

Chapter 4: Electrostatics

Lesson #22

Chapter — Section: 4-1 to 4-3

Topics: Charge and current distributions, Coulomb's law

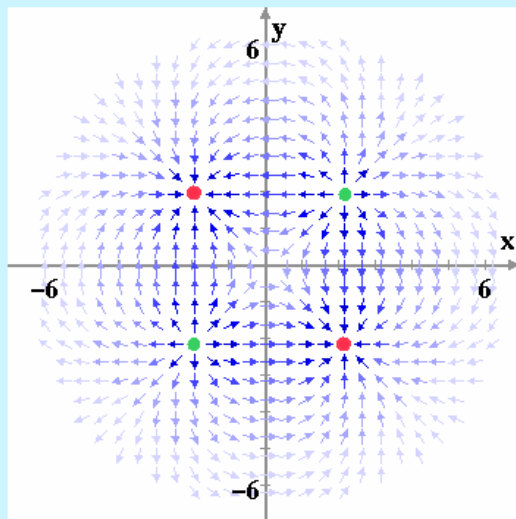
Highlights:

- Maxwell's Equations reduce to *uncoupled* electrostatics and magnetostatics when charges are either fixed in space or move at constant speed.
- Line, surface and volume charge distributions
- Coulomb's law for various charge distributions

Special Illustrations:

- Examples 4-3 and 4-4
- CD-ROM Modules 4.1-4.5
- CD-ROM Demos 4.1-4.8

Demo 4.5: Square with Diagonal Symmetry



Given: Four point charges on the corners of a square, with $Q_1 = Q_3 = 1\text{C}$, and $Q_2 = Q_4 = -1\text{C}$, as shown.

In this demo, arrows are used to sketch the electric field pattern in the x - y plane.

to display the graphical and analytical solution.

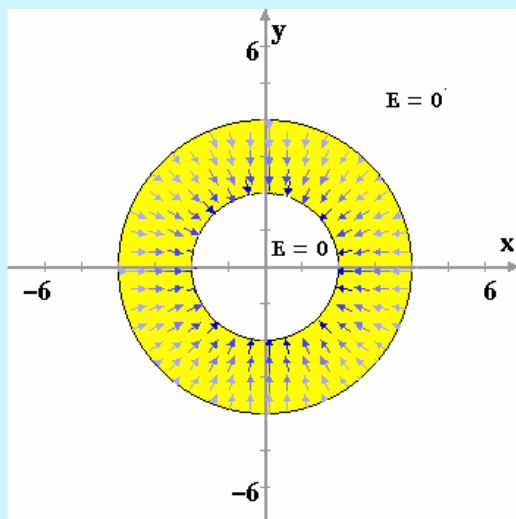
Note: Color intensity is proportional to the strength of the Electric field.

Lesson #23**Chapter — Section:** 4-4**Topics:** Gauss's law**Highlights:**

- Gauss's law in differential and integral form
- The need for symmetry to apply Gauss's law in practice
- Coulomb's law for various charge distributions

Special Illustrations:

- Example 4-6
- CD-ROM Module 4.6
- CD-ROM Demos 4.9 and 4.10

Demo 4.10: Two Concentric Spherical Shells of Opposite Polarity

Given: Two thin, concentric spherical shells, of radii a and $2a$. Positive charge Q is distributed uniformly over the outer shell and negative charge $-Q$ is distributed uniformly over the inner shell. Sketch the electric field pattern in the x - y plane.

to display the graphical and

analytical solution.

Note: Color intensity is proportional to the strength of the Electric field.

Lesson #24**Chapter — Section:** 4-5**Topics:** Electric potential**Highlights:**

- Concept of “potential”
- Relation to electric field
- Relation to charges
- Poisson’s and Laplace’s equations

Special Illustrations:

- Example 4-7

Lesson #25

Chapter — Section: 4-6 and 4-7

Topics: Electrical materials and conductors

Highlights:

- Conductivity ranges for conductors, semiconductors, and insulators
- Ohm's law
- Resistance of a wire
- Joule's law

Special Illustrations:

- Example 4-9
- Technology Brief on “Resistive Sensors” (CD-ROM)

Resistive Sensors

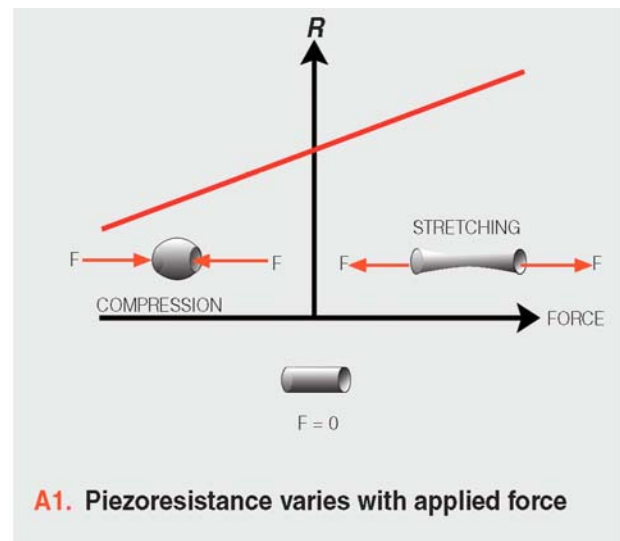
An **electrical sensor** is a device capable of responding to an applied **stimulus** by generating an electrical signal whose voltage, current, or some other attribute is related to the intensity of the stimulus. The family of possible stimuli encompasses a wide array of physical, chemical, and biological quantities including temperature, pressure, position, distance, motion, velocity, acceleration, concentration (of a gas or liquid), blood flow, etc. The sensing process relies on measuring resistance, capacitance, inductance, induced electromotive force (emf), oscillation frequency or time delay, among others. This Technology Brief covers resistive sensors.

Capacitive, **inductive**, and **emf sensors** are covered separately (in this and later chapters).

Piezoresistivity

According to Eq. (4.70), the resistance of a cylindrical resistor or wire conductor is given by $R = l/(\sigma A)$, where l is the cylinder's length, A is its cross-sectional area, and σ is the conductivity of its material. Stretching the wire by an applied external force causes l to increase and A to decrease. Consequently, R increases (**A**).

Conversely, compressing the wire causes R to decrease. The Greek word *piezein* means to press, from which the term piezoresistivity is derived. This should not be confused with piezoelectricity, which is an emf effect (see EMF Sensors).



Lesson #26**Chapter — Section:** 4-8, 4-9**Topics:** Dielectrics, boundary conditions**Highlights:**

- Relative permittivity and dielectric strength
- Electrostatic boundary conditions for various dielectric and conductor combinations

Special Illustrations:

- Example 4-10

Lesson #27

Chapter — Section: 4-10

Topics: Capacitance

Highlights:

- Capacitor as “charge accumulator”
- General expression for C
- Capacitance of parallel-plate and coaxial capacitors
- Joule’s law

Special Illustrations:

- Examples 4-11 and 4-12
- Technology Brief on “Capacitive Sensors” (CD-ROM)

Capacitive Sensors

To **sense** is to respond to a stimulus (see Resistive Sensors). A capacitor can function as a sensor if the stimulus changes the capacitor’s **geometry**—usually the spacing between its conductive elements—or the **dielectric properties** of the insulating material situated between them.

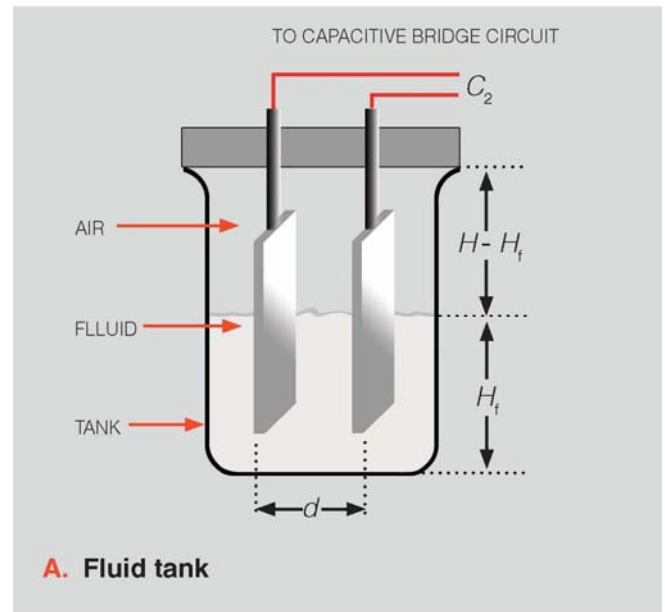
Capacitive sensors are used in a multitude of applications. A few examples follow.

Fluid Gauge

The two metal electrodes in (A), usually rods or plates, form a capacitor whose capacitance is directly proportional to the **permittivity** of the material between them. If the fluid section is of height H_f and the height of the empty space above it is $(H - H_f)$, then the overall capacitance is equivalent to two capacitors in parallel:

$$C_2 = C_f + C_a = \epsilon_f \frac{(wH_f)}{d} + \epsilon_a \frac{(H - H_f)}{d}$$

where w is the electrode plate width, d is the spacing between electrodes, and ϵ_f and ϵ_a are the permittivities of the fluid and air, respectively.



Lesson #28

Chapter — Section: 4-11

Topics: Energy

Highlights:

- A charged capacitor is an energy storage device
- Energy density

Special Illustrations:

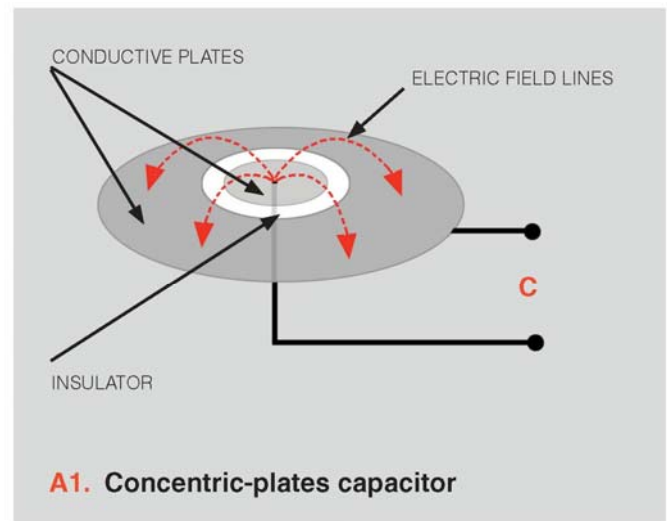
- Technology Brief on “Non-Contact Sensors” (CD-ROM)

Non-Contact Sensors

Precision positioning is a critical ingredient of semiconductor device fabrication, as well as the operation and control of many mechanical systems. **Non-contact capacitive sensors** are used to sense the position of silicon wafers during the deposition, etching, and cutting processes, without coming in direct contact with the wafers. They are also used to sense and control robot arms in equipment manufacturing and to position hard disc drives, photocopier rollers, printing presses, and other similar systems.

Basic Principle

The concentric plate capacitor (**A1**) consists of two metal plates, sharing the same plane, but electrically isolated from each other by an insulating material. When connected to a voltage source, charges of opposite polarity will form on the two plates, resulting in the creation of electric-field lines between them. The same principle applies to the adjacent-plates capacitor in (**A2**). In both cases, the capacitance is determined by the shapes and sizes of the conductive elements and by the permittivity of the dielectric medium containing the electric field lines between them.

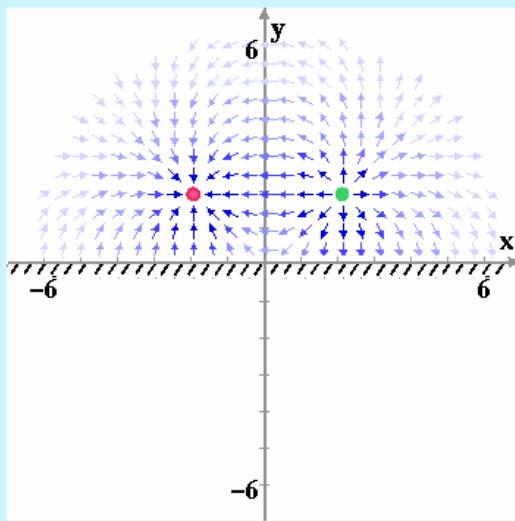


Lesson #29**Chapter — Section:** 4-12**Topics:** Image method**Highlights:**

- Image method useful for solving problems involving charges next to conducting planes
- Remove conducting plane and replace with mirror images for the charges (with opposite polarity)

Special Illustrations:

- Example 4-13
- CD-ROM Demos 4.11-4.13

Demo 4.12: Two Charges of Opposite Polarity Above a Conducting Plane

Given: $Q_1 = 1\text{C}$ and $Q_2 = -1\text{C}$, with both located above a conducting plane situated in the x - y plane, as shown. Sketch the electric field pattern in the x - y plane.

to display the graphical and analytical solution.

Note: Color intensity is proportional to the strength of the Electric field.

Chapter 4

Sections 4-2: Charge and Current Distributions

Problem 4.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2e^{-2z}$ (mC/m³).

Solution: For the cube shown in Fig. P4.1, application of Eq. (4.5) gives

$$\begin{aligned} Q &= \int_V \rho_v dV = \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 xy^2e^{-2z} dx dy dz \\ &= \left(\frac{-1}{12} x^2 y^3 e^{-2z} \right) \Big|_{x=0}^2 \Big|_{y=0}^2 \Big|_{z=0}^2 = \frac{8}{3} (1 - e^{-4}) = 2.62 \text{ mC}. \end{aligned}$$

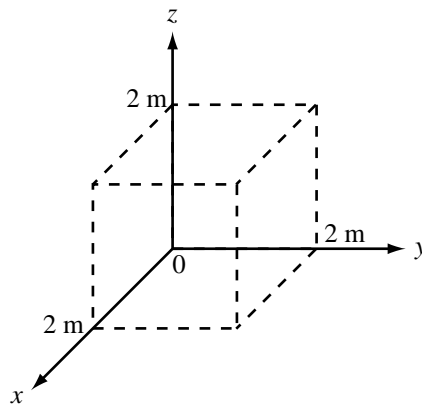


Figure P4.1: Cube of Problem 4.1.

Problem 4.2 Find the total charge contained in a cylindrical volume defined by $r \leq 2$ m and $0 \leq z \leq 3$ m if $\rho_v = 20rz$ (mC/m³).

Solution: For the cylinder shown in Fig. P4.2, application of Eq. (4.5) gives

$$\begin{aligned} Q &= \int_{z=0}^3 \int_{\phi=0}^{2\pi} \int_{r=0}^2 20rz r dr d\phi dz \\ &= \left(\frac{10}{3} r^3 \phi z^2 \right) \Big|_{r=0}^2 \Big|_{\phi=0}^{2\pi} \Big|_{z=0}^3 = 480\pi \text{ (mC)} = 1.5 \text{ C}. \end{aligned}$$

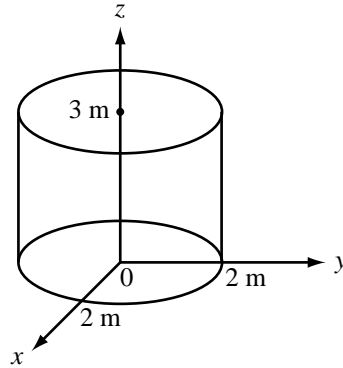


Figure P4.2: Cylinder of Problem 4.2.

Problem 4.3 Find the total charge contained in a cone defined by $R \leq 2$ m and $0 \leq \theta \leq \pi/4$, given that $\rho_v = 10R^2 \cos^2 \theta$ (mC/m³).

Solution: For the cone of Fig. P4.3, application of Eq. (4.5) gives

$$\begin{aligned}
 Q &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{R=0}^2 10R^2 \cos^2 \theta R^2 \sin \theta dR d\theta d\phi \\
 &= \left(\frac{-2}{3} R^5 \phi \cos^3 \theta \right) \Big|_{R=0}^2 \Big|_{\theta=0}^{\pi/4} \Big|_{\phi=0}^{2\pi} \\
 &= \frac{128\pi}{3} \left(1 - \left(\frac{\sqrt{2}}{2} \right)^3 \right) = 86.65 \text{ (mC)}.
 \end{aligned}$$

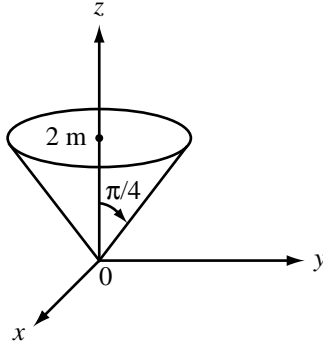


Figure P4.3: Cone of Problem 4.3.

Problem 4.4 If the line charge density is given by $\rho_l = 24y^2$ (mC/m), find the total charge distributed on the y -axis from $y = -5$ to $y = 5$.

Solution:

$$Q = \int_{-5}^5 \rho_l dy = \int_{-5}^5 24y^2 dy = \frac{24y^3}{3} \Big|_{-5}^5 = 2000 \text{ mC} = 2 \text{ C}.$$

Problem 4.5 Find the total charge on a circular disk defined by $r \leq a$ and $z = 0$ if:

- (a) $\rho_s = \rho_{s0} \cos \phi$ (C/m²),
- (b) $\rho_s = \rho_{s0} \sin^2 \phi$ (C/m²),
- (c) $\rho_s = \rho_{s0} e^{-r}$ (C/m²),
- (d) $\rho_s = \rho_{s0} e^{-r} \sin^2 \phi$ (C/m²),

where ρ_{s0} is a constant.

Solution:

(a)

$$Q = \int \rho_s ds = \int_{r=0}^a \int_{\phi=0}^{2\pi} \rho_{s0} \cos \phi r dr d\phi = \rho_{s0} \frac{r^2}{2} \Big|_0^a \sin \phi \Big|_0^{2\pi} = 0.$$

(b)

$$\begin{aligned} Q &= \int_{r=0}^a \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2 \phi r dr d\phi = \rho_{s0} \frac{r^2}{2} \Big|_0^a \int_0^{2\pi} \left(\frac{1 - \cos 2\phi}{2} \right) d\phi \\ &= \frac{\rho_{s0} a^2}{4} \left(\phi - \frac{\sin 2\phi}{2} \right) \Big|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}. \end{aligned}$$

(c)

$$\begin{aligned}
 Q &= \int_{r=0}^a \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} r dr d\phi = 2\pi \rho_{s0} \int_0^a r e^{-r} dr \\
 &= 2\pi \rho_{s0} [-r e^{-r} - e^{-r}]_0^a \\
 &= 2\pi \rho_{s0} [1 - e^{-a}(1+a)].
 \end{aligned}$$

(d)

$$\begin{aligned}
 Q &= \int_{r=0}^a \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} \sin^2 \phi r dr d\phi \\
 &= \rho_{s0} \int_{r=0}^a r e^{-r} dr \int_{\phi=0}^{2\pi} \sin^2 \phi d\phi \\
 &= \rho_{s0} [1 - e^{-a}(1+a)] \cdot \pi = \pi \rho_{s0} [1 - e^{-a}(1+a)].
 \end{aligned}$$

Problem 4.6 If $\mathbf{J} = \hat{\mathbf{y}}4xz$ (A/m²), find the current I flowing through a square with corners at $(0, 0, 0)$, $(2, 0, 0)$, $(2, 0, 2)$, and $(0, 0, 2)$.

Solution: Using Eq. (4.12), the net current flowing through the square shown in Fig. P4.6 is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{x=0}^2 \int_{z=0}^2 (\hat{\mathbf{y}}4xz) \Big|_{y=0} \cdot (\hat{\mathbf{y}} dx dz) = (x^2 z^2) \Big|_{x=0}^2 \Big|_{z=0}^2 = 16 \text{ A.}$$

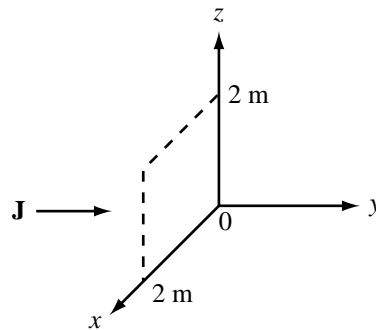


Figure P4.6: Square surface.

Problem 4.7 If $\mathbf{J} = \hat{\mathbf{R}}5/R$ (A/m²), find I through the surface $R = 5$ m.

Solution: Using Eq. (4.12), we have

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\hat{\mathbf{R}} \frac{5}{R} \right) \cdot (\hat{\mathbf{R}} R^2 \sin \theta \, d\theta \, d\phi) \\ &= -5R\phi \cos \theta \Big|_{R=5} \Big|_{\theta=0}^{\pi} \Big|_{\phi=0}^{2\pi} = 100\pi = 314.2 \quad (\text{A}). \end{aligned}$$

Problem 4.8 An electron beam shaped like a circular cylinder of radius r_0 carries a charge density given by

$$\rho_v = \left(\frac{-\rho_0}{1+r^2} \right) \quad (\text{C/m}^3),$$

where ρ_0 is a positive constant and the beam's axis is coincident with the z -axis.

- (a) Determine the total charge contained in length L of the beam.
 (b) If the electrons are moving in the $+z$ -direction with uniform speed u , determine the magnitude and direction of the current crossing the z -plane.

Solution:

(a)

$$\begin{aligned} Q &= \int_{r=0}^{r_0} \int_{z=0}^L \rho_v \, d\mathcal{V} = \int_{r=0}^{r_0} \int_{z=0}^L \left(\frac{-\rho_0}{1+r^2} \right) 2\pi r \, dr \, dz \\ &= -2\pi\rho_0 L \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi\rho_0 L \ln(1+r_0^2). \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{J} &= \rho_v \mathbf{u} = -\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \quad (\text{A/m}^2), \\ I &= \int \mathbf{J} \cdot d\mathbf{s} \\ &= \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \left(-\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \right) \cdot \hat{\mathbf{z}} r \, dr \, d\phi \\ &= -2\pi u \rho_0 \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi u \rho_0 \ln(1+r_0^2) \quad (\text{A}). \end{aligned}$$

Current direction is along $-\hat{\mathbf{z}}$.

Section 4-3: Coulomb's Law

Problem 4.9 A square with sides 2 m each has a charge of $40 \mu\text{C}$ at each of its four corners. Determine the electric field at a point 5 m above the center of the square.

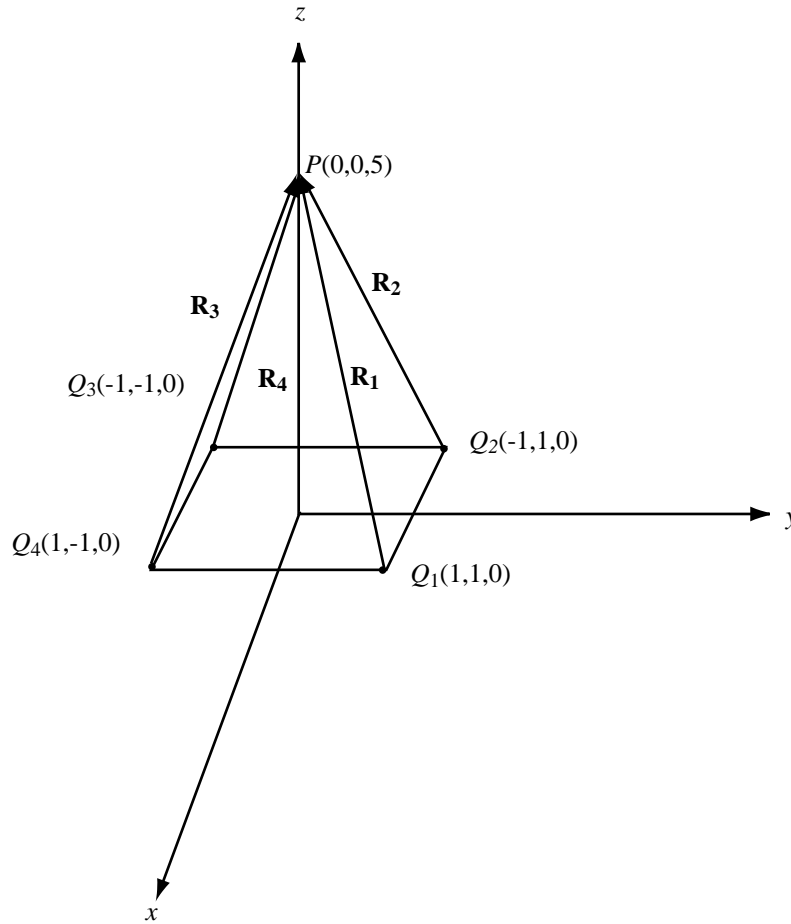


Figure P4.9: Square with charges at the corners.

Solution: The distance $|R|$ between any of the charges and point P is

$$|R| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}.$$

$$\begin{aligned} \mathbf{E} &= \frac{Q}{4\pi\epsilon_0} \left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3} + \frac{\mathbf{R}_2}{|\mathbf{R}|^3} + \frac{\mathbf{R}_3}{|\mathbf{R}|^3} + \frac{\mathbf{R}_4}{|\mathbf{R}|^3} \right] \\ &= \frac{Q}{4\pi\epsilon_0} \left[\frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} \right] \\ &= \hat{\mathbf{z}} \frac{5Q}{(27)^{3/2}\pi\epsilon_0} = \hat{\mathbf{z}} \frac{5 \times 40 \mu\text{C}}{(27)^{3/2}\pi\epsilon_0} = \frac{1.42}{\pi\epsilon_0} \times 10^{-6} \text{ (V/m)} = \hat{\mathbf{z}} 51.2 \text{ (kV/m)}. \end{aligned}$$

Problem 4.10 Three point charges, each with $q = 3 \text{ nC}$, are located at the corners of a triangle in the x - y plane, with one corner at the origin, another at $(2 \text{ cm}, 0, 0)$, and the third at $(0, 2 \text{ cm}, 0)$. Find the force acting on the charge located at the origin.

Solution: Use Eq. (4.19) to determine the electric field at the origin due to the other two point charges [Fig. P4.10]:

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \left[\frac{3 \text{ nC} (-\hat{\mathbf{x}}0.02)}{(0.02)^3} \right] + \frac{3 \text{ nC} (-\hat{\mathbf{y}}0.02)}{(0.02)^3} = -67.4(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \text{ (kV/m) at } \mathbf{R} = 0.$$

Employ Eq. (4.14) to find the force $\mathbf{F} = q\mathbf{E} = -202.2(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \text{ } (\mu\text{N})$.

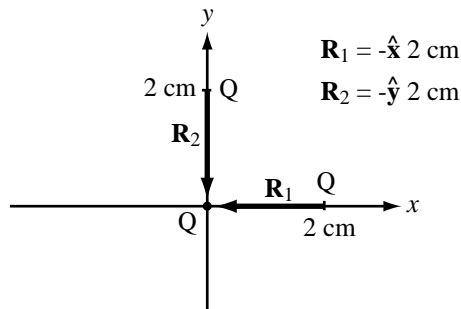


Figure P4.10: Locations of charges in Problem 4.10.

Problem 4.11 Charge $q_1 = 6 \mu\text{C}$ is located at $(1 \text{ cm}, 1 \text{ cm}, 0)$ and charge q_2 is located at $(0, 0, 4 \text{ cm})$. What should q_2 be so that \mathbf{E} at $(0, 2 \text{ cm}, 0)$ has no y -component?

Solution: For the configuration of Fig. P4.11, use of Eq. (4.19) gives

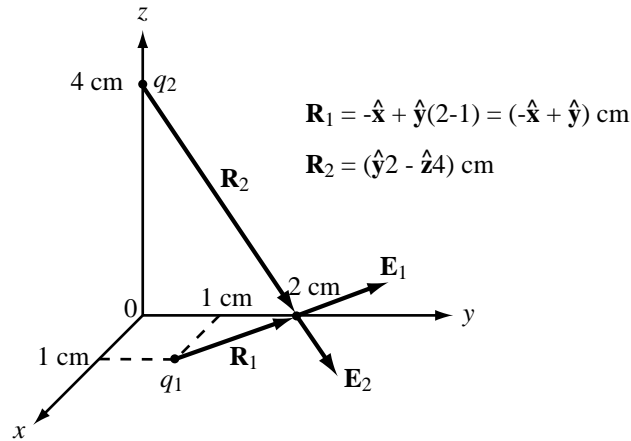


Figure P4.11: Locations of charges in Problem 4.11.

$$\begin{aligned} \mathbf{E}(\mathbf{R} = \hat{\mathbf{y}}2\text{cm}) &= \frac{1}{4\pi\epsilon} \left[\frac{6\mu\text{C}(-\hat{\mathbf{x}} + \hat{\mathbf{y}}) \times 10^{-2}}{(2 \times 10^{-2})^{3/2}} + \frac{q_2(\hat{\mathbf{y}}2 - \hat{\mathbf{z}}4) \times 10^{-2}}{(20 \times 10^{-2})^{3/2}} \right] \\ &= \frac{1}{4\pi\epsilon} [-\hat{\mathbf{x}}21.21 \times 10^{-6} + \hat{\mathbf{y}}(21.21 \times 10^{-6} + 0.224q_2) \\ &\quad - \hat{\mathbf{z}}0.447q_2] \quad (\text{V/m}). \end{aligned}$$

If $E_y = 0$, then $q_2 = -21.21 \times 10^{-6} / 0.224 \approx -94.69 \mu\text{C}$.

Problem 4.12 A line of charge with uniform density $\rho_l = 8 \mu\text{C/m}$ exists in air along the z -axis between $z = 0$ and $z = 5$ cm. Find \mathbf{E} at $(0, 10 \text{ cm}, 0)$.

Solution: Use of Eq. (4.21c) for the line of charge shown in Fig. P4.12 gives

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_l dl'}{R'^2}, \\ R' &= \hat{\mathbf{y}}0.1 - \hat{\mathbf{z}}z \\ &= \frac{1}{4\pi\epsilon_0} \int_{z=0}^{0.05} (8 \times 10^{-6}) \frac{(\hat{\mathbf{y}}0.1 - \hat{\mathbf{z}}z)}{[(0.1)^2 + z^2]^{3/2}} dz \\ &= \frac{8 \times 10^{-6}}{4\pi\epsilon_0} \left[\frac{\hat{\mathbf{y}}10z + \hat{\mathbf{z}}}{\sqrt{(0.1)^2 + z^2}} \right]_{z=0}^{0.05} \\ &= 71.86 \times 10^3 [\hat{\mathbf{y}}4.47 - \hat{\mathbf{z}}1.06] = \hat{\mathbf{y}}321.4 \times 10^3 - \hat{\mathbf{z}}76.2 \times 10^3 \quad (\text{V/m}). \end{aligned}$$

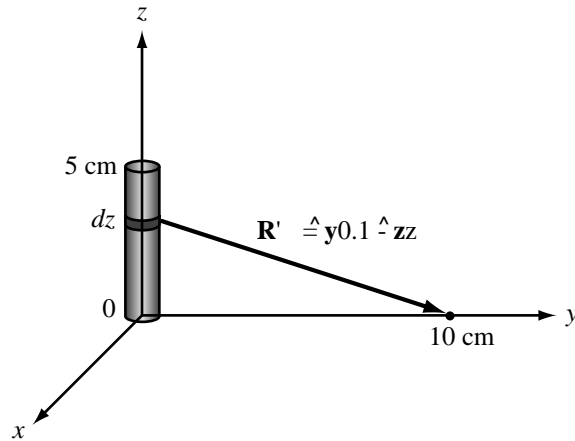


Figure P4.12: Line charge.

Problem 4.13 Electric charge is distributed along an arc located in the x - y plane and defined by $r = 2$ cm and $0 \leq \phi \leq \pi/4$. If $\rho_l = 5$ ($\mu\text{C}/\text{m}$), find \mathbf{E} at $(0, 0, z)$ and then evaluate it at (a) the origin, (b) $z = 5$ cm, and (c) $z = -5$ cm.

Solution: For the arc of charge shown in Fig. P4.13, $dl = r d\phi = 0.02 d\phi$, and $\mathbf{R}' = -\hat{\mathbf{x}}0.02 \cos \phi - \hat{\mathbf{y}}0.02 \sin \phi + \hat{\mathbf{z}}z$. Use of Eq. (4.21c) gives

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_l dl'}{R'^2} \\ &= \frac{1}{4\pi\epsilon_0} \int_{\phi=0}^{\pi/4} \rho_l \frac{(-\hat{\mathbf{x}}0.02 \cos \phi - \hat{\mathbf{y}}0.02 \sin \phi + \hat{\mathbf{z}}z)}{((0.02)^2 + z^2)^{3/2}} 0.02 d\phi \\ &= \frac{898.8}{((0.02)^2 + z^2)^{3/2}} [-\hat{\mathbf{x}}0.014 - \hat{\mathbf{y}}0.006 + \hat{\mathbf{z}}0.78z] \quad (\text{V/m}). \end{aligned}$$

- (a) At $z = 0$, $\mathbf{E} = -\hat{\mathbf{x}}1.6 - \hat{\mathbf{y}}0.66$ (MV/m).
 (b) At $z = 5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}81.4 - \hat{\mathbf{y}}33.7 + \hat{\mathbf{z}}226$ (kV/m).
 (c) At $z = -5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}81.4 - \hat{\mathbf{y}}33.7 - \hat{\mathbf{z}}226$ (kV/m).

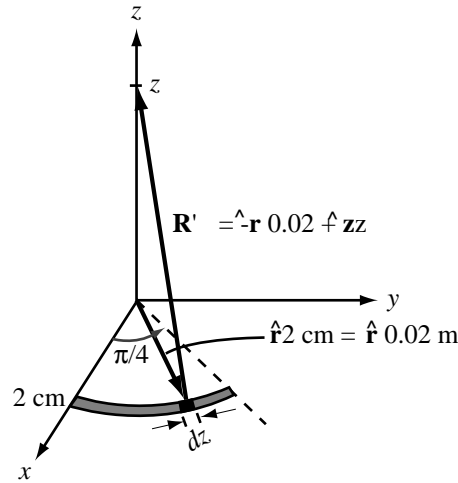


Figure P4.13: Line charge along an arc.

Problem 4.14 A line of charge with uniform density ρ_l extends between $z = -L/2$ and $z = L/2$ along the z -axis. Apply Coulomb's law to obtain an expression for the electric field at any point $P(r, \phi, 0)$ on the x - y plane. Show that your result reduces to the expression given by Eq. (4.33) as the length L is extended to infinity.

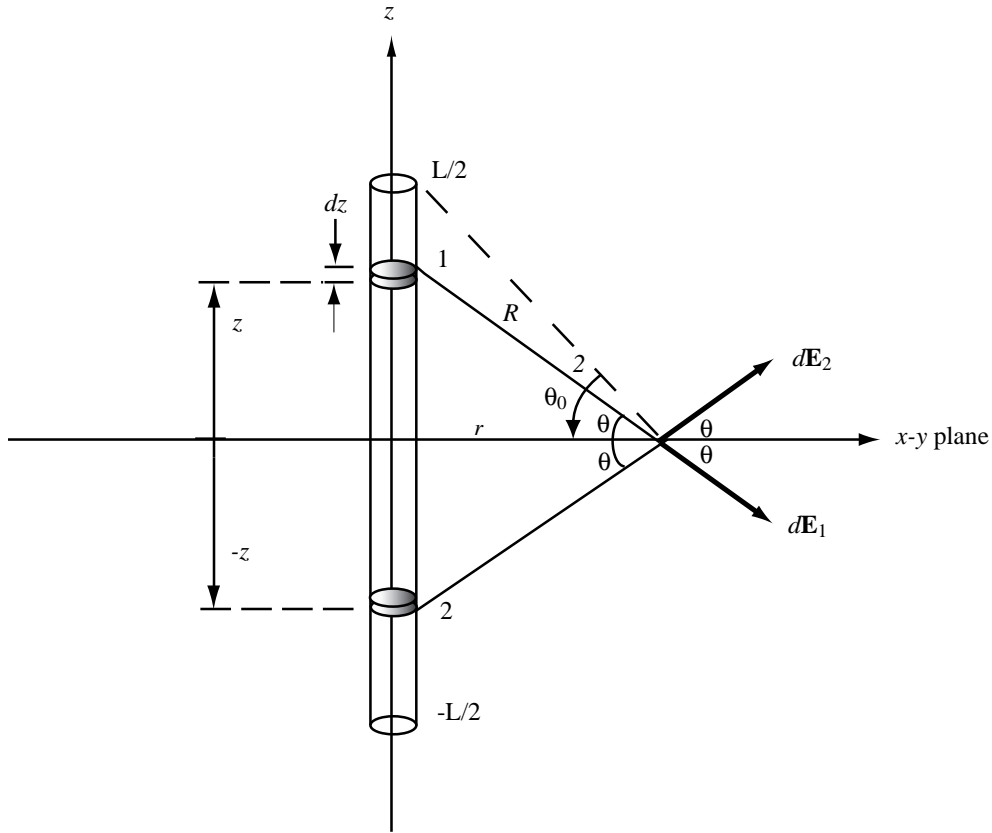
Solution: Consider an element of charge of height dz at height z . Call it element 1. The electric field at P due to this element is $d\mathbf{E}_1$. Similarly, an element at $-z$ produces $d\mathbf{E}_2$. These two electric fields have equal z -components, but in opposite directions, and hence they will cancel. Their components along $\hat{\mathbf{r}}$ will add. Thus, the net field due to both elements is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{r}} \frac{2\rho_l \cos \theta dz}{4\pi\epsilon_0 R^2} = \frac{\hat{\mathbf{r}}\rho_l \cos \theta dz}{2\pi\epsilon_0 R^2}.$$

where the $\cos \theta$ factor provides the components of $d\mathbf{E}_1$ and $d\mathbf{E}_2$ along $\hat{\mathbf{r}}$.

Our integration variable is z , but it will be easier to integrate over the variable θ from $\theta = 0$ to

$$\theta_0 = \sin^{-1} \frac{L/2}{\sqrt{r^2 + (L/2)^2}}.$$

Figure P4.14: Line charge of length L .

Hence, with $R = r/\cos\theta$, and $z = r\tan\theta$ and $dz = r\sec^2\theta d\theta$, we have

$$\begin{aligned} \mathbf{E} &= \int_{z=0}^{L/2} d\mathbf{E} = \int_{\theta=0}^{\theta_0} d\mathbf{E} = \int_0^{\theta_0} \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0} \frac{\cos^3\theta}{r^2} r\sec^2\theta d\theta \\ &= \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 r} \int_0^{\theta_0} \cos\theta d\theta \\ &= \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 r} \sin\theta_0 = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 r} \frac{L/2}{\sqrt{r^2 + (L/2)^2}}. \end{aligned}$$

For $L \gg r$,

$$\frac{L/2}{\sqrt{r^2 + (L/2)^2}} \approx 1,$$

and

$$\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 r} \quad (\text{infinite line of charge}).$$

Problem 4.15 Repeat Example 4-5 for the circular disk of charge of radius a , but in the present case assume the surface charge density to vary with r as

$$\rho_s = \rho_{s0} r^2 \quad (\text{C/m}^2),$$

where ρ_{s0} is a constant.

Solution: We start with the expression for $d\mathbf{E}$ given in Example 4-5 but we replace ρ_s with $\rho_{s0} r^2$:

$$d\mathbf{E} = \hat{\mathbf{z}} \frac{h}{4\pi\epsilon_0 (r^2 + h^2)^{3/2}} (2\pi\rho_{s0} r^3 dr),$$

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_{s0} h}{2\epsilon_0} \int_0^a \frac{r^3 dr}{(r^2 + h^2)^{3/2}}.$$

To perform the integration, we use

$$R^2 = r^2 + h^2,$$

$$2R dR = 2r dr,$$

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_{s0} h}{2\epsilon_0} \int_h^{(a^2+h^2)^{1/2}} \frac{(R^2 - h^2) dR}{R^2}$$

$$= \hat{\mathbf{z}} \frac{\rho_{s0} h}{2\epsilon_0} \left[\int_h^{(a^2+h^2)^{1/2}} dR - \int_h^{(a^2+h^2)^{1/2}} \frac{h^2}{R^2} dR \right]$$

$$= \hat{\mathbf{z}} \frac{\rho_{s0} h}{2\epsilon_0} \left[\sqrt{a^2 + h^2} + \frac{h^2}{\sqrt{a^2 + h^2}} - 2h \right].$$

Problem 4.16 Multiple charges at different locations are said to be in equilibrium if the force acting on any one of them is identical in magnitude and direction to the force acting on any of the others. Suppose we have two negative charges, one located at the origin and carrying charge $-9e$, and the other located on the positive x -axis at a distance d from the first one and carrying charge $-36e$. Determine the location, polarity and magnitude of a third charge whose placement would bring the entire system into equilibrium.

Solution: If

$$\mathbf{F}_1 = \text{force on } Q_1,$$

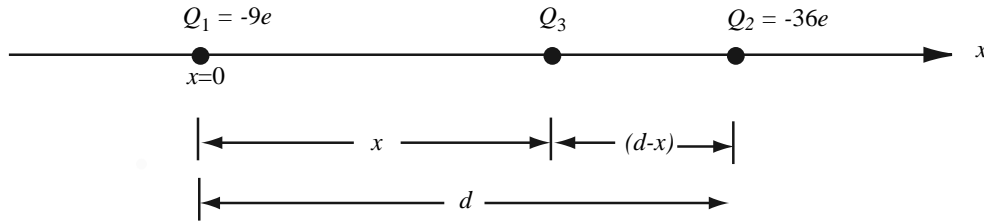


Figure P4.16: Three collinear charges.

$\mathbf{F}_2 =$ force on Q_2 ,

$\mathbf{F}_3 =$ force on Q_3 ,

then equilibrium means that

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3.$$

The two original charges are both negative, which mean they would repel each other. The third charge has to be positive and has to lie somewhere between them in order to counteract their repulsion force. The forces acting on charges Q_1 , Q_2 , and Q_3 are respectively

$$\begin{aligned} \mathbf{F}_1 &= \frac{\hat{\mathbf{R}}_{21} Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2} + \frac{\hat{\mathbf{R}}_{31} Q_1 Q_3}{4\pi\epsilon_0 R_{31}^2} = -\hat{\mathbf{x}} \frac{324e^2}{4\pi\epsilon_0 d^2} + \hat{\mathbf{x}} \frac{9eQ_3}{4\pi\epsilon_0 x^2}, \\ \mathbf{F}_2 &= \frac{\hat{\mathbf{R}}_{12} Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} + \frac{\hat{\mathbf{R}}_{32} Q_3 Q_2}{4\pi\epsilon_0 R_{32}^2} = \hat{\mathbf{x}} \frac{324e^2}{4\pi\epsilon_0 d^2} - \hat{\mathbf{x}} \frac{36eQ_3}{4\pi\epsilon_0 (d-x)^2}, \\ \mathbf{F}_3 &= \frac{\hat{\mathbf{R}}_{13} Q_1 Q_3}{4\pi\epsilon_0 R_{13}^2} + \frac{\hat{\mathbf{R}}_{23} Q_2 Q_3}{4\pi\epsilon_0 R_{23}^2} = -\hat{\mathbf{x}} \frac{9eQ_3}{4\pi\epsilon_0 x^2} + \hat{\mathbf{x}} \frac{36eQ_3}{4\pi\epsilon_0 (d-x)^2}. \end{aligned}$$

Hence, equilibrium requires that

$$-\frac{324e}{d^2} + \frac{9Q_3}{x^2} = \frac{324e}{d^2} - \frac{36Q_3}{(d-x)^2} = -\frac{9Q_3}{x^2} + \frac{36Q_3}{(d-x)^2}.$$

Solution of the above equations yields

$$Q_3 = 4e, \quad x = \frac{d}{3}.$$

Section 4-4: Gauss's Law

Problem 4.17 Three infinite lines of charge, all parallel to the z -axis, are located at the three corners of the kite-shaped arrangement shown in Fig. 4-29 (P4.17). If the

two right triangles are symmetrical and of equal corresponding sides, show that the electric field is zero at the origin.

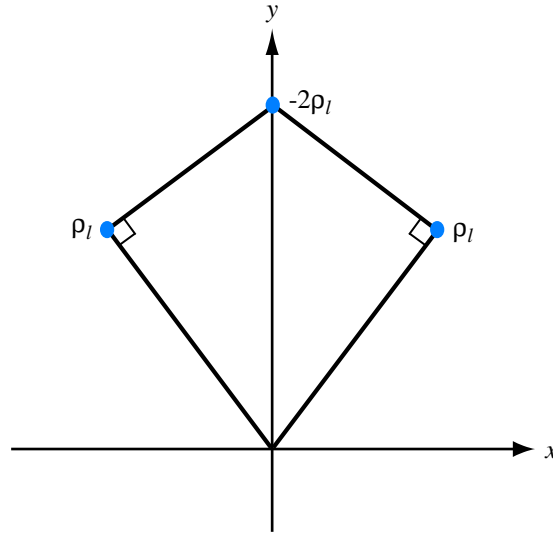


Figure P4.17: Kite-shaped arrangement of line charges for Problem 4.17.

Solution: The field due to an infinite line of charge is given by Eq. (4.33). In the present case, the total \mathbf{E} at the origin is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.$$

The components of \mathbf{E}_1 and \mathbf{E}_2 along $\hat{\mathbf{x}}$ cancel and their components along $-\hat{\mathbf{y}}$ add. Also, \mathbf{E}_3 is along $\hat{\mathbf{y}}$ because the line charge on the y -axis is negative. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}} \frac{2\rho_l \cos \theta}{2\pi\epsilon_0 R_1} + \hat{\mathbf{y}} \frac{2\rho_l}{2\pi\epsilon_0 R_2}.$$

But $\cos \theta = R_1/R_2$. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}} \frac{\rho_l}{\pi\epsilon_0 R_1} \frac{R_1}{R_2} + \hat{\mathbf{y}} \frac{\rho_l}{\pi\epsilon_0 R_2} = 0.$$

Problem 4.18 Three infinite lines of charge, $\rho_{l_1} = 3$ (nC/m), $\rho_{l_2} = -3$ (nC/m), and $\rho_{l_3} = 3$ (nC/m), are all parallel to the z -axis. If they pass through the respective points

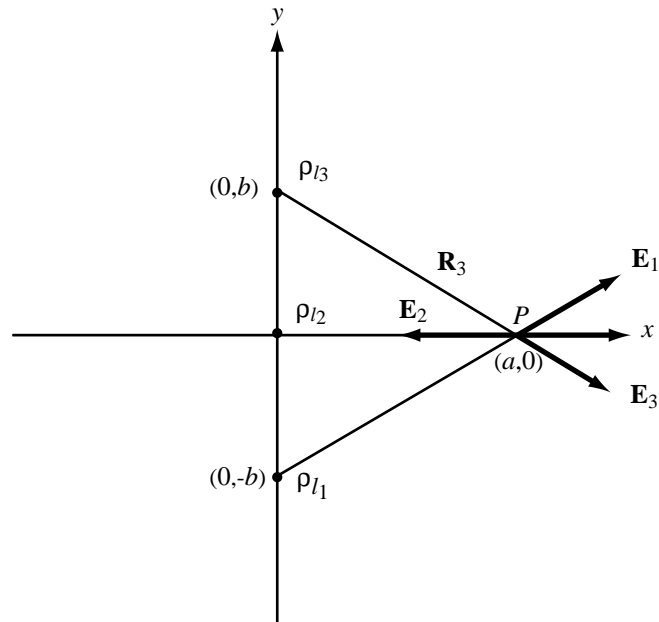


Figure P4.18: Three parallel line charges.

$(0, -b)$, $(0, 0)$, and $(0, b)$ in the x - y plane, find the electric field at $(a, 0, 0)$. Evaluate your result for $a = 2$ cm and $b = 1$ cm.

Solution:

$$\begin{aligned}\rho_{l1} &= 3 \text{ (nC/m)}, \\ \rho_{l2} &= -3 \text{ (nC/m)}, \\ \rho_{l3} &= \rho_{l1}, \\ \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.\end{aligned}$$

Components of line charges 1 and 3 along y cancel and components along x add. Hence, using Eq. (4.33),

$$\mathbf{E} = \hat{\mathbf{x}} \frac{2\rho_{l1}}{2\pi\epsilon_0 R_1} \cos\theta + \hat{\mathbf{x}} \frac{\rho_{l2}}{2\pi\epsilon_0 a}.$$

with $\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$ and $R_1 = \sqrt{a^2 + b^2}$,

$$\mathbf{E} = \frac{\hat{\mathbf{x}} 3}{2\pi\epsilon_0} \left[\frac{2a}{a^2 + b^2} - \frac{1}{a} \right] \times 10^{-9} \text{ (V/m)}.$$

For $a = 2$ cm and $b = 1$ cm,

$$\mathbf{E} = \hat{\mathbf{x}} 1.62 \text{ (kV/m)}.$$

Problem 4.19 A horizontal strip lying in the x - y plane is of width d in the y -direction and infinitely long in the x -direction. If the strip is in air and has a uniform charge distribution ρ_s , use Coulomb's law to obtain an explicit expression for the electric field at a point P located at a distance h above the centerline of the strip. Extend your result to the special case where d is infinite and compare it with Eq. (4.25).

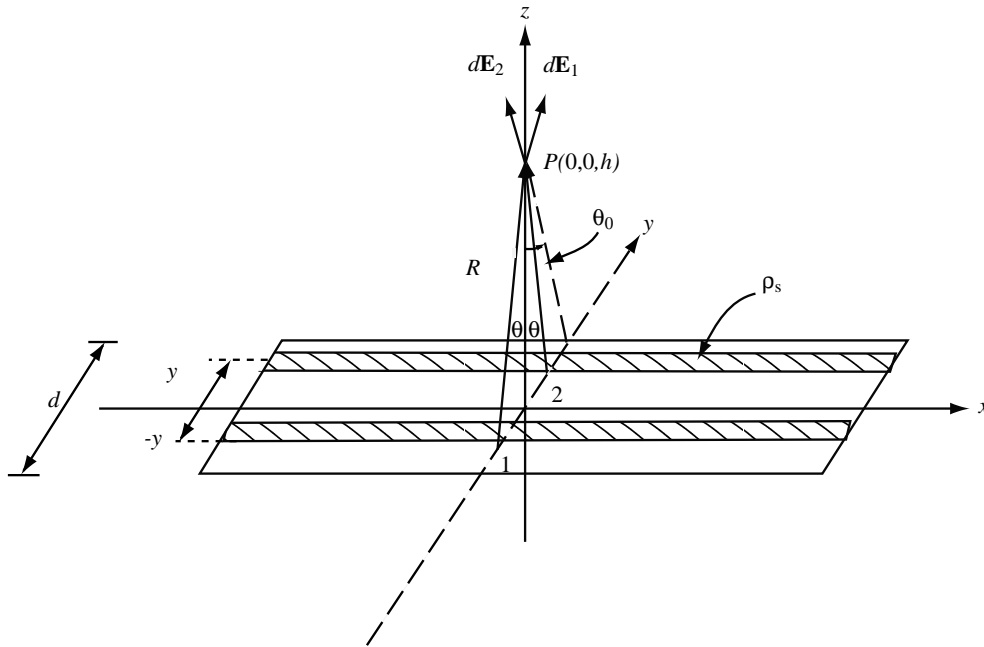


Figure P4.19: Horizontal strip of charge.

Solution: The strip of charge density ρ_s (C/m^2) can be treated as a set of adjacent line charges each of charge $\rho_l = \rho_s dy$ and width dy . At point P , the fields of line charge at distance y and line charge at distance $-y$ give contributions that cancel each other along $\hat{\mathbf{y}}$ and add along $\hat{\mathbf{z}}$. For each such pair,

$$d\mathbf{E} = \hat{\mathbf{z}} \frac{2\rho_s dy \cos \theta}{2\pi\epsilon_0 R}.$$

With $R = h/\cos\theta$, we integrate from $y = 0$ to $d/2$, which corresponds to $\theta = 0$ to $\theta_0 = \sin^{-1}[(d/2)/(h^2 + (d/2)^2)^{1/2}]$. Thus,

$$\begin{aligned}\mathbf{E} &= \int_0^{d/2} d\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{\pi\epsilon_0} \int_0^{d/2} \frac{\cos\theta}{R} dy = \hat{\mathbf{z}} \frac{\rho_s}{\pi\epsilon_0} \int_0^{\theta_0} \frac{\cos^2\theta}{h} \cdot \frac{h}{\cos^2\theta} d\theta \\ &= \hat{\mathbf{z}} \frac{\rho_s}{\pi\epsilon_0} \theta_0.\end{aligned}$$

For an infinitely wide sheet, $\theta_0 = \pi/2$ and $\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0}$, which is identical with Eq. (4.25).

Problem 4.20 Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2),$$

determine

- ρ_v by applying Eq. (4.26),
- the total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x -, y -, and z -axes and one of its corners at the origin, and
- the total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

- By applying Eq. (4.26)

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

- Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathcal{V}'} \nabla \cdot \mathbf{D} dv = \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 0 dx dy dz = 0.$$

- Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$\begin{aligned}Q &= \oint \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}, \\ F_{\text{front}} &= \int_{y=0}^2 \int_{z=0}^2 (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)) \Big|_{x=2} \cdot (\hat{\mathbf{x}} dz dy) \\ &= \int_{y=0}^2 \int_{z=0}^2 2(x+y) \Big|_{x=2} dz dy = \left(2z \left(2y + \frac{1}{2}y^2 \right) \Big|_{z=0}^2 \right) \Big|_{y=0}^2 = 24,\end{aligned}$$

$$\begin{aligned}
F_{\text{back}} &= \int_{y=0}^2 \int_{z=0}^2 (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)) \Big|_{x=0} \cdot (-\hat{\mathbf{x}} dz dy) \\
&= - \int_{y=0}^2 \int_{z=0}^2 2(x+y) \Big|_{x=0} dz dy = - \left(zy^2 \Big|_{z=0}^2 \right) \Big|_{y=0}^2 = -8, \\
F_{\text{right}} &= \int_{x=0}^2 \int_{z=0}^2 (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)) \Big|_{y=2} \cdot (\hat{\mathbf{y}} dz dx) \\
&= \int_{x=0}^2 \int_{z=0}^2 (3x-2y) \Big|_{y=2} dz dx = \left(z \left(\frac{3}{2}x^2 - 4x \right) \Big|_{z=0}^2 \right) \Big|_{x=0}^2 = -4, \\
F_{\text{left}} &= \int_{x=0}^2 \int_{z=0}^2 (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)) \Big|_{y=0} \cdot (-\hat{\mathbf{y}} dz dx) \\
&= - \int_{x=0}^2 \int_{z=0}^2 (3x-2y) \Big|_{y=0} dz dx = - \left(z \left(\frac{3}{2}x^2 \right) \Big|_{z=0}^2 \right) \Big|_{x=0}^2 = -12, \\
F_{\text{top}} &= \int_{x=0}^2 \int_{z=0}^2 (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)) \Big|_{z=2} \cdot (\hat{\mathbf{z}} dy dx) \\
&= \int_{x=0}^2 \int_{z=0}^2 0 \Big|_{z=2} dy dx = 0, \\
F_{\text{bottom}} &= \int_{x=0}^2 \int_{z=0}^2 (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)) \Big|_{z=0} \cdot (\hat{\mathbf{z}} dy dx) \\
&= \int_{x=0}^2 \int_{z=0}^2 0 \Big|_{z=0} dy dx = 0.
\end{aligned}$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 24 - 8 - 4 - 12 + 0 + 0 = 0$.

Problem 4.21 Repeat Problem 4.20 for $\mathbf{D} = \hat{\mathbf{x}}xy^3z^3$ (C/m²).

Solution:

(a) From Eq. (4.26), $\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(xy^3z^3) = y^3z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_V \nabla \cdot \mathbf{D} d\mathcal{V} = \int_{z=0}^2 \int_{y=0}^2 \int_{x=0}^2 y^3 z^3 dx dy dz = \frac{xy^4z^4}{16} \Big|_{x=0}^2 \Big|_{y=0}^2 \Big|_{z=0}^2 = 32 \text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}}D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$\begin{aligned} F_{\text{front}} &= \int_{z=0}^2 \int_{y=0}^2 (\hat{\mathbf{x}}xy^3z^3) \Big|_{x=2} \cdot (\hat{\mathbf{x}} dy dz) \\ &= \int_{z=0}^2 \int_{y=0}^2 xy^3z^3 \Big|_{x=2} dy dz = \left(2 \left(\frac{y^4 z^4}{16} \right) \Big|_{y=0}^2 \right) \Big|_{z=0}^2 = 32, \\ F_{\text{back}} &= \int_{z=0}^2 \int_{y=0}^2 (\hat{\mathbf{x}}xy^3z^3) \Big|_{x=0} \cdot (-\hat{\mathbf{x}} dy dz) = - \int_{z=0}^2 \int_{y=0}^2 xy^3z^3 \Big|_{x=0} dy dz = 0. \end{aligned}$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 32 + 0 + 0 + 0 + 0 + 0 = 32 \text{ C}$.

Problem 4.22 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a , and charge Q_2 is uniformly distributed over a second spherical shell of radius b , with $b > a$. Apply Gauss's law to find \mathbf{E} in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}}D_R$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}}R^2 \sin\theta d\theta d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S . For a spherical surface of radius R ,

$$\begin{aligned} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}}D_R) \cdot (\hat{\mathbf{R}}R^2 \sin\theta d\theta d\phi) &= Q_{\text{tot}}, \\ D_R R^2 (2\pi) [-\cos\theta]_0^{\pi} &= Q_{\text{tot}}, \\ D_R &= \frac{Q_{\text{tot}}}{4\pi R^2}. \end{aligned}$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \epsilon\mathbf{E}$. Thus, we find \mathbf{E} from \mathbf{D} .

(a) In the region $R < a$,

$$Q_{\text{tot}} = 0, \quad \mathbf{E} = \hat{\mathbf{R}}E_R = \frac{\hat{\mathbf{R}}Q_{\text{tot}}}{4\pi R^2\epsilon} = 0 \quad (\text{V/m}).$$

(b) In the region $a < R < b$,

$$Q_{\text{tot}} = Q_1, \quad \mathbf{E} = \hat{\mathbf{R}}E_R = \frac{\hat{\mathbf{R}}Q_1}{4\pi R^2\epsilon} \quad (\text{V/m}).$$

(c) In the region $R > b$,

$$Q_{\text{tot}} = Q_1 + Q_2, \quad \mathbf{E} = \hat{\mathbf{R}}E_R = \frac{\hat{\mathbf{R}}(Q_1 + Q_2)}{4\pi R^2\epsilon} \quad (\text{V/m}).$$

Problem 4.23 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \quad (\text{C/m}^2),$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$\begin{aligned} Q &= \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R \cdot \hat{\mathbf{R}}R^2 \sin\theta d\theta d\phi \Big|_{R=a} \\ &= 2\pi\rho_0 a^3 \int_0^{\pi} \sin\theta d\theta = -2\pi\rho_0 a^3 \cos\theta \Big|_0^{\pi} = 4\pi\rho_0 a^3 \quad (\text{C}). \end{aligned}$$

Problem 4.24 In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_v = 50re^{-r} \quad (\text{C/m}^3).$$

Apply Gauss's law to find \mathbf{D} .

Solution:

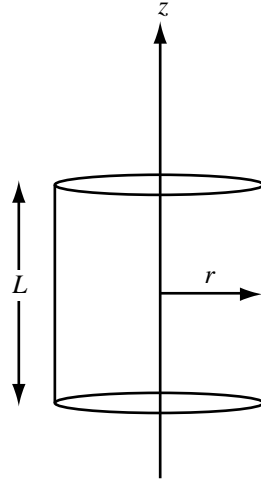


Figure P4.24: Gaussian surface.

Method 1: Integral Form of Gauss's Law

Since ρ_v varies as a function of r only, so will \mathbf{D} . Hence, we construct a cylinder of radius r and length L , coincident with the z -axis. Symmetry suggests that \mathbf{D} has the functional form $\mathbf{D} = \hat{\mathbf{r}}D$. Hence,

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{s} &= Q, \\ \int \hat{\mathbf{r}}D \cdot d\mathbf{s} &= D(2\pi rL), \\ Q &= 2\pi L \int_0^r 50re^{-r} \cdot r dr \\ &= 100\pi L[-r^2e^{-r} + 2(1 - e^{-r}(1+r))], \\ \mathbf{D} = \hat{\mathbf{r}}D &= \hat{\mathbf{r}}50 \left[\frac{2}{r}(1 - e^{-r}(1+r)) - re^{-r} \right].\end{aligned}$$

Method 2: Differential Method

$$\nabla \cdot \mathbf{D} = \rho_v, \quad \mathbf{D} = \hat{\mathbf{r}}D_r,$$

with D_r being a function of r .

$$\frac{1}{r} \frac{\partial}{\partial r} (rD_r) = 50re^{-r},$$

$$\begin{aligned}\frac{\partial}{\partial r}(rD_r) &= 50r^2e^{-r}, \\ \int_0^r \frac{\partial}{\partial r}(rD_r) dr &= \int_0^r 50r^2e^{-r} dr, \\ rD_r &= 50[2(1 - e^{-r}(1+r)) - r^2e^{-r}], \\ \mathbf{D} = \hat{\mathbf{r}}rD_r &= \hat{\mathbf{r}}50 \left[\frac{2}{r}(1 - e^{-r}(1+r)) - re^{-r} \right].\end{aligned}$$

Problem 4.25 An infinitely long cylindrical shell extending between $r = 1$ m and $r = 3$ m contains a uniform charge density ρ_{v0} . Apply Gauss's law to find \mathbf{D} in all regions.

Solution: For $r < 1$ m, $\mathbf{D} = 0$.

For $1 \leq r \leq 3$ m,

$$\begin{aligned}\oint_S \hat{\mathbf{r}}D_r \cdot d\mathbf{s} &= Q, \\ D_r \cdot 2\pi rL &= \rho_{v0} \cdot \pi L(r^2 - 1^2), \\ \mathbf{D} = \hat{\mathbf{r}}D_r &= \hat{\mathbf{r}} \frac{\rho_{v0}\pi L(r^2 - 1)}{2\pi rL} = \hat{\mathbf{r}} \frac{\rho_{v0}(r^2 - 1)}{2r}, \quad 1 \leq r \leq 3 \text{ m}.\end{aligned}$$

For $r \geq 3$ m,

$$\begin{aligned}D_r \cdot 2\pi rL &= \rho_{v0}\pi L(3^2 - 1^2) = 8\rho_{v0}\pi L, \\ \mathbf{D} = \hat{\mathbf{r}}D_r &= \hat{\mathbf{r}} \frac{4\rho_{v0}}{r}, \quad r \geq 3 \text{ m}.\end{aligned}$$

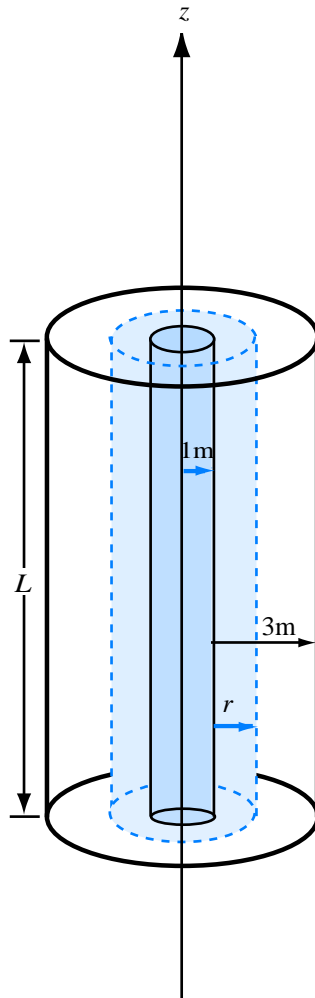


Figure P4.25: Cylindrical shell.

Problem 4.26 If the charge density increases linearly with distance from the origin such that $\rho_v = 0$ at the origin and $\rho_v = 40 \text{ C/m}^3$ at $R = 2 \text{ m}$, find the corresponding variation of \mathbf{D} .

Solution:

$$\rho_v(R) = a + bR,$$

$$\rho_v(0) = a = 0,$$

$$\rho_v(2) = 2b = 40.$$

Hence, $b = 20$.

$$\rho_v(R) = 20R \quad (\text{C/m}^3).$$

Applying Gauss's law to a spherical surface of radius R ,

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{s} &= \int_V \rho_v d\mathcal{V}, \\ D_R \cdot 4\pi R^2 &= \int_0^R 20R \cdot 4\pi R^2 dR = 80\pi \frac{R^4}{4}, \\ D_R &= 5R^2 \quad (\text{C/m}^2), \\ \mathbf{D} &= \hat{\mathbf{R}}D_R = \hat{\mathbf{R}}5R^2 \quad (\text{C/m}^2). \end{aligned}$$

Section 4-5: Electric Potential

Problem 4.27 A square in the x - y plane in free space has a point charge of $+Q$ at corner $(a/2, a/2)$ and the same at corner $(a/2, -a/2)$ and a point charge of $-Q$ at each of the other two corners.

(a) Find the electric potential at any point P along the x -axis.

(b) Evaluate V at $x = a/2$.

Solution: $R_1 = R_2$ and $R_3 = R_4$.

$$V = \frac{Q}{4\pi\epsilon_0 R_1} + \frac{Q}{4\pi\epsilon_0 R_2} + \frac{-Q}{4\pi\epsilon_0 R_3} + \frac{-Q}{4\pi\epsilon_0 R_4} = \frac{Q}{2\pi\epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_3} \right)$$

with

$$\begin{aligned} R_1 &= \sqrt{\left(x - \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}, \\ R_3 &= \sqrt{\left(x + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}. \end{aligned}$$

At $x = a/2$,

$$\begin{aligned} R_1 &= \frac{a}{2}, \\ R_3 &= \frac{a\sqrt{5}}{2}, \\ V &= \frac{Q}{2\pi\epsilon_0} \left(\frac{2}{a} - \frac{2}{\sqrt{5}a} \right) = \frac{0.55Q}{\pi\epsilon_0 a}. \end{aligned}$$

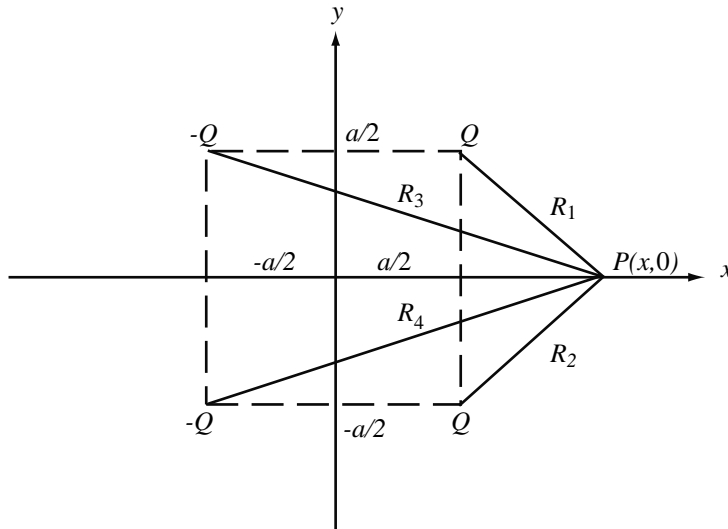


Figure P4.27: Potential due to four point charges.

Problem 4.28 The circular disk of radius a shown in Fig. 4-7 (P4.28) has uniform charge density ρ_s across its surface.

- (a) Obtain an expression for the electric potential V at a point $P(0, 0, z)$ on the z -axis.
- (b) Use your result to find \mathbf{E} and then evaluate it for $z = h$. Compare your final expression with Eq. (4.24), which was obtained on the basis of Coulomb's law.

Solution:

(a) Consider a ring of charge at a radial distance r . The charge contained in width dr is

$$dq = \rho_s(2\pi r dr) = 2\pi\rho_s r dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{2\pi\rho_s r dr}{4\pi\epsilon_0(r^2 + z^2)^{1/2}}.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\epsilon_0} \int_0^a \frac{r dr}{(r^2 + z^2)^{1/2}} = \frac{\rho_s}{2\epsilon_0} (r^2 + z^2)^{1/2} \Big|_0^a = \frac{\rho_s}{2\epsilon_0} \left[(a^2 + z^2)^{1/2} - z \right].$$

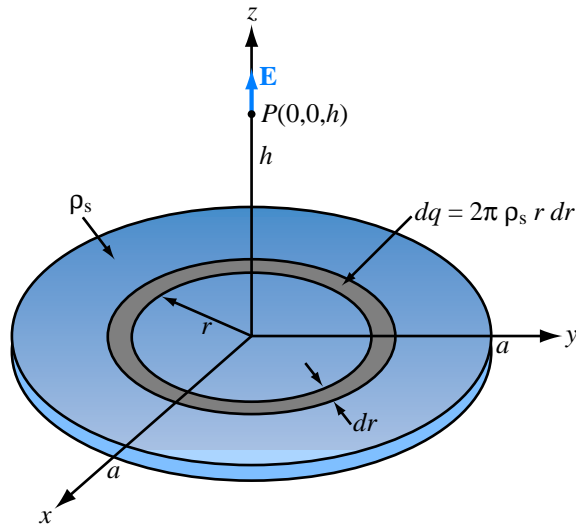


Figure P4.28: Circular disk of charge.

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}} \frac{\partial V}{\partial x} - \hat{\mathbf{y}} \frac{\partial V}{\partial y} - \hat{\mathbf{z}} \frac{\partial V}{\partial z} = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{a^2 + z^2}} \right].$$

The expression for \mathbf{E} reduces to Eq. (4.24) when $z = h$.

Problem 4.29 A circular ring of charge of radius a lies in the x - y plane and is centered at the origin. If the ring is in air and carries a uniform density ρ_l , (a) show that the electrical potential at $(0, 0, z)$ is given by $V = \rho_l a / [2\epsilon_0 (a^2 + z^2)^{1/2}]$, and (b) find the corresponding electric field \mathbf{E} .

Solution:

(a) For the ring of charge shown in Fig. P4.29, using Eq. (3.67) in Eq. (4.48c) gives

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon_0} \int_l \frac{\rho_l}{R'} dl' = \frac{1}{4\pi\epsilon_0} \int_{\phi'=0}^{2\pi} \frac{\rho_l}{\sqrt{a^2 + r^2 - 2ar\cos(\phi' - \phi) + z^2}} a d\phi'.$$

Point $(0, 0, z)$ in Cartesian coordinates corresponds to $(r, \phi, z) = (0, \phi, z)$ in cylindrical coordinates. Hence, for $r = 0$,

$$V(0, 0, z) = \frac{1}{4\pi\epsilon_0} \int_{\phi'=0}^{2\pi} \frac{\rho_l}{\sqrt{a^2 + z^2}} a d\phi' = \frac{\rho_l a}{2\epsilon_0 \sqrt{a^2 + z^2}}.$$

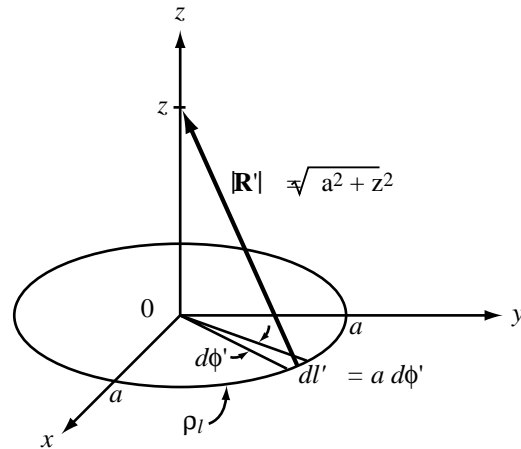


Figure P4.29: Ring of charge.

(b) From Eq. (4.51),

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{z}} \frac{\rho_l a}{2\epsilon_0} \frac{\partial}{\partial z} (a^2 + z^2)^{-1/2} = \hat{\mathbf{z}} \frac{\rho_l a}{2\epsilon_0} \frac{z}{(a^2 + z^2)^{3/2}} \quad (\text{V/m}).$$

Problem 4.30 Show that the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_l along the z -axis is $V_{12} = (\rho_l/2\pi\epsilon_0) \ln(r_2/r_1)$.

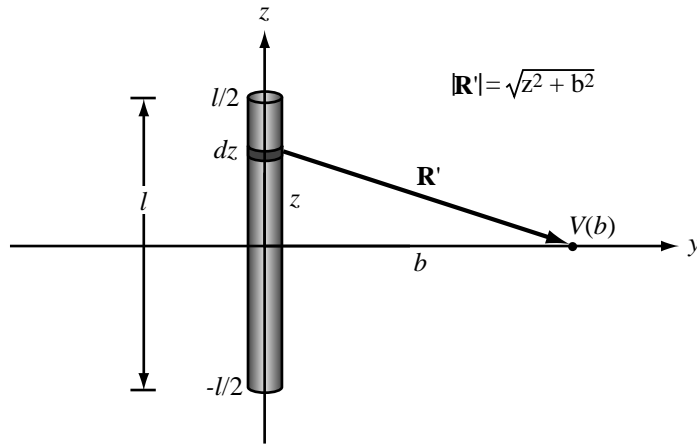
Solution: From Eq. (4.33), the electric field due to an infinite line of charge is

$$\mathbf{E} = \hat{\mathbf{r}} E_r = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 r}.$$

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1} \mathbf{E} \cdot d\mathbf{l} = -\int_{r_2}^{r_1} \frac{\hat{\mathbf{r}} \rho_l}{2\pi\epsilon_0 r} \cdot \hat{\mathbf{r}} dr = \frac{\rho_l}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right).$$

Problem 4.31 Find the electric potential V at a location a distance b from the origin in the x - y plane due to a line charge with charge density ρ_l and of length l . The line charge is coincident with the z -axis and extends from $z = -l/2$ to $z = l/2$.

Figure P4.31: Line of charge of length ℓ .

Solution: From Eq. (4.48c), we can find the voltage at a distance b away from a line of charge [Fig. P4.31]:

$$V(b) = \frac{1}{4\pi\epsilon} \int_{l'} \frac{\rho_l}{R'} dl' = \frac{\rho_l}{4\pi\epsilon} \int_{-l/2}^{l/2} \frac{dz}{\sqrt{z^2 + b^2}} = \frac{\rho_l}{4\pi\epsilon} \ln \left(\frac{l + \sqrt{l^2 + 4b^2}}{-l + \sqrt{l^2 + 4b^2}} \right).$$

Problem 4.32 For the electric dipole shown in Fig. 4-13, $d = 1$ cm and $|\mathbf{E}| = 4$ (mV/m) at $R = 1$ m and $\theta = 0^\circ$. Find \mathbf{E} at $R = 2$ m and $\theta = 90^\circ$.

Solution: For $R = 1$ m and $\theta = 0^\circ$, $|\mathbf{E}| = 4$ mV/m, we can solve for q using Eq. (4.56):

$$\mathbf{E} = \frac{qd}{4\pi\epsilon_0 R^3} (\hat{\mathbf{R}} 2 \cos \theta + \hat{\boldsymbol{\theta}} \sin \theta).$$

Hence,

$$|\mathbf{E}| = \left(\frac{qd}{4\pi\epsilon_0} \right) 2 = 4 \text{ mV/m} \quad \text{at } \theta = 0^\circ,$$

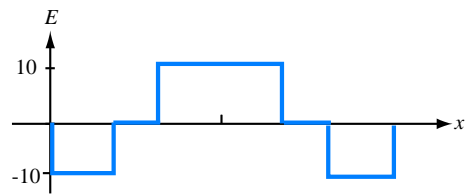
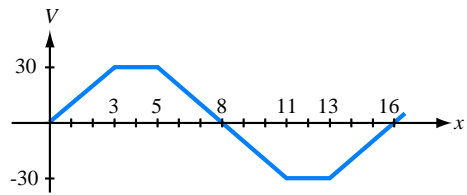
$$q = \frac{10^{-3} \times 8\pi\epsilon_0}{d} = \frac{10^{-3} \times 8\pi\epsilon_0}{10^{-2}} = 0.8\pi\epsilon_0 \quad (\text{C}).$$

Again using Eq. (4.56) to find \mathbf{E} at $R = 2$ m and $\theta = 90^\circ$, we have

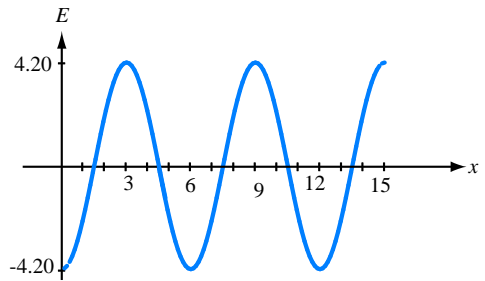
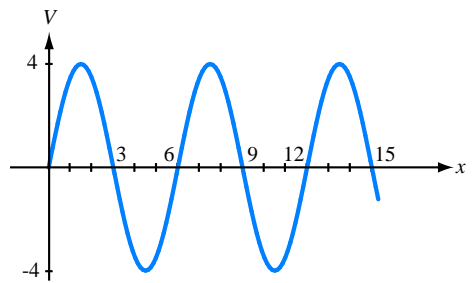
$$\mathbf{E} = \frac{0.8\pi\epsilon_0 \times 10^{-2}}{4\pi\epsilon_0 \times 2^3} (\hat{\mathbf{R}}(0) + \hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}} \frac{1}{4} \quad (\text{mV/m}).$$

Problem 4.33 For each of the following distributions of the electric potential V , sketch the corresponding distribution of \mathbf{E} (in all cases, the vertical axis is in volts and the horizontal axis is in meters):

Solution:



(a)



(b)

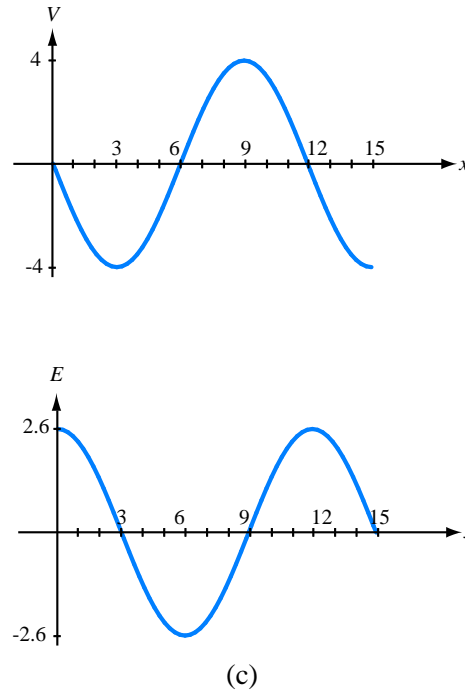


Figure P4.33: Electric potential distributions of Problem 4.33.

Problem 4.34 Given the electric field

$$\mathbf{E} = \hat{\mathbf{R}} \frac{18}{R^2} \quad (\text{V/m}),$$

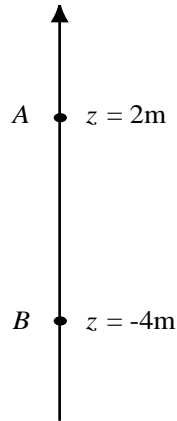
find the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z -axis.

Solution:

$$V_{AB} = V_A - V_B = - \int_B^A \mathbf{E} \cdot d\mathbf{l}.$$

Along z -direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}} \frac{18}{z^2}$ for $z \geq 0$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$ and $\mathbf{E} = -\hat{\mathbf{z}} \frac{18}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = - \int_{-4}^2 \hat{\mathbf{R}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} dz = - \left[\int_{-4}^0 -\hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} dz + \int_0^2 \hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} dz \right] = 4 \text{ V}.$$

Figure P4.34: Potential between B and A .

Problem 4.35 An infinitely long line of charge with uniform density $\rho_l = 9$ (nC/m) lies in the x - y plane parallel to the y -axis at $x = 2$ m. Find the potential V_{AB} at point $A(3 \text{ m}, 0, 4 \text{ m})$ in Cartesian coordinates with respect to point $B(0, 0, 0)$ by applying the result of Problem 4.30.

Solution: According to Problem 4.30,

$$V = \frac{\rho_l}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

where r_1 and r_2 are the distances of A and B . In this case,

$$r_1 = \sqrt{(3-2)^2 + 4^2} = \sqrt{17} \text{ m},$$

$$r_2 = 2 \text{ m}.$$

Hence,

$$V_{AB} = \frac{9 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}} \ln\left(\frac{2}{\sqrt{17}}\right) = -117.09 \text{ V}.$$

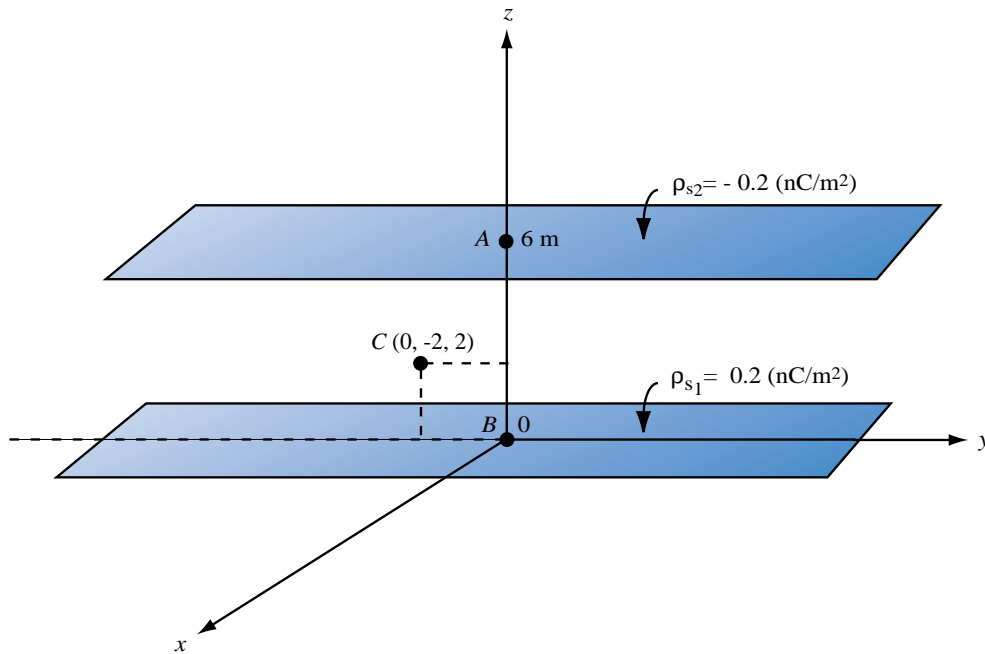


Figure P4.36: Two parallel planes of charge.

The voltage at C depends only on the z -coordinate of C . Hence, with point A being at the lowest potential and B at the highest potential,

$$V_{BC} = \frac{-2}{6} V_{AB} = -\frac{(-135.59)}{3} = 45.20 \text{ V},$$

$$V_{AC} = V_{AB} + V_{BC} = -135.59 + 45.20 = -90.39 \text{ V}.$$

Section 4-7: Conductors

Problem 4.37 A cylindrical bar of silicon has a radius of 4 mm and a length of 8 cm. If a voltage of 5 V is applied between the ends of the bar and $\mu_e = 0.13 \text{ (m}^2/\text{V}\cdot\text{s)}$, $\mu_h = 0.05 \text{ (m}^2/\text{V}\cdot\text{s)}$, $N_e = 1.5 \times 10^{16} \text{ electrons/m}^3$, and $N_h = N_e$, find

- the conductivity of silicon,
- the current I flowing in the bar,
- the drift velocities \mathbf{u}_e and \mathbf{u}_h ,
- the resistance of the bar, and
- the power dissipated in the bar.

Solution:

(a) Conductivity is given in Eq. (4.65),

$$\begin{aligned}\sigma &= (N_e\mu_e + N_h\mu_h)e \\ &= (1.5 \times 10^{16})(0.13 + 0.05)(1.6 \times 10^{-19}) = 4.32 \times 10^{-4} \quad (\text{S/m}).\end{aligned}$$

(b) Similarly to Example 4.8, parts b and c,

$$I = JA = \sigma EA = (4.32 \times 10^{-4}) \left(\frac{5\text{V}}{0.08} \right) (\pi(4 \times 10^{-3})^2) = 1.36 \quad (\mu\text{A}).$$

(c) From Eqs. (4.62a) and (4.62b),

$$\begin{aligned}\mathbf{u}_e &= -\mu_e\mathbf{E} = -(0.13) \left(\frac{5}{0.08} \right) \frac{\mathbf{E}}{|\mathbf{E}|} = -8.125 \frac{\mathbf{E}}{|\mathbf{E}|} \quad (\text{m/s}), \\ \mathbf{u}_h &= \mu_h\mathbf{E} = +(0.05) \left(\frac{5}{0.08} \right) \frac{\mathbf{E}}{|\mathbf{E}|} = 3.125 \frac{\mathbf{E}}{|\mathbf{E}|} \quad (\text{m/s}).\end{aligned}$$

(d) To find the resistance, we use what we calculated above,

$$R = \frac{V}{I} = \frac{5\text{V}}{1.36 \mu\text{A}} = 3.68 \quad (\text{M}\Omega).$$

(e) Power dissipated in the bar is $P = IV = (5\text{V})(1.36 \mu\text{A}) = 6.8 \quad (\mu\text{W})$.

Problem 4.38 Repeat Problem 4.37 for a bar of germanium with $\mu_e = 0.4 \text{ (m}^2/\text{V}\cdot\text{s)}$, $\mu_h = 0.2 \text{ (m}^2/\text{V}\cdot\text{s)}$, and $N_e = N_h = 2.4 \times 10^{19} \text{ electrons or holes/m}^3$.

Solution:

(a) Conductivity is given in Eq. (4.65),

$$\sigma = (N_e\mu_e + N_h\mu_h)e = (2.4 \times 10^{19})(0.4 + 0.2)(1.6 \times 10^{-19}) = 2.3 \quad (\text{S/m}).$$

(b) Similarly to Example 4.8, parts b and c,

$$I = JA = \sigma EA = (2.3) \left(\frac{5\text{V}}{0.08} \right) (\pi(4 \times 10^{-3})^2) = 7.225 \quad (\text{mA}).$$

(c) From Eqs. (4.62a) and (4.62b),

$$\begin{aligned}\mathbf{u}_e &= -\mu_e\mathbf{E} = -(0.4) \left(\frac{5}{0.08} \right) \frac{\mathbf{E}}{|\mathbf{E}|} = -25 \frac{\mathbf{E}}{|\mathbf{E}|} \quad (\text{m/s}), \\ \mathbf{u}_h &= \mu_h\mathbf{E} = (0.2) \left(\frac{5}{0.08} \right) \frac{\mathbf{E}}{|\mathbf{E}|} = 12.5 \frac{\mathbf{E}}{|\mathbf{E}|} \quad (\text{m/s}).\end{aligned}$$

(d) To find the resistance, we use what we calculated above,

$$R = \frac{V}{I} = \frac{5\text{V}}{7.225\text{ mA}} = 0.69 \text{ (k}\Omega\text{)}.$$

(e) Power dissipated in the bar is $P = IV = (5\text{V})(7.225\text{ mA}) = 36.125\text{ (mW)}$.

Problem 4.39 A 100-m-long conductor of uniform cross section has a voltage drop of 4 V between its ends. If the density of the current flowing through it is 1.4×10^6 (A/m²), identify the material of the conductor.

Solution: We know that conductivity characterizes a material:

$$\mathbf{J} = \sigma \mathbf{E}, \quad 1.4 \times 10^6 \text{ (A/m}^2\text{)} = \sigma \left(\frac{4 \text{ (V)}}{100 \text{ (m)}} \right), \quad \sigma = 3.5 \times 10^7 \text{ (S/m)}.$$

From Table B-2, we find that aluminum has $\sigma = 3.5 \times 10^7$ (S/m).

Problem 4.40 A coaxial resistor of length l consists of two concentric cylinders. The inner cylinder has radius a and is made of a material with conductivity σ_1 , and the outer cylinder, extending between $r = a$ and $r = b$, is made of a material with conductivity σ_2 . If the two ends of the resistor are capped with conducting plates, show that the resistance between the two ends is $R = l/[\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))]$.

Solution: Due to the conducting plates, the ends of the coaxial resistor are each uniform at the same potential. Hence, the electric field everywhere in the resistor will be parallel to the axis of the resistor, in which case the two cylinders can be considered to be two separate resistors in parallel. Then, from Eq. (4.70),

$$\frac{1}{R} = \frac{1}{R_{\text{inner}}} + \frac{1}{R_{\text{outer}}} = \frac{\sigma_1 A_1}{l_1} + \frac{\sigma_2 A_2}{l_2} = \frac{\sigma_1 \pi a^2}{l} + \frac{\sigma_2 \pi (b^2 - a^2)}{l},$$

or

$$R = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))} \quad (\Omega).$$

Problem 4.41 Apply the result of Problem 4.40 to find the resistance of a 20-cm-long hollow cylinder (Fig. P4.41) made of carbon with $\sigma = 3 \times 10^4$ (S/m).

Solution: From Problem 4.40, we know that for two concentric cylinders,

$$R = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))} \quad (\Omega).$$

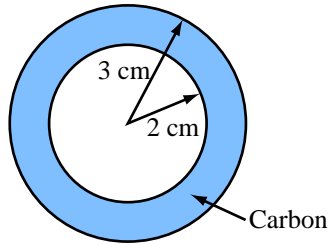


Figure P4.41: Cross section of hollow cylinder of Problem 4.41.

For air $\sigma_1 = 0$ (S/m), $\sigma_2 = 3 \times 10^4$ (S/m); hence,

$$R = \frac{0.2}{3\pi \times 10^4((0.03)^2 - (0.02)^2)} = 4.2 \text{ (m}\Omega\text{)}.$$

Problem 4.42 A 2×10^{-3} -mm-thick square sheet of aluminum has $5 \text{ cm} \times 5 \text{ cm}$ faces. Find:

- the resistance between opposite edges on a square face, and
- the resistance between the two square faces. (See Appendix B for the electrical constants of materials).

Solution:

(a)

$$R = \frac{l}{\sigma A}.$$

For aluminum, $\sigma = 3.5 \times 10^7$ (S/m) [Appendix B].

$$l = 5 \text{ cm}, \quad A = 5 \text{ cm} \times 2 \times 10^{-3} \text{ mm} = 10 \times 10^{-2} \times 10^{-6} = 1 \times 10^{-7} \text{ m}^2,$$

$$R = \frac{5 \times 10^{-2}}{3.5 \times 10^7 \times 1 \times 10^{-7}} = 14 \text{ (m}\Omega\text{)}.$$

(b) Now, $l = 2 \times 10^{-3} \text{ mm}$ and $A = 5 \text{ cm} \times 5 \text{ cm} = 2.5 \times 10^{-3} \text{ m}^2$.

$$R = \frac{2 \times 10^{-6}}{3.5 \times 10^7 \times 2.5 \times 10^{-3}} = 22.8 \text{ p}\Omega.$$

Section 4-9: Boundary Conditions

Problem 4.43 With reference to Fig. 4-19, find \mathbf{E}_1 if $\mathbf{E}_2 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}2$ (V/m), $\epsilon_1 = 2\epsilon_0$, $\epsilon_2 = 18\epsilon_0$, and the boundary has a surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m²). What angle does \mathbf{E}_2 make with the z -axis?

Solution: We know that $\mathbf{E}_{1t} = \mathbf{E}_{2t}$ for any 2 media. Hence, $\mathbf{E}_{1t} = \mathbf{E}_{2t} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$. Also, $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_s$ (from Table 4.3). Hence, $\epsilon_1(\mathbf{E}_1 \cdot \hat{\mathbf{n}}) - \epsilon_2(\mathbf{E}_2 \cdot \hat{\mathbf{n}}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \epsilon_2 E_{2z}}{\epsilon_1} = \frac{3.54 \times 10^{-11}}{2\epsilon_0} + \frac{18(2)}{2} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \text{ (V/m)}.$$

Hence, $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}20$ (V/m). Finding the angle \mathbf{E}_2 makes with the z -axis:

$$\mathbf{E}_2 \cdot \hat{\mathbf{z}} = |\mathbf{E}_2| \cos \theta, \quad 2 = \sqrt{9+4+4} \cos \theta, \quad \theta = \cos^{-1} \left(\frac{2}{\sqrt{17}} \right) = 61^\circ.$$

Problem 4.44 An infinitely long conducting cylinder of radius a has a surface charge density ρ_s . The cylinder is surrounded by a dielectric medium with $\epsilon_r = 4$ and contains no free charges. If the tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\phi} \cos^2 \phi / r^2$, find ρ_s .

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$\mathbf{E}_2 = \hat{\mathbf{r}}E_r - \hat{\phi} \frac{1}{r^2} \cos^2 \phi,$$

with E_r , the normal component of \mathbf{E}_2 , unknown. The surface charge density is related to E_r . To find E_r , we invoke Gauss's law in medium 2:

$$\nabla \cdot \mathbf{D}_2 = 0,$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} (rE_r) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(-\frac{1}{r^2} \cos^2 \phi \right) = 0,$$

which leads to

$$\frac{\partial}{\partial r} (rE_r) = \frac{\partial}{\partial \phi} \left(\frac{1}{r^2} \cos^2 \phi \right) = -\frac{2}{r^2} \sin \phi \cos \phi.$$

Integrating both sides with respect to r ,

$$\int \frac{\partial}{\partial r} (rE_r) dr = -2 \sin \phi \cos \phi \int \frac{1}{r^2} dr$$

$$rE_r = \frac{2}{r} \sin \phi \cos \phi,$$

or

$$E_r = \frac{2}{r^2} \sin \phi \cos \phi.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{r}} \frac{2}{r^2} \sin \phi \cos \phi - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos^2 \phi.$$

According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s,$$

where $\hat{\mathbf{n}}_2$ is the normal to the boundary and points away from medium 1. Hence, $\hat{\mathbf{n}}_2 = \hat{\mathbf{r}}$. Also, $\mathbf{D}_1 = 0$ because the cylinder is a conductor. Consequently,

$$\begin{aligned} \rho_s &= -\hat{\mathbf{r}} \cdot \mathbf{D}_2|_{r=a} \\ &= -\hat{\mathbf{r}} \cdot \epsilon_2 \mathbf{E}_2|_{r=a} \\ &= -\hat{\mathbf{r}} \cdot \epsilon_r \epsilon_0 \left[\hat{\mathbf{r}} \frac{2}{r^2} \sin \phi \cos \phi - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos^2 \phi \right] \Big|_{r=a} \\ &= -\frac{8\epsilon_0}{a^2} \sin \phi \cos \phi \quad (\text{C/m}^2). \end{aligned}$$

Problem 4.45 A 2-cm conducting sphere is embedded in a charge-free dielectric medium with $\epsilon_{2r} = 9$. If $\mathbf{E}_2 = \hat{\mathbf{R}} 3 \cos \theta - \hat{\boldsymbol{\theta}} 3 \sin \theta$ (V/m) in the surrounding region, find the charge density on the sphere's surface.

Solution: According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s.$$

In the present case, $\hat{\mathbf{n}}_2 = \hat{\mathbf{R}}$ and $\mathbf{D}_1 = 0$. Hence,

$$\begin{aligned} \rho_s &= -\hat{\mathbf{R}} \cdot \mathbf{D}_2|_{r=2 \text{ cm}} \\ &= -\hat{\mathbf{R}} \cdot \epsilon_2 (\hat{\mathbf{R}} 3 \cos \theta - \hat{\boldsymbol{\theta}} 3 \sin \theta) \\ &= -27\epsilon_0 \cos \theta \quad (\text{C/m}^2). \end{aligned}$$

Problem 4.46 If $\mathbf{E} = \hat{\mathbf{R}} 150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. \mathbf{E}_2 inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1R} = \rho_s, \quad E_{1R} = \frac{\rho_s}{\epsilon_0} = \frac{Q}{S\epsilon_0},$$

$$Q = E_R S \epsilon_0 = (150)4\pi(0.05)^2 \epsilon_0 = \frac{3\pi\epsilon_0}{2} \quad (C).$$

Problem 4.47 Figure 4-34(a) (P4.47) shows three planar dielectric slabs of equal thickness but with different dielectric constants. If \mathbf{E}_0 in air makes an angle of 45° with respect to the z -axis, find the angle of \mathbf{E} in each of the other layers.

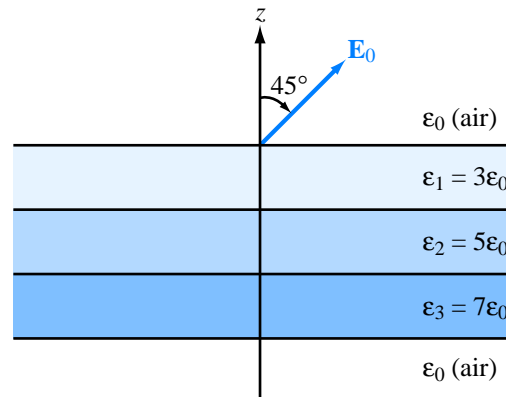


Figure P4.47: Dielectric slabs in Problem 4.47.

Solution: Labeling the upper air region as region 0 and using Eq. (4.99),

$$\begin{aligned}\theta_1 &= \tan^{-1} \left(\frac{\epsilon_1}{\epsilon_0} \tan \theta_0 \right) = \tan^{-1} (3 \tan 45^\circ) = 71.6^\circ, \\ \theta_2 &= \tan^{-1} \left(\frac{\epsilon_2}{\epsilon_1} \tan \theta_1 \right) = \tan^{-1} \left(\frac{5}{3} \tan 71.6^\circ \right) = 78.7^\circ, \\ \theta_3 &= \tan^{-1} \left(\frac{\epsilon_3}{\epsilon_2} \tan \theta_2 \right) = \tan^{-1} \left(\frac{7}{5} \tan 78.7^\circ \right) = 81.9^\circ.\end{aligned}$$

In the lower air region, the angle is again 45° .

Sections 4-10 and 4-11: Capacitance and Electrical Energy

Problem 4.48 Determine the force of attraction in a parallel-plate capacitor with $A = 5 \text{ cm}^2$, $d = 2 \text{ cm}$, and $\epsilon_r = 4$ if the voltage across it is 50 V.

Solution: From Eq. (4.131),

$$\mathbf{F} = -\hat{\mathbf{z}} \frac{\epsilon A |\mathbf{E}|^2}{2} = -\hat{\mathbf{z}} 2\epsilon_0 (5 \times 10^{-4}) \left(\frac{50}{0.02} \right)^2 = -\hat{\mathbf{z}} 55.3 \times 10^{-9} \quad (\text{N}).$$

Problem 4.49 Dielectric breakdown occurs in a material whenever the magnitude of the field \mathbf{E} exceeds the dielectric strength anywhere in that material. In the coaxial capacitor of Example 4-12,

- (a) At what value of r is $|E|$ maximum?
- (b) What is the breakdown voltage if $a = 1$ cm, $b = 2$ cm, and the dielectric material is mica with $\epsilon_r = 6$?

Solution:

(a) From Eq. (4.114), $\mathbf{E} = -\hat{\mathbf{r}}\rho_l/2\pi\epsilon r$ for $a < r < b$. Thus, it is evident that $|\mathbf{E}|$ is maximum at $r = a$.

(b) The dielectric breaks down when $|\mathbf{E}| > 200$ (MV/m) (see Table 4-2), or

$$|\mathbf{E}| = \frac{\rho_l}{2\pi\epsilon r} = \frac{\rho_l}{2\pi(6\epsilon_0)(10^{-2})} = 200 \quad (\text{MV/m}),$$

which gives $\rho_l = (200 \text{ MV/m})(2\pi)6(8.854 \times 10^{-12})(0.01) = 667.6 \text{ } (\mu\text{C/m})$.

From Eq. (4.115), we can find the voltage corresponding to that charge density,

$$V = \frac{\rho_l}{2\pi\epsilon} \ln\left(\frac{b}{a}\right) = \frac{(667.6 \mu\text{C/m})}{12\pi(8.854 \times 10^{-12} \text{ F/m})} \ln(2) = 1.39 \quad (\text{MV}).$$

Thus, $V = 1.39$ (MV) is the breakdown voltage for this capacitor.

Problem 4.50 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area Fig. 4-33 (P4.50). If the voltage across the capacitor is 10 V, find

- (a) the force acting on the electron,
- (b) the acceleration of the electron, and
- (c) the time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

Solution:

(a) The electric force acting on a charge Q_e is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_e E = Q_e \frac{V}{d} = -1.6 \times 10^{-19} \frac{10}{0.01} = -1.6 \times 10^{-16} \quad (\text{N}).$$

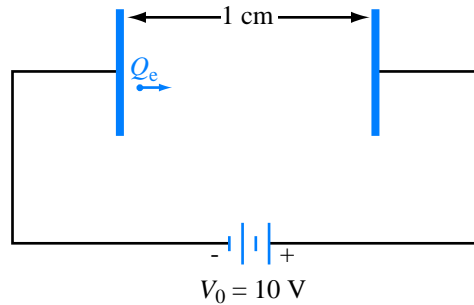


Figure P4.50: Electron between charged plates of Problem 4.50.

The force is directed from the negatively charged plate towards the positively charged plate.

(b)

$$a = \frac{F}{m} = \frac{1.6 \times 10^{-16}}{9.1 \times 10^{-31}} = 1.76 \times 10^{14} \quad (\text{m/s}^2).$$

(c) The electron does not get fast enough at the end of its short trip for relativity to manifest itself; classical mechanics is adequate to find the transit time. From classical mechanics, $d = d_0 + u_0 t + \frac{1}{2} a t^2$, where in the present case the start position is $d_0 = 0$, the total distance traveled is $d = 1$ cm, the initial velocity $u_0 = 0$, and the acceleration is given by part (b). Solving for the time t ,

$$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2 \times 0.01}{1.76 \times 10^{14}}} = 10.7 \times 10^{-9} \text{ s} = 10.7 \quad (\text{ns}).$$

Problem 4.51 In a dielectric medium with $\epsilon_r = 4$, the electric field is given by

$$\mathbf{E} = \hat{\mathbf{x}}(x^2 + 2z) + \hat{\mathbf{y}}x^2 - \hat{\mathbf{z}}(y + z) \quad (\text{V/m}).$$

Calculate the electrostatic energy stored in the region $-1 \text{ m} \leq x \leq 1 \text{ m}$, $0 \leq y \leq 2 \text{ m}$, and $0 \leq z \leq 3 \text{ m}$.

Solution: Electrostatic potential energy is given by Eq. (4.124),

$$\begin{aligned} W_e &= \frac{1}{2} \int_{\mathcal{V}} \epsilon |\mathbf{E}|^2 d\mathcal{V} = \frac{\epsilon}{2} \int_{z=0}^3 \int_{y=0}^2 \int_{x=-1}^1 [(x^2 + 2z)^2 + x^4 + (y+z)^2] dx dy dz \\ &= \frac{4\epsilon_0}{2} \left(\left(\left(\frac{2}{5} x^5 y z + \frac{2}{3} z^2 x^3 y + \frac{4}{3} z^3 x y + \frac{1}{12} (y+z)^4 x \right) \Big|_{x=-1}^1 \right) \Big|_{y=0}^2 \right) \Big|_{z=0}^3 \\ &= \frac{4\epsilon_0}{2} \left(\frac{1304}{5} \right) = 4.62 \times 10^{-9} \text{ (J)}. \end{aligned}$$

Problem 4.52 Figure 4-34a (P4.52(a)) depicts a capacitor consisting of two parallel, conducting plates separated by a distance d . The space between the plates

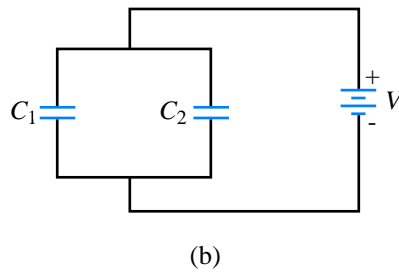
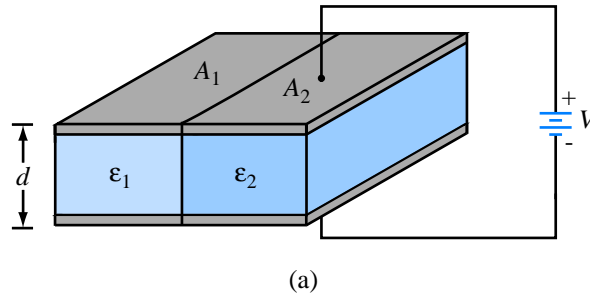


Figure P4.52: (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

contains two adjacent dielectrics, one with permittivity ϵ_1 and surface area A_1 and another with ϵ_2 and A_2 . The objective of this problem is to show that the capacitance C of the configuration shown in Fig. 4-34a (P4.52(a)) is equivalent to two capacitances in parallel, as illustrated in Fig. 4-34b (P4.52(b)), with

$$C = C_1 + C_2, \quad (4.132)$$

where

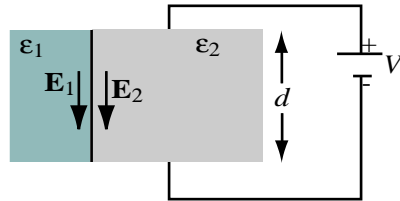
$$C_1 = \frac{\epsilon_1 A_1}{d}, \quad (4.133)$$

$$C_2 = \frac{\epsilon_2 A_2}{d}. \quad (4.134)$$

To this end, you are asked to proceed as follows:

- Find the electric fields \mathbf{E}_1 and \mathbf{E}_2 in the two dielectric layers.
- Calculate the energy stored in each section and use the result to calculate C_1 and C_2 .
- Use the total energy stored in the capacitor to obtain an expression for C . Show that Eq. (4.132) is indeed a valid result.

Solution:



(c)

Figure P4.52: (c) Electric field inside of capacitor.

(a) Within each dielectric section, \mathbf{E} will point from the plate with positive voltage to the plate with negative voltage, as shown in Fig. P4-52(c). From $V = Ed$,

$$E_1 = E_2 = \frac{V}{d}.$$

(b)

$$W_{e1} = \frac{1}{2} \epsilon_1 E_1^2 \cdot \nu = \frac{1}{2} \epsilon_1 \frac{V^2}{d^2} \cdot A_1 d = \frac{1}{2} \epsilon_1 V^2 \frac{A_1}{d}.$$

But, from Eq. (4.121),

$$W_{e1} = \frac{1}{2} C_1 V^2.$$

Hence $C_1 = \epsilon_1 \frac{A_1}{d}$. Similarly, $C_2 = \epsilon_2 \frac{A_2}{d}$.

(c) Total energy is

$$W_e = W_{e1} + W_{e2} = \frac{1}{2} \frac{V^2}{d} (\epsilon_1 A_1 + \epsilon_2 A_2) = \frac{1}{2} C V^2.$$

Hence,

$$C = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d} = C_1 + C_2.$$

Problem 4.53 Use the result of Problem 4.52 to determine the capacitance for each of the following configurations:

- (a) conducting plates are on top and bottom faces of rectangular structure in Fig. 4-35(a) (P4.53(a)),
- (b) conducting plates are on front and back faces of structure in Fig. 4-35(a) (P4.53(a)),
- (c) conducting plates are on top and bottom faces of the cylindrical structure in Fig. 4-35(b) (P4.53(b)).

Solution:

- (a) The two capacitors share the same voltage; hence they are in parallel.

$$C_1 = \epsilon_1 \frac{A_1}{d} = 2\epsilon_0 \frac{(5 \times 1) \times 10^{-4}}{2 \times 10^{-2}} = 5\epsilon_0 \times 10^{-2},$$

$$C_2 = \epsilon_2 \frac{A_2}{d} = 4\epsilon_0 \frac{(5 \times 3) \times 10^{-4}}{2 \times 10^{-2}} = 30\epsilon_0 \times 10^{-2},$$

$$C = C_1 + C_2 = (5\epsilon_0 + 30\epsilon_0) \times 10^{-2} = 0.35\epsilon_0 = 3.1 \times 10^{-12} \text{ F}.$$

- (b)

$$C_1 = \epsilon_1 \frac{A_1}{d} = 2\epsilon_0 \frac{(2 \times 1) \times 10^{-4}}{5 \times 10^{-2}} = 0.8\epsilon_0 \times 10^{-2},$$

$$C_2 = \epsilon_2 \frac{A_2}{d} = 4\epsilon_0 \frac{(3 \times 2) \times 10^{-4}}{5 \times 10^{-2}} = \frac{24}{5} \epsilon_0 \times 10^{-2},$$

$$C = C_1 + C_2 = 0.5 \times 10^{-12} \text{ F}.$$

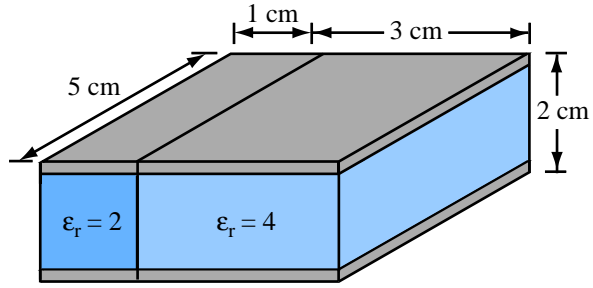
- (c)

$$C_1 = \epsilon_1 \frac{A_1}{d} = 8\epsilon_0 \frac{(\pi r_1^2)}{2 \times 10^{-2}} = \frac{4\pi\epsilon_0}{10^{-2}} (2 \times 10^{-3})^2 = 0.04 \times 10^{-12} \text{ F},$$

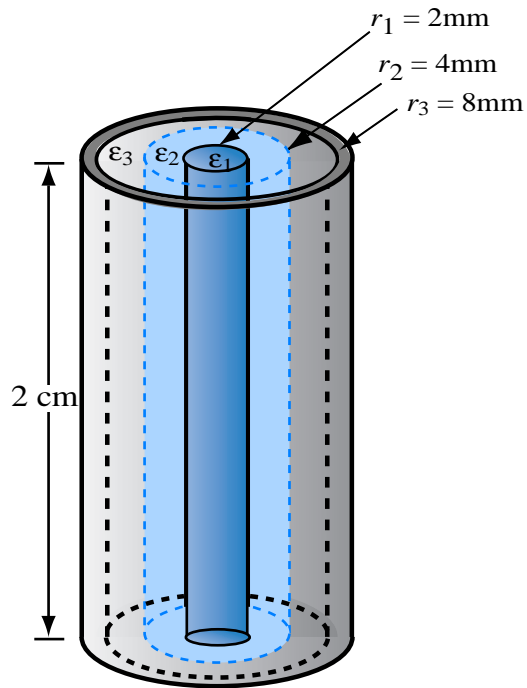
$$C_2 = \epsilon_2 \frac{A_2}{d} = 4\epsilon_0 \frac{(\pi(r_2^2 - r_1^2))}{2 \times 10^{-2}} = \frac{2\pi\epsilon_0}{10^{-2}} [(4 \times 10^{-3})^2 - (2 \times 10^{-3})^2] = 0.06 \times 10^{-12} \text{ F},$$

$$C_3 = \epsilon_3 \frac{A_3}{d} = 2\epsilon_0 \frac{(\pi(r_3^2 - r_2^2))}{2 \times 10^{-2}} = \frac{\pi\epsilon_0}{10^{-2}} [(8 \times 10^{-3})^2 - (4 \times 10^{-3})^2] = 0.12 \times 10^{-12} \text{ F},$$

$$C = C_1 + C_2 + C_3 = 0.22 \times 10^{-12} \text{ F}.$$



(a)



$$\epsilon_1 = 8\epsilon_0; \epsilon_2 = 4\epsilon_0; \epsilon_3 = 2\epsilon_0$$

(b)

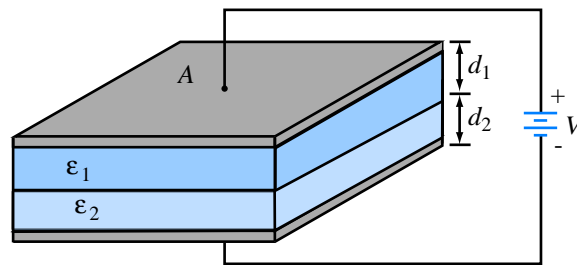
Figure P4.53: Dielectric sections for Problems 4.53 and 4.55.

Problem 4.54 The capacitor shown in Fig. 4-36 (P4.54) consists of two parallel dielectric layers. We wish to use energy considerations to show that the equivalent capacitance of the overall capacitor, C , is equal to the series combination of the capacitances of the individual layers, C_1 and C_2 , namely

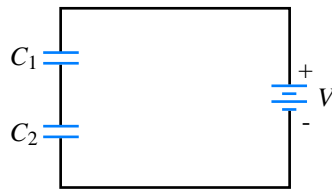
$$C = \frac{C_1 C_2}{C_1 + C_2}, \quad (4.136)$$

where

$$C_1 = \epsilon_1 \frac{A}{d_1}, \quad C_2 = \epsilon_2 \frac{A}{d_2}.$$



(a)



(b)

Figure P4.54: (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.54).

- (a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2 ? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ϵ_1 , ϵ_2 , V , and the indicated dimensions of the capacitor.
- (b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C .

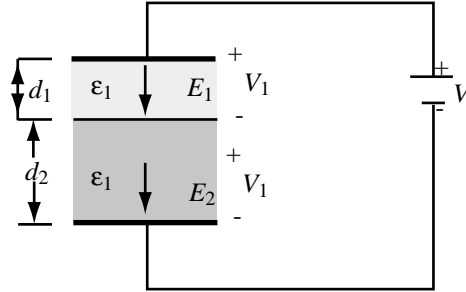


Figure P4.54: (c) Electric fields inside of capacitor.

(c) Show that C is given by Eq. (4.136).

Solution:

(a) If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \quad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of \mathbf{D} is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\epsilon_1 E_1 = \epsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\epsilon_1 E_1}{\epsilon_2} d_2,$$

which can be solved for E_1 :

$$E_1 = \frac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \frac{\epsilon_2}{\epsilon_1} d_1}.$$

(b)

$$W_{e_1} = \frac{1}{2} \epsilon_1 E_1^2 \cdot \nu_1 = \frac{1}{2} \epsilon_1 \left(\frac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[\frac{\epsilon_1 \epsilon_2^2 A d_1}{(\epsilon_2 d_1 + \epsilon_1 d_2)^2} \right],$$

$$W_{e_2} = \frac{1}{2} \epsilon_2 E_2^2 \cdot \nu_2 = \frac{1}{2} \epsilon_2 \left(\frac{V}{d_2 + \frac{\epsilon_2}{\epsilon_1} d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[\frac{\epsilon_1^2 \epsilon_2 A d_2}{(\epsilon_1 d_2 + \epsilon_2 d_1)^2} \right],$$

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[\frac{\epsilon_1 \epsilon_2^2 A d_1 + \epsilon_1^2 \epsilon_2 A d_2}{(\epsilon_1 d_2 + \epsilon_2 d_1)^2} \right].$$

But $W_e = \frac{1}{2} C V^2$, hence,

$$C = \frac{\epsilon_1 \epsilon_2^2 A d_1 + \epsilon_1^2 \epsilon_2 A d_2}{(\epsilon_2 d_1 + \epsilon_1 d_2)^2} = \epsilon_1 \epsilon_2 A \frac{(\epsilon_2 d_1 + \epsilon_1 d_2)}{(\epsilon_2 d_1 + \epsilon_1 d_2)^2} = \frac{\epsilon_1 \epsilon_2 A}{\epsilon_2 d_1 + \epsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for C by $A/d_1 d_2$, we have

$$C = \frac{\frac{\epsilon_1 A}{d_1} \cdot \frac{\epsilon_2 A}{d_2}}{\frac{\epsilon_1 A}{d_1} + \frac{\epsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\epsilon_1 A}{d_1}, \quad C_2 = \frac{\epsilon_2 A}{d_2}.$$

Problem 4.55 Use the expressions given in Problem 4.54 to determine the capacitance for the configurations in Fig. 4.35(a) (P4.55) when the conducting plates are placed on the right and left faces of the structure.

Solution:

$$C_1 = \epsilon_1 \frac{A}{d_1} = 2\epsilon_0 \frac{(2 \times 5) \times 10^{-4}}{1 \times 10^{-2}} = 20\epsilon_0 \times 10^{-2} = 1.77 \times 10^{-12} \text{ F},$$

$$C_2 = \epsilon_2 \frac{A}{d_2} = 4\epsilon_0 \frac{(2 \times 5) \times 10^{-4}}{3 \times 10^{-2}} = 1.18 \times 10^{-12} \text{ F},$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1.77 \times 1.18}{1.77 + 1.18} \times 10^{-12} = 0.71 \times 10^{-12} \text{ F}.$$

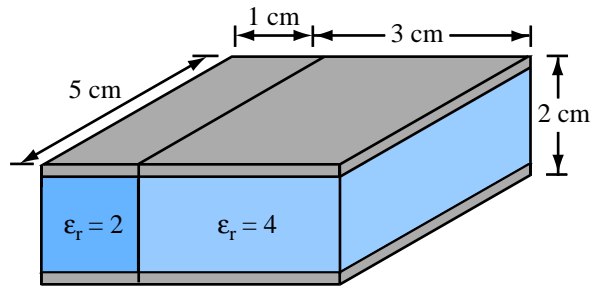
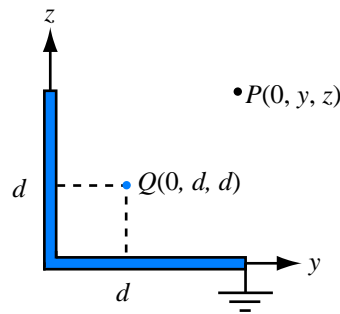


Figure P4.55: Dielectric section for Problem 4.55.

Section 4-12: Image Method

Problem 4.56 With reference to Fig. 4-37 (P4.56), charge Q is located at a distance d above a grounded half-plane located in the x - y plane and at a distance d from another grounded half-plane in the x - z plane. Use the image method to

- establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent), and
- then find the electric potential and electric field at an arbitrary point $P(0, y, z)$.

Figure P4.56: Charge Q next to two perpendicular, grounded, conducting half planes.

Solution:

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative y -axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location

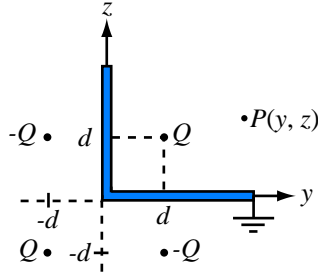


Figure P4.56: (a) Image charges.

$(0, d, -d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the $x-z$ plane gives rise to the image charges of $-Q$ at $(0, -d, d)$ and $+Q$ at $(0, -d, -d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$\begin{aligned}
 V(x, y, z) &= \frac{Q}{4\pi\epsilon} \left(\frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z-d)|} - \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z-d)|} \right. \\
 &\quad \left. + \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z+d)|} - \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z+d)|} \right) \\
 &= \frac{Q}{4\pi\epsilon} \left(\frac{1}{\sqrt{x^2 + (y-d)^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + (y+d)^2 + (z-d)^2}} \right. \\
 &\quad \left. + \frac{1}{\sqrt{x^2 + (y+d)^2 + (z+d)^2}} - \frac{1}{\sqrt{x^2 + (y-d)^2 + (z+d)^2}} \right) \\
 &= \frac{Q}{4\pi\epsilon} \left(\frac{1}{\sqrt{x^2 + y^2 - 2yd + z^2 - 2zd + 2d^2}} \right. \\
 &\quad - \frac{1}{\sqrt{x^2 + y^2 + 2yd + z^2 - 2zd + 2d^2}} \\
 &\quad + \frac{1}{\sqrt{x^2 + y^2 + 2yd + z^2 + 2zd + 2d^2}} \\
 &\quad \left. - \frac{1}{\sqrt{x^2 + y^2 - 2yd + z^2 + 2zd + 2d^2}} \right) \quad (\text{V}).
 \end{aligned}$$

From Eq. (4.51),

$$\begin{aligned}
 \mathbf{E} &= -\nabla V \\
 &= \frac{Q}{4\pi\epsilon} \left(\nabla \frac{1}{\sqrt{x^2 + (y-d)^2 + (z-d)^2}} - \nabla \frac{1}{\sqrt{x^2 + (y+d)^2 + (z-d)^2}} \right. \\
 &\quad \left. + \nabla \frac{1}{\sqrt{x^2 + (y+d)^2 + (z+d)^2}} - \nabla \frac{1}{\sqrt{x^2 + (y-d)^2 + (z+d)^2}} \right) \\
 &= \frac{Q}{4\pi\epsilon} \left(\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z-d)}{(x^2 + (y-d)^2 + (z-d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z-d)}{(x^2 + (y+d)^2 + (z-d)^2)^{3/2}} \right. \\
 &\quad \left. + \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z+d)}{(x^2 + (y+d)^2 + (z+d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z+d)}{(x^2 + (y-d)^2 + (z+d)^2)^{3/2}} \right) \quad (\text{V/m}).
 \end{aligned}$$

Problem 4.57 Conducting wires above a conducting plane carry currents I_1 and I_2 in the directions shown in Fig. 4-38 (P4.57). Keeping in mind that the direction

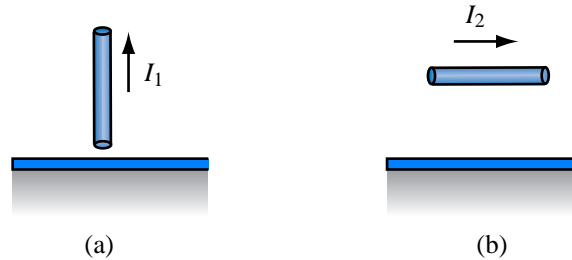


Figure P4.57: Currents above a conducting plane (Problem 4.57).

of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to I_1 and I_2 ?

Solution:

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of I_1 is same as I_1 .

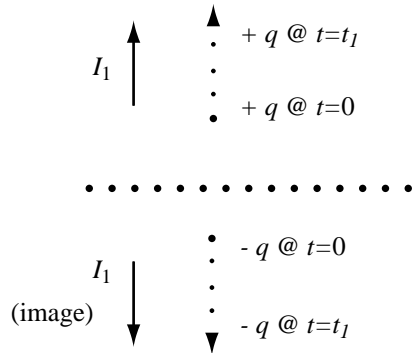


Figure P4.57: (a) Solution for part (a).

(b) In the image current, movement of negative charges to right = movement of positive charges to left.

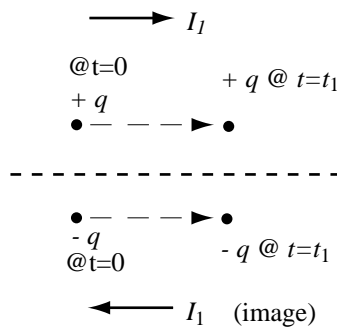


Figure P4.57: (b) Solution for part (b).

Problem 4.58 Use the image method to find the capacitance per unit length of an infinitely long conducting cylinder of radius a situated at a distance d from a parallel conducting plane, as shown in Fig. 4-39 (P4.58).

Solution: Let us distribute charge ρ_l (C/m) on the conducting cylinder. Its image cylinder at $z = -d$ will have charge density $-\rho_l$.

For the line at $z = d$, the electric field at any point z (at a distance of $d - z$ from the center of the cylinder) is, from Eq. (4.33),

$$\mathbf{E}_1 = -\hat{z} \frac{\rho_l}{2\pi\epsilon_0(d-z)}$$

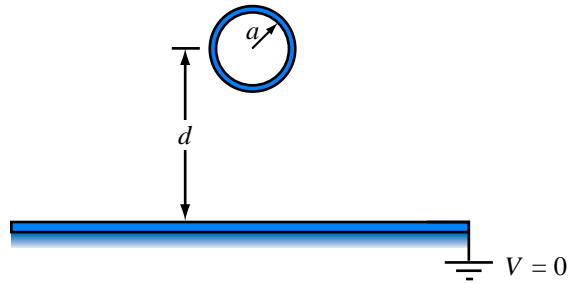


Figure P4.58: Conducting cylinder above a conducting plane (Problem 4.58).

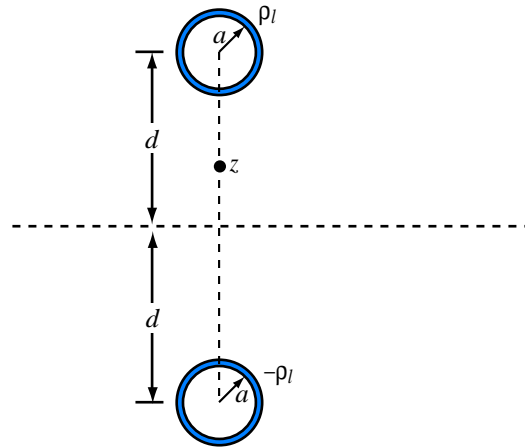


Figure P4.58: (a) Cylinder and its image.

where $-\hat{\mathbf{z}}$ is the direction away from the cylinder. Similarly for the image cylinder at distance $(d+z)$ and carrying charge $-\rho_l$,

$$\mathbf{E}_2 = \hat{\mathbf{z}} \frac{(-\rho_l)}{2\pi\epsilon_0(d+z)} = -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\epsilon_0(d+z)}.$$

The potential difference between the cylinders is obtained by integrating the total electric field from $z = -(d-a)$ to $z = (d-a)$:

$$\begin{aligned} V &= -\int_2^1 (\mathbf{E}_1 + \mathbf{E}_2) \cdot \hat{\mathbf{z}} dz \\ &= -\int_{-(d-a)}^{d-a} -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\epsilon_0} \left(\frac{1}{d-z} + \frac{1}{d+z} \right) \cdot \hat{\mathbf{z}} dz \end{aligned}$$

$$\begin{aligned}
&= \frac{\rho_l}{2\pi\epsilon_0} \int_{-(d-a)}^{d-a} \left(\frac{1}{d-z} + \frac{1}{d+z} \right) dz \\
&= \frac{\rho_l}{2\pi\epsilon_0} [-\ln(d-z) + \ln(d+z)]_{-(d-a)}^{d-a} \\
&= \frac{\rho_l}{2\pi\epsilon_0} [-\ln(a) + \ln(2d-a) + \ln(2d-a) - \ln(a)] \\
&= \frac{\rho_l}{\pi\epsilon_0} \ln \left(\frac{2d-a}{a} \right).
\end{aligned}$$

For a length L , $Q = \rho_l L$ and

$$C = \frac{Q}{V} = \frac{\rho_l L}{(\rho_l/\pi\epsilon_0) \ln[(2d-a)/a]},$$

and the capacitance per unit length is

$$C' = \frac{C}{L} = \frac{\pi\epsilon_0}{\ln[(2d/a) - 1]} \quad (\text{C/m}).$$

Problem 4.59 A circular beam of charge of radius a consists of electrons moving with a constant speed u along the $+z$ direction. The beam's axis is coincident with the z -axis and the electron charge density is given by

$$\rho_v = -cr^2 \quad (\text{C/m}^3)$$

where c is a constant and r is the radial distance from the axis of the beam.

- (a) Determine the charge density per unit length.
- (b) Determine the current crossing the z -plane.

Solution:

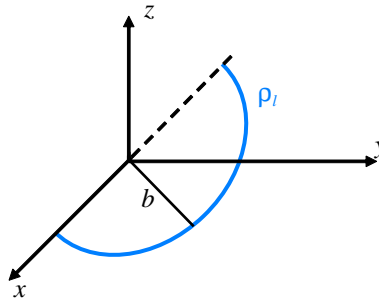
(a)

$$\begin{aligned}
\rho_l &= \int \rho_v ds \\
&= \int_{r=0}^a \int_{\phi=0}^{2\pi} -cr^2 \cdot r dr d\phi = -2\pi c \left. \frac{r^4}{4} \right|_0^a = -\frac{\pi c a^4}{2} \quad (\text{C/m}).
\end{aligned}$$

(b)

$$\begin{aligned}
 \mathbf{J} &= \rho_v \mathbf{u} = -\hat{\mathbf{z}} cr^2 u \quad (\text{A/m}^2) \\
 I &= \int \mathbf{J} \cdot d\mathbf{s} \\
 &= \int_{r=0}^a \int_{\phi=0}^{2\pi} (-\hat{\mathbf{z}} cr^2) \cdot \hat{\mathbf{z}} r \, dr \, d\phi \\
 &= -2\pi cu \int_0^a r^3 \, dr = -\frac{\pi c u a^4}{2} = \rho_l u. \quad (\text{A}).
 \end{aligned}$$

Problem 4.60 A line of charge of uniform density ρ_l occupies a semicircle of radius b as shown in the figure. Use the material presented in Example 4-4 to determine the electric field at the origin.



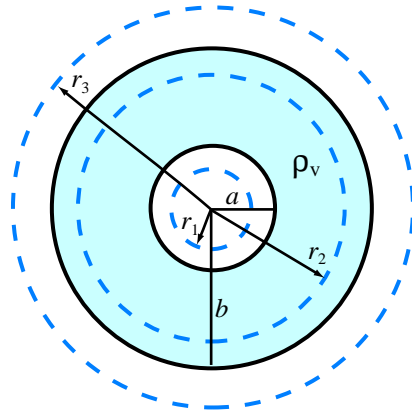
Solution: Since we have only half of a circle, we need to integrate the expression for $d\mathbf{E}_1$ given in Example 4-4 over ϕ from 0 to π . Before we do that, however, we need to set $h = 0$ (the problem asks for \mathbf{E} at the origin). Hence,

$$\begin{aligned}
 d\mathbf{E}_1 &= \frac{\rho_l b}{4\pi\epsilon_0} \frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}} d\phi \Big|_{h=0} \\
 &= \frac{-\hat{\mathbf{r}}\rho_l}{4\pi\epsilon_0 b} d\phi \\
 \mathbf{E}_1 &= \int_{\phi=0}^{\pi} d\mathbf{E}_1 = -\frac{\hat{\mathbf{r}}\rho_l}{4\epsilon_0 b}.
 \end{aligned}$$

Problem 4.61 A spherical shell with outer radius b surrounds a charge-free cavity of radius $a < b$. If the shell contains a charge density given by

$$\rho_v = -\frac{\rho_{v0}}{R^2}, \quad a \leq R \leq b,$$

where ρ_{v0} is a positive constant, determine \mathbf{D} in all regions.



Solution: Symmetry dictates that \mathbf{D} is radially oriented. Thus,

$$\mathbf{D} = \hat{\mathbf{R}}D_R.$$

At any R , Gauss's law gives

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{s} &= Q \\ \int_S \hat{\mathbf{R}}D_R \cdot \hat{\mathbf{R}} ds &= Q \\ 4\pi R^2 D_R &= Q \\ D_R &= \frac{Q}{4\pi R^2}. \end{aligned}$$

(a) For $R < a$, no charge is contained in the cavity. Hence, $Q = 0$, and

$$D_R = 0, \quad R \leq a.$$

(b) For $a \leq R \leq b$,

$$\begin{aligned} Q &= \int_{R=a}^R \rho_v dV = \int_{R=a}^R -\frac{\rho_{v0}}{R^2} \cdot 4\pi R^2 dR \\ &= -4\pi\rho_{v0}(R-a). \end{aligned}$$

Hence,

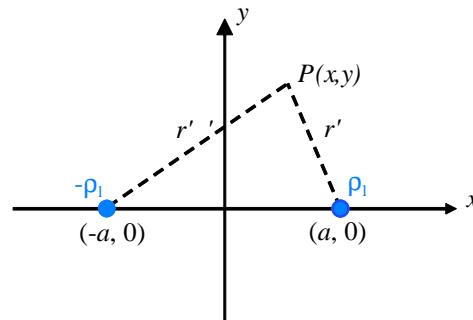
$$D_R = -\frac{\rho_{v0}(R-a)}{R^2}, \quad a \leq R \leq b.$$

(c) For $R \geq b$,

$$Q = \int_{R=a}^b \rho_v dV = -4\pi\rho_{v0}(b-a)$$

$$D_R = -\frac{\rho_{v0}(b-a)}{R^2}, \quad R \geq b.$$

Problem 4.62 Two infinite lines of charge, both parallel to the z -axis, lie in the x - z plane, one with density ρ_l and located at $x = a$ and the other with density $-\rho_l$ and located at $x = -a$. Obtain an expression for the electric potential $V(x, y)$ at a point $P(x, y)$ relative to the potential at the origin.



Solution: According to the result of Problem 4.30, the electric potential difference between a point at a distance r_1 and another at a distance r_2 from a line charge of density ρ_l is

$$V = \frac{\rho_l}{2\pi\epsilon_0} \ln \left(\frac{r_2}{r_1} \right).$$

Applying this result to the line charge at $x = a$, which is at a distance a from the origin:

$$\begin{aligned} V' &= \frac{\rho_l}{2\pi\epsilon_0} \ln \left(\frac{a}{r'} \right) \quad (r_2 = a \text{ and } r_1 = r') \\ &= \frac{\rho_l}{2\pi\epsilon_0} \ln \left(\frac{a}{\sqrt{(x-a)^2 + y^2}} \right). \end{aligned}$$

Similarly, for the negative line charge at $x = -a$,

$$\begin{aligned} V'' &= \frac{-\rho_l}{2\pi\epsilon_0} \ln \left(\frac{a}{r''} \right) \quad (r_2 = a \text{ and } r_1 = r'') \\ &= \frac{-\rho_l}{2\pi\epsilon_0} \ln \left(\frac{a}{\sqrt{(x+a)^2 + y^2}} \right). \end{aligned}$$

The potential due to both lines is

$$V = V' + V'' = \frac{\rho_l}{2\pi\epsilon_0} \left[\ln \left(\frac{a}{\sqrt{(x-a)^2 + y^2}} \right) - \ln \left(\frac{a}{\sqrt{(x+a)^2 + y^2}} \right) \right].$$

At the origin, $V = 0$, as it should be since the origin is the reference point. The potential is also zero along all points on the y -axis ($x = 0$).

Problem 4.63 A cylinder-shaped carbon resistor is 8 cm in length and its circular cross section has a diameter $d = 1$ mm.

- (a) Determine the resistance R .
- (b) To reduce its resistance by 40%, the carbon resistor is coated with a layer of copper of thickness t . Use the result of Problem 4.40 to determine t .

Solution:

- (a) From (4.70), and using the value of σ for carbon from Appendix B,

$$R = \frac{l}{\sigma A} = \frac{l}{\sigma \pi (d/2)^2} = \frac{8 \times 10^{-2}}{3 \times 10^4 \pi (10^{-3}/2)^2} = 3.4 \Omega.$$

- (b) The 40%-reduced resistance is:

$$R' = 0.6R = 0.6 \times 3.4 = 2.04 \Omega.$$

Using the result of Problem 4.40:

$$R' = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2 (b^2 - a^2))} = 2.04 \Omega.$$

With $\sigma_1 = 3.4 \times 10^4$ S/m (carbon), $\sigma_2 = 5.8 \times 10^7$ S/m (copper), $a = 1 \text{ mm}/2 = 5 \times 10^{-4}$ m, and b unknown, we have

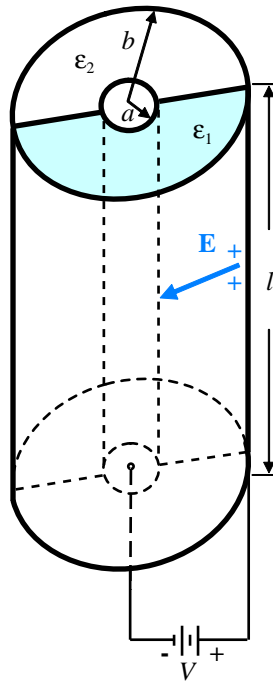
$$b = 5.00086 \times 10^{-4} \text{ m}$$

and

$$\begin{aligned} t = b - a &= (5.00086 - 5) \times 10^{-4} \\ &= 0.00086 \times 10^{-4} \text{ m} = 0.086 \mu\text{m}. \end{aligned}$$

Thus, the addition of a copper coating less than $0.1 \mu\text{m}$ in thickness reduces the resistance by 40%.

Problem 4.64 A coaxial capacitor consists of two concentric, conducting, cylindrical surfaces, one of radius a and another of radius b , as shown in the figure. The insulating layer separating the two conducting surfaces is divided equally into two semi-cylindrical sections, one filled with dielectric ϵ_1 and the other filled with dielectric ϵ_2 .



- (a) Develop an expression for C in terms of the length l and the given quantities.
- (b) Evaluate the value of C for $a = 2$ mm, $b = 6$ mm, $\epsilon_{r1} = 2$, $\epsilon_{r2} = 4$, and $l = 4$ cm.

Solution:

(a) For the indicated voltage polarity, the \mathbf{E} field inside the capacitor exists in only the dielectric materials and points radially inward. Let \mathbf{E}_1 be the field in dielectric ϵ_1 and \mathbf{E}_2 be the field in dielectric ϵ_2 . At the interface between the two dielectric sections, \mathbf{E}_1 is parallel to \mathbf{E}_2 and both are tangential to the interface. Since boundary conditions require that the tangential components of \mathbf{E}_1 and \mathbf{E}_2 be the same, it follows that:

$$\mathbf{E}_1 = \mathbf{E}_2 = -\hat{\mathbf{r}}E.$$

At $r = a$ (surface of inner conductor), in medium 1, the boundary condition on \mathbf{D} , as stated by (4.101), leads to

$$\begin{aligned}\mathbf{D}_1 &= \epsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}} \rho_{s1} \\ -\hat{\mathbf{r}} \epsilon_1 E &= \hat{\mathbf{r}} \rho_{s1}\end{aligned}$$

or

$$\rho_{s1} = -\epsilon_1 E.$$

Similarly, in medium 2

$$\rho_{s2} = -\epsilon_2 E.$$

Thus, the \mathbf{E} fields will be the same in the two dielectrics, but the charge densities will be different along the two sides of the inner conducting cylinder.

Since the same voltage applies for the two sections of the capacitor, we can treat them as two capacitors in parallel. For the capacitor half that includes dielectric ϵ_1 , we can apply the results of Eqs. (4.114)–(4.116), but we have to keep in mind that Q is now the charge on only one half of the inner cylinder. Hence,

$$C_1 = \frac{\pi \epsilon_1 l}{\ln(b/a)}.$$

Similarly,

$$C_2 = \frac{\pi \epsilon_2 l}{\ln(b/a)},$$

and

$$C = C_1 + C_2 = \frac{\pi l (\epsilon_1 + \epsilon_2)}{\ln(b/a)}.$$

(b)

$$\begin{aligned}C &= \frac{\pi \times 4 \times 10^{-2} (2 + 4) \times 8.85 \times 10^{-12}}{\ln(6/2)} \\ &= 6.07 \text{ pF.}\end{aligned}$$