^{j(\omega t - k_1 z - \pi/2)})\right] \\
&= \hat{\mathbf{x}}\, a_0 \cos(\omega t - k_1 z) + \hat{\mathbf{y}}\, a_0 \cos(\omega t - k_1 z - \pi/2) \\
&= \hat{\mathbf{x}}\, a_0 \cos(\omega t - k_1 z) + \hat{\mathbf{y}}\, a_0 \sin(\omega t - k_1 z) \\
|\mathbf{E}^{\mathrm{i}}| &= \left[a_0^2 \cos^2(\omega t - k_1 z) + a_0^2 \sin^2(\omega t - k_1 z)\right]^{1/2} = a_0 = 30 \text{ V/m}.
\end{aligned}$$

Hence,

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = 30(x_0 - jy_0)e^{-j\pi z/3} \quad \text{(V/m)}.$$

**(b)**

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \frac{120\pi}{\sqrt{9}} = 40\pi \quad (\Omega).$$
$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{40\pi - 120\pi}{40\pi + 120\pi} = -0.5$$
$$\tau = 1 + \Gamma = 1 - 0.5 = 0.5.$$

**(c)**

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{r}} &= \Gamma a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{jk_1 z} \\
&= -0.5 \times 30(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{jk_1 z} \\
&= -15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{j\pi z/3} \quad \text{(V/m)}. \\
\widetilde{\mathbf{E}}^{\mathrm{t}} &= \tau a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-jk_2 z} \\
&= 15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-j\pi z} \quad \text{(V/m)}. \\
\widetilde{\mathbf{E}}_1 &= \widetilde{\mathbf{E}}^{\mathrm{i}} + \widetilde{\mathbf{E}}^{\mathrm{r}} \\
&= 30(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-j\pi z/3} - 15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{j\pi z/3} \\
&= 15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})[2e^{-j\pi z/3} - e^{j\pi z/3}] \quad \text{(V/m)}.
\end{aligned}$$

**(d)**

$$\text{\% of reflected power} = 100 \times |\Gamma|^2 = 100 \times (0.5)^2 = 25\%$$

$$\text{\% of transmitted power} = 100|\tau|^2 \frac{\eta_1}{\eta_2} = 100 \times (0.5)^2 \times \frac{120\pi}{40\pi} = 75\%.$$

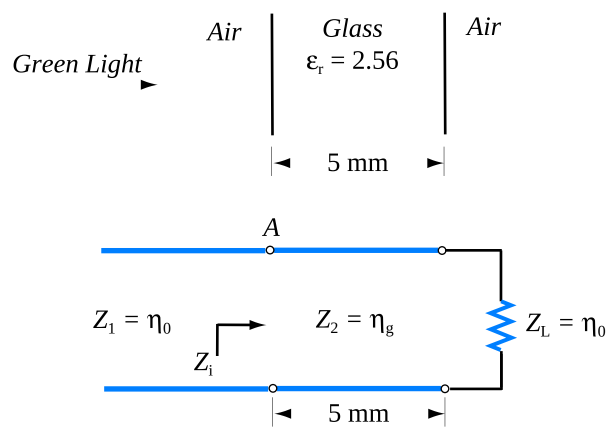
---

**Problem 8.45** Consider a flat 5-mm-thick slab of glass with $\varepsilon_{\mathrm{r}} = 2.56$.

**(a)** If a beam of green light ($\lambda_0 = 0.52$ $\mu$m) is normally incident upon one of the sides of the slab, what percentage of the incident power is reflected back by the glass?

**(b)** To eliminate reflections, it is desired to add a thin layer of antireflection coating material on each side of the glass. If you are at liberty to specify the thickness of the antireflection material as well as its relative permittivity, what would these specifications be?

**Solution:**




**(a)** Representing the wave propagation process by an equivalent transmission line model, the input impedance at the left-hand side of the air-glass interface is (from 2.63):

$$Z_{\rm i} = Z_0 \left( \frac{Z_{\rm L} + jZ_0 \tan \beta l}{Z_0 + jZ_{\rm L} \tan \beta l} \right)$$

For the glass,

$$Z_0 = \eta_{\rm g} = \frac{\eta_0}{\sqrt{\varepsilon_{\rm r}}} = \frac{\eta_0}{\sqrt{2.56}} = \frac{\eta_0}{1.6}$$

$$Z_{\rm L} = \eta_0$$

$$\beta l = \frac{2\pi}{\lambda} l = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_{\rm r}}\, l = \frac{2\pi}{0.52 \times 10^{-6}} \times \sqrt{2.56} \times 5 \times 10^{-3} = 30769.23\pi.$$

Subtracting the maximum possible multiples of $2\pi$, namely $30768\pi$, leaves a remainder of

$$\beta l = 1.23\pi \text{ rad}.$$

Hence,

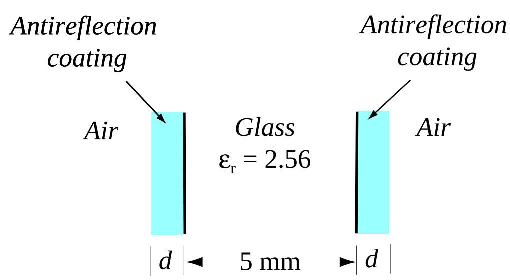
$$\begin{aligned} Z_{\rm i} &= \frac{\eta_0}{1.6} \left( \frac{\eta_0 + j(\eta_0/1.6) \tan 1.23\pi}{(\eta_0/1.6) + j\eta_0 \tan 1.23\pi} \right) \\ &= \left( \frac{1.6 + j \tan 1.23\pi}{1 + j1.6 \tan 1.23\pi} \right) \frac{120\pi}{1.6} \\ &= \left( \frac{1.6 + j0.882}{1 + j1.41} \right) \frac{120\pi}{1.6} = 249\angle{-25.8^\circ} = (224.2 - j108.4)\ \Omega. \end{aligned}$$

With $Z_i$ now representing the input impedance of the glass, the reflection coefficient at point $A$ is:

$$\begin{aligned}\Gamma &= \frac{Z_i - \eta_0}{Z_i + \eta_0} \\ &= \frac{224.2 - j108.4 - 120\pi}{224.2 - j108.4 + 120\pi} = \frac{187.34\angle{-144.6^\circ}}{610.89\angle{-10.2^\circ}} = 0.3067\angle{-154.8^\circ}.\end{aligned}$$

% of reflected power $= |\Gamma|^2 \times 100 = 9.4\%$.

**(b)** To avoid reflections, we can add a quarter-wave transformer on each side of the glass.




This requires that $d$ be:

$$d = \frac{\lambda}{4} + 2n\lambda, \qquad n = 0, 1, 2, \ldots$$

where $\lambda$ is the wavelength in that material; i.e., $\lambda = \lambda_0/\sqrt{\varepsilon_{rc}}$, where $\varepsilon_{rc}$ is the relative permittivity of the coating material. It is also required that $\eta_c$ of the coating material be:

$$\eta_c^2 = \eta_0 \eta_g.$$

Thus

$$\frac{\eta_0^2}{\varepsilon_{rc}} = \eta_0 \frac{\eta_0}{\sqrt{\varepsilon_r}},$$

or

$$\varepsilon_{rc} = \sqrt{\varepsilon_r} = \sqrt{2.56} = 1.6.$$

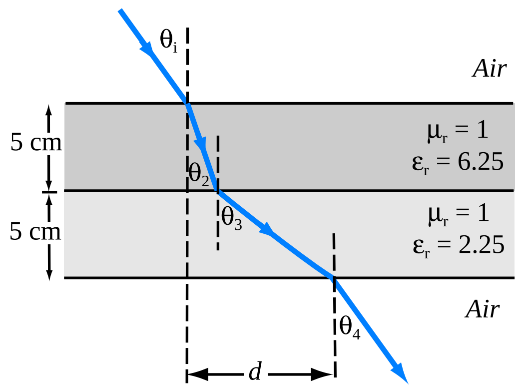
Hence,

$$\begin{aligned}\lambda &= \frac{\lambda_0}{\sqrt{\varepsilon_{rc}}} = \frac{0.52\ \mu\text{m}}{\sqrt{1.6}} = 0.411\ \mu\text{m}, \\ d &= \frac{\lambda}{4} + 2n\lambda \\ &= (0.103 + 0.822n) \quad (\mu\text{m}), \qquad n = 0, 1, 2, \ldots\end{aligned}$$

**Problem 8.46** A parallel-polarized plane wave is incident from air at an angle $\theta_i = 30°$ onto a pair of dielectric layers as shown in the figure.

**(a)** Determine the angles of transmission $\theta_2$, $\theta_3$, and $\theta_4$.

**(b)** Determine the lateral distance $d$.




**Solution:**

**(a)** Application of Snell's law of refraction given by (8.31) leads to:

$$\sin\theta_2 = \sin\theta_1\sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}} = \sin 30^\circ\sqrt{\frac{1}{6.25}} = 0.2$$
$$\theta_2 = 11.54^\circ.$$

Similarly,

$$\sin\theta_3 = \sin\theta_2\sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r3}}} = \sin 11.54^\circ\sqrt{\frac{6.25}{2.25}} = 0.33$$
$$\theta_3 = 19.48^\circ.$$

And,

$$\sin\theta_4 = \sin\theta_3\sqrt{\frac{\varepsilon_{r3}}{\varepsilon_{r4}}} = \sin 19.48^\circ\sqrt{\frac{2.25}{1}} = 0.5$$
$$\theta_4 = 30^\circ.$$

As expected, the exit ray back into air will be at the same angle as $\theta_i$.

**(b)**

$$\begin{aligned} d &= (5 \text{ cm}) \tan\theta_2 + (5 \text{ cm}) \tan\theta_3 \\ &= 5\tan 11.54^\circ + 5\tan 19.48^\circ = 2.79 \text{ cm}. \end{aligned}$$

---

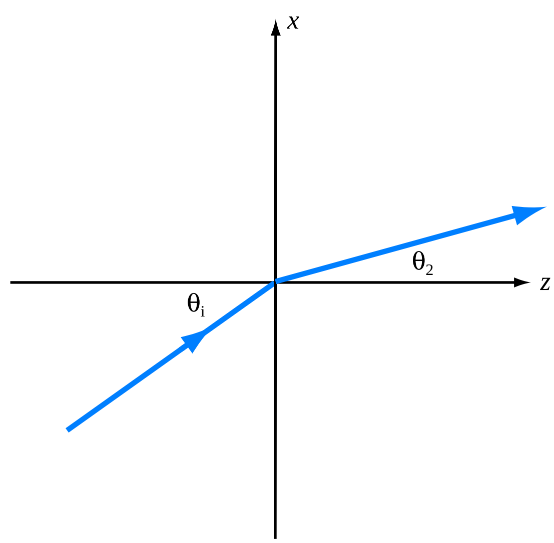
**Problem 8.47** A plane wave in air with

$$\widetilde{\mathbf{E}}^{\text{i}} = (\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}6)e^{-j(2x+3z)} \quad \text{(V/m)}$$

is incident upon the planar surface of a dielectric material, with $\varepsilon_{\text{r}} = 2.25$, occupying the half-space $z \geq 0$. Determine

**(a)** The incidence angle $\theta_{\text{i}}$.
**(b)** The frequency of the wave.
**(c)** The field $\widetilde{\mathbf{E}}^{\text{r}}$ of the reflected wave.
**(d)** The field $\widetilde{\mathbf{E}}^{\text{t}}$ of the wave transmitted into the dielectric medium.
**(e)** The average power density carried by the wave into the dielectric medium.

**Solution:**




**(a)** From the exponential of the given expression, it is clear that the wave direction of travel is in the $x$–$z$ plane. By comparison with the expressions in (8.48a) for perpendicular polarization or (8.65a) for parallel polarization, both of which have the same phase factor, we conclude that:

$$k_1 \sin\theta_\mathrm{i} = 2,$$
$$k_1 \cos\theta_\mathrm{i} = 3.$$

Hence,

$$k_1 = \sqrt{2^2 + 3^2} = 3.6 \quad \text{(rad/m)}$$
$$\theta_\mathrm{i} = \tan^{-1}(2/3) = 33.7^\circ.$$

Also,

$$k_2 = k_1\sqrt{\varepsilon_{\mathrm{r}_2}} = 3.6\sqrt{2.25} = 5.4 \quad \text{(rad/m)}$$
$$\theta_2 = \sin^{-1}\left[\sin\theta_\mathrm{i}\sqrt{\frac{1}{2.25}}\right] = 21.7^\circ.$$

**(b)**

$$k_1 = \frac{2\pi f}{c}$$
$$f = \frac{k_1 c}{2\pi} = \frac{3.6 \times 3 \times 10^8}{2\pi} = 172 \text{ MHz}.$$

**(c)** In order to determine the electric field of the reflected wave, we first have to determine the polarization of the wave. The vector argument in the given expression for $\widetilde{\mathbf{E}}^\mathrm{i}$ indicates that the incident wave is a mixture of parallel and perpendicular polarization components. Perpendicular polarization has a $\hat{\mathbf{y}}$-component only (see 8.46a), whereas parallel polarization has only $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components (see 8.65a). Hence, we shall decompose the incident wave accordingly:

$$\widetilde{\mathbf{E}}^\mathrm{i} = \widetilde{\mathbf{E}}^\mathrm{i}_\perp + \widetilde{\mathbf{E}}^\mathrm{i}_\parallel$$

with

$$\widetilde{\mathbf{E}}^\mathrm{i}_\perp = -\hat{\mathbf{y}}\,4e^{-j(2x+3z)} \quad \text{(V/m)}$$
$$\widetilde{\mathbf{E}}^\mathrm{i}_\parallel = (\hat{\mathbf{x}}\,2 - \hat{\mathbf{z}}\,6)e^{-j(2x+3z)} \quad \text{(V/m)}$$

From the above expressions, we deduce:

$$E^\mathrm{i}_{\perp 0} = -4 \text{ V/m}$$
$$E^\mathrm{i}_{\parallel 0} = \sqrt{2^2 + 6^2} = 6.32 \text{ V/m}.$$

Next, we calculate $\Gamma$ and $\tau$ for each of the two polarizations:

$$\Gamma_\perp = \frac{\cos\theta_\mathrm{i} - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}{\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}$$

Using $\theta_\mathrm{i} = 33.7^\circ$ and $\varepsilon_2/\varepsilon_1 = 2.25/1 = 2.25$ leads to:

$$\Gamma_\perp = -0.25$$
$$\tau_\perp = 1 + \Gamma_\perp = 0.75.$$

Similarly,

$$\Gamma_\perp = \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}} = -0.15,$$
$$\tau_\parallel = (1 + \Gamma_\parallel)\frac{\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{t}} = (1 - 0.15)\frac{\cos 33.7^\circ}{\cos 21.7^\circ} = 0.76.$$

The electric fields of the reflected and transmitted waves for the two polarizations are given by (8.49a), (8.49c), (8.65c), and (8.65e):

$$\widetilde{\mathbf{E}}^\mathrm{r}_\perp = \hat{\mathbf{y}} E^\mathrm{r}_{\perp 0} e^{-jk_1(x\sin\theta_\mathrm{r} - z\cos\theta_\mathrm{r})}$$
$$\widetilde{\mathbf{E}}^\mathrm{t}_\perp = \hat{\mathbf{y}} E^\mathrm{t}_{\perp 0} e^{-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})}$$
$$\widetilde{\mathbf{E}}^\mathrm{r}_\parallel = (\hat{\mathbf{x}}\cos\theta_\mathrm{r} + \hat{\mathbf{z}}\sin\theta_\mathrm{r}) E^\mathrm{r}_{\parallel 0} e^{-jk_1(x\sin\theta_\mathrm{r} - z\cos\theta_\mathrm{r})}$$
$$\widetilde{\mathbf{E}}^\mathrm{t}_\parallel = (\hat{\mathbf{x}}\cos\theta_\mathrm{t} - \hat{\mathbf{z}}\sin\theta_\mathrm{t}) E^\mathrm{t}_{\parallel 0} e^{-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})}$$

Based on our earlier calculations:

$$\theta_\mathrm{r} = \theta_\mathrm{i} = 33.7^\circ$$
$$\theta_\mathrm{t} = 21.7^\circ$$
$$k_1 = 3.6 \text{ rad/m}, \qquad k_2 = 5.4 \text{ rad/m},$$
$$E^\mathrm{r}_{\perp 0} = \Gamma_\perp E^\mathrm{i}_{\perp 0} = (-0.25) \times (-4) = 1 \text{ V/m}.$$
$$E^\mathrm{t}_{\perp 0} = \tau_\perp E^\mathrm{i}_{\perp 0} = 0.75 \times (-4) = -3 \text{ V/m}.$$
$$E^\mathrm{r}_{\parallel 0} = \Gamma_\parallel E^\mathrm{i}_{\parallel 0} = (-0.15) \times 6.32 = -0.95 \text{ V/m}.$$
$$E^\mathrm{t}_{\parallel 0} = \tau_\parallel E^\mathrm{i}_{\parallel 0} = 0.76 \times 6.32 = 4.8 \text{ V/m}.$$

Using the above values, we have:

$$\widetilde{\mathbf{E}}^\mathrm{r} = \widetilde{\mathbf{E}}^\mathrm{r}_\perp + \widetilde{\mathbf{E}}^\mathrm{r}_\parallel$$
$$= (-\hat{\mathbf{x}}0.79 + \hat{\mathbf{y}} - \hat{\mathbf{z}}0.53) e^{-j(2x - 3z)} \quad \text{(V/m)}.$$

**(d)**

$$
\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{t}} &= \widetilde{\mathbf{E}}^{\mathrm{t}}_{\perp} + \widetilde{\mathbf{E}}^{\mathrm{t}}_{\parallel} \\
&= (\hat{\mathbf{x}}4.46 - \hat{\mathbf{y}}3 - \hat{\mathbf{z}}1.78)e^{-j(2x+5z)} \quad \text{(V/m)}.
\end{aligned}
$$

**(e)**

$$
\begin{aligned}
S^{\mathrm{t}} &= \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} \\
|E_0^{\mathrm{t}}|^2 &= (4.46)^2 + 3^2 + (1.78)^2 = 32.06 \\
\eta_2 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{377}{1.5} = 251.3\ \Omega \\
S^{\mathrm{t}} &= \frac{32.06}{2 \times 251.3} = 63.8 \quad (\mathrm{mW/m^2}).
\end{aligned}
$$