Chapter 3: Vector Analysis

Lesson #14

Chapter — Section: 3-1

Topics: Basic laws of vector algebra

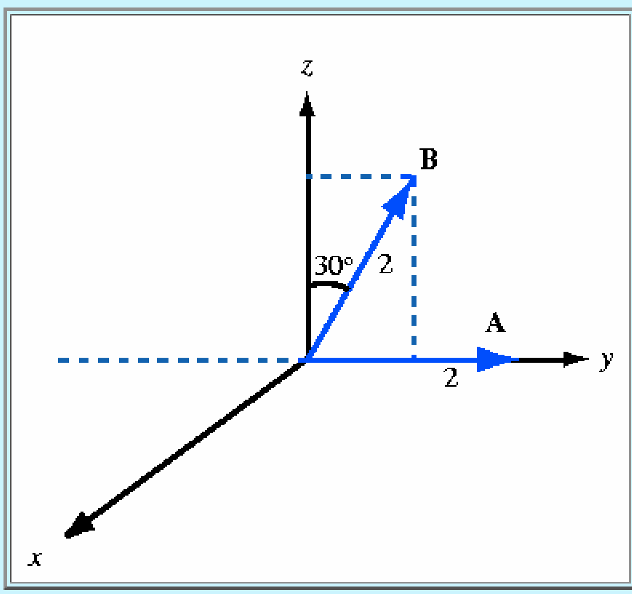
Highlights:

- Vector magnitude, direction, unit vector
- Position and distance vectors
- Vector addition and multiplication
    - Dot product
    - Vector product
    - Triple product

Special Illustrations:

- CD-ROM Module 3.2

Module 3.2 Two Intersecting Vectors



Given: Vectors **A** and **B** both lie in the *y*-*z* plane and they have the same magnitude of 2.

Q1. What is the value of the dot product of **A** and **B**?
Choose one answer.

- select $\mathbf{A} \cdot \mathbf{B} = 3.46$
- select $\mathbf{A} \cdot \mathbf{B} = 2$
- select $\mathbf{A} \cdot \mathbf{B} = 1.73$

Q2. What is the cross product of **A** and **B**?
Choose one answer.

- select $\mathbf{A} \times \mathbf{B} = \hat{x}\, 1.73$
- select $\mathbf{A} \times \mathbf{B} = \hat{x}\, 3.46$
- select $\mathbf{A} \times \mathbf{B} = -\hat{x}\, 2$

Lessons #15 and 16

Chapter — Section: 3-2

Topics: Coordinate systems

Highlights:

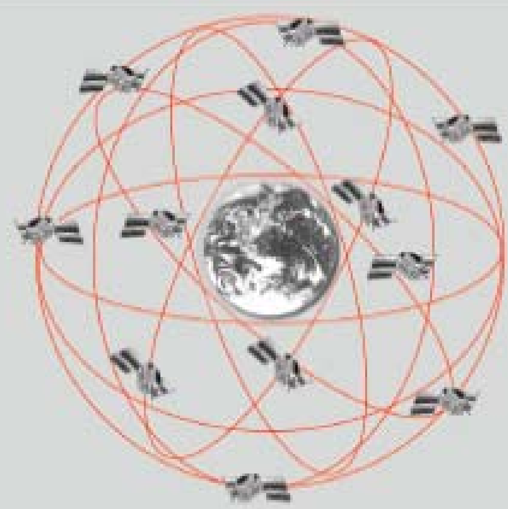
- Commonly used coordinate systems: Cartesian, cylindrical, spherical
- Choice is based on which one best suits problem geometry
- Differential surface vectors and differential volumes

Special Illustrations:

- Examples 3-3 to 3-5
- Technology Brief on "GPS" (CD-ROM)

Global Positioning System

The Global Positioning System (GPS), initially developed in the 1980s by the U.S. Department of Defense as a navigation tool for military use, has evolved into a system with numerous civilian applications including vehicle tracking, aircraft navigation, map displays in automobiles, and topographic mapping. The overall GPS is composed of 3 segments. The space segment consists of 24 satellites (A), each circling Earth every 12 hours at an orbital altitude of about 12,000 miles and transmitting continuous coded time signals. The user segment consists of hand-held or vehicle-mounted receivers that determine their own locations by receiving and processing multiple satellite signals. The third segment is a network of five ground stations, distributed around the world, that monitor the satellites and provide them with updates on their precise orbital information. GPS provides a location inaccuracy of about 30 m, both horizontally and vertically, but it can be improved to within 1 m by differential GPS (see illustration).



A. GPS nominal satellite constellation
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination

Lesson #17

Chapter — Section: 3-3

Topics: Coordinate transformations

Highlights:

- Basic logic for decomposing a vector in one coordinate system into the coordinate variables of another system
- Transformation relations (Table 3-2)

Special Illustrations:

- Example 3-8

Lesson #18

Chapter — Section: 3-4

Topics: Gradient operator

Highlights:

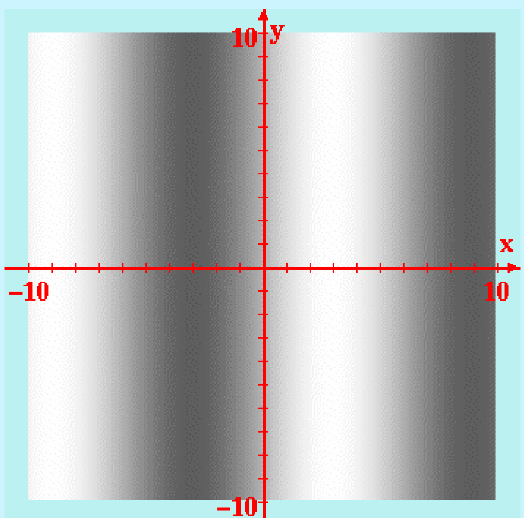
- Derivation of ∇T in Cartesian coordinates
- Directional derivative
- ∇T in cylindrical and spherical coordinates

Special Illustrations:

- Example 3-10(b)
- CD-ROM Modules 3.5 or 3.6
- CD-ROM Demos 3.1-3.9 (any 2)

Demo 3.6: Gradient of Scalar Fields

Given: A scalar field defined by:



$$T = 1 + \sin(2\pi x/6) \qquad \text{for } -10 \leq x \leq 10.$$

The field T is displayed graphically in the figure, wherein the brightness of the image at a given location is proportional to the magnitude of T at that location.

Display the graphical and analytical solution for ∇ T

Lesson #19

Chapter — Section: 3-5

Topics: Divergence operator

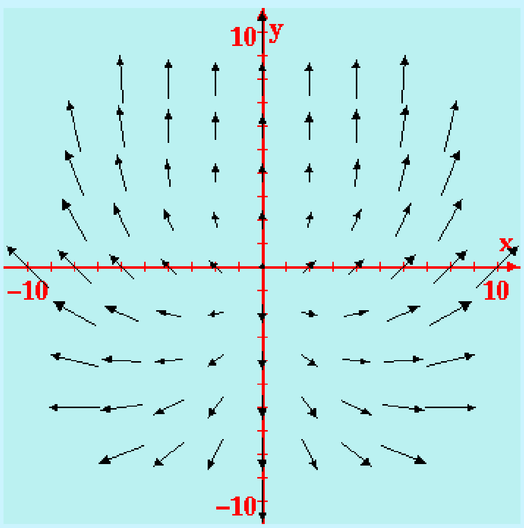
Highlights:

- Concept of "flux"
- Derivation of $\nabla . \mathbf{E}$
- Divergence theorem

Special Illustrations:

- CD-ROM Modules 3.7-3.11 (any 2)
- CD-ROM Demos 3.10-3.15 (any 1 or 2)

Demo 3.14: Divergence of Vector Fields



Given: A vector field defined by:

$$\mathbf{A} = \hat{\mathbf{r}} r + \hat{\boldsymbol{\phi}} r \cos\phi \qquad \text{for} \begin{cases} 0 \le r \le 10 \text{ and} \\ 0 \le \phi \le 2\pi \end{cases}$$

The vector **A** is displayed graphically in the figure, wherein vectors are used to depict the direction and magnitude of **A** at any given location.

Display the graphical and analytical solution for $\nabla \cdot \mathbf{A}$

Lesson #20

Chapter — Section: 3-6

Topics: Curl operator

Highlights:

- Concept of "circulation"
- Derivation of $\nabla \times \mathbf{B}$
- Stokes's theorem

Special Illustrations:

- Example 3-12

Lesson #21

Chapter — Section: 3-7

Topics: Laplacian operator

Highlights:

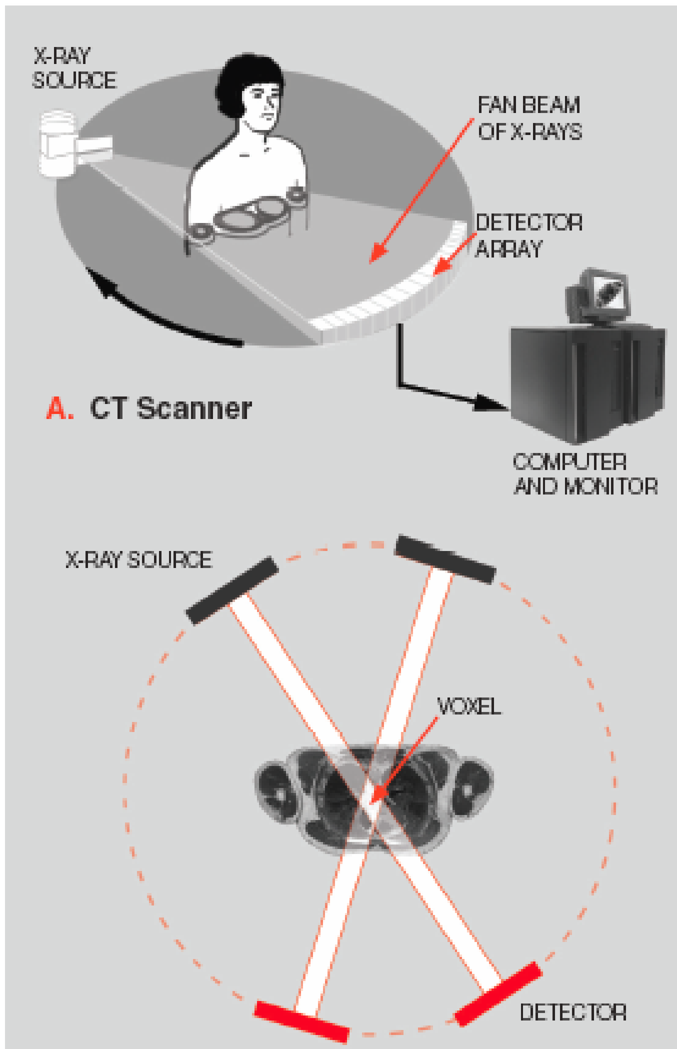
- Definition of $\nabla^2 V$
- Definition of $\nabla^2 \mathbf{E}$

Special Illustrations:

- Technology Brief on "X-Ray Computed Tomography"

X-Ray Computed Tomography

Tomography is derived from the Greek words *tome*, meaning section or slice, and *graphia*, meaning writing. Computed tomography, also known as CT scan or CAT scan (for computed axial tomography), refers to a technique capable of generating 3-D images of the x-ray attenuation (absorption) properties of an object. This is in contrast with the traditional x-ray technique which produces only a 2-D profile of the object. CT was invented in 1972 by British electrical engineer Godfrey Hounsfield, and independently by Allan Cormack, a South African-born American physicist. The two inventors shared the 1979 Nobel Prize for Physiology or Medicine. Among diagnostic imaging techniques, CT has the decided advantage in having the sensitivity to image body parts on a wide range of densities, from soft tissue to blood vessels and bones.



Chapter 3

Section 3-1: Vector Algebra

Problem 3.1 Vector **A** starts at point $(1,-1,-3)$ and ends at point $(2,-1,0)$. Find a unit vector in the direction of **A**.

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{x}}(2-1)+\hat{\mathbf{y}}(-1-(-1))+\hat{\mathbf{z}}(0-(-3)) = \hat{\mathbf{x}}+\hat{\mathbf{z}}3, \\
|\mathbf{A}| &= \sqrt{1+9} = 3.16, \\
\hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}3}{3.16} = \hat{\mathbf{x}}0.32+\hat{\mathbf{z}}0.95.
\end{aligned}$$

Problem 3.2 Given vectors $\mathbf{A}=\hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}$, $\mathbf{B}=\hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3$, and $\mathbf{C}=\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2$, show that **C** is perpendicular to both **A** and **B**.

Solution:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{C} &= (\hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}})\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-6-2 = 0, \\
\mathbf{B}\cdot\mathbf{C} &= (\hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3)\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-2-6 = 0.
\end{aligned}$$

Problem 3.3 In Cartesian coordinates, the three corners of a triangle are $P_1(0,4,4)$, $P_2(4,-4,4)$, and $P_3(2,2,-4)$. Find the area of the triangle.

Solution: Let $\mathbf{B}=\overrightarrow{P_1P_2}=\hat{\mathbf{x}}4-\hat{\mathbf{y}}8$ and $\mathbf{C}=\overrightarrow{P_1P_3}=\hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}8$ represent two sides of the triangle. Since the magnitude of the cross product is the area of the parallelogram (see the definition of cross product in Section 3-1.4), half of this is the area of the triangle:

$$\begin{aligned}
A = \tfrac{1}{2}|\mathbf{B}\times\mathbf{C}| &= \tfrac{1}{2}|(\hat{\mathbf{x}}4-\hat{\mathbf{y}}8)\times(\hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}8)| \\
&= \tfrac{1}{2}|\hat{\mathbf{x}}(-8)(-8)+\hat{\mathbf{y}}(-(4)(-8))+\hat{\mathbf{z}}(4(-2)-(-8)2)| \\
&= \tfrac{1}{2}|\hat{\mathbf{x}}64+\hat{\mathbf{y}}32+\hat{\mathbf{z}}8| = \tfrac{1}{2}\sqrt{64^2+32^2+8^2} = \tfrac{1}{2}\sqrt{5184} = 36,
\end{aligned}$$

where the cross product is evaluated with Eq. (3.27).

Problem 3.4 Given $\mathbf{A}=\hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}1$ and $\mathbf{B}=\hat{\mathbf{x}}B_x+\hat{\mathbf{y}}2+\hat{\mathbf{z}}B_z$:

(a) find B_x and B_z if **A** is parallel to **B**;

(b) find a relation between B_x and B_z if **A** is perpendicular to **B**.

Solution:

(a) If **A** is parallel to **B**, then their directions are equal or opposite: $\hat{\mathbf{a}}_A = \pm\hat{\mathbf{a}}_B$, or

$$\mathbf{A}/|\mathbf{A}| = \pm\mathbf{B}/|\mathbf{B}|,$$
$$\frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}}{\sqrt{14}} = \pm\frac{\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}B_z}{\sqrt{4 + B_x^2 + B_z^2}}\,.$$

From the y-component,

$$\frac{-3}{\sqrt{14}} = \frac{\pm 2}{\sqrt{4 + B_x^2 + B_z^2}}$$

which can only be solved for the minus sign (which means that **A** and **B** must point in opposite directions for them to be parallel). Solving for $B_x^2 + B_z^2$,

$$B_x^2 + B_z^2 = \left(\frac{-2}{-3}\sqrt{14}\right)^2 - 4 = \frac{20}{9}\,.$$

From the x-component,

$$\frac{2}{\sqrt{14}} = \frac{-B_x}{\sqrt{56/9}}\,, \qquad B_x = \frac{-2\sqrt{56}}{3\sqrt{14}} = \frac{-4}{3}$$

and, from the z-component,

$$B_z = \frac{-2}{3}\,.$$

This is consistent with our result for $B_x^2 + B_z^2$.

These results could also have been obtained by assuming θ_{AB} was 0° or 180° and solving $|\mathbf{A}||\mathbf{B}| = \pm\mathbf{A}\cdot\mathbf{B}$, or by solving $\mathbf{A}\times\mathbf{B} = 0$.

(b) If **A** is perpendicular to **B**, then their dot product is zero (see Section 3-1.4). Using Eq. (3.17),

$$0 = \mathbf{A}\cdot\mathbf{B} = 2B_x - 6 + B_z,$$

or

$$B_z = 6 - 2B_x.$$

There are an infinite number of vectors which could be **B** and be perpendicular to **A**, but their x- and z-components must satisfy this relation.

This result could have also been obtained by assuming $\theta_{AB} = 90°$ and calculating $|\mathbf{A}||\mathbf{B}| = |\mathbf{A}\times\mathbf{B}|$.

Problem 3.5 Given vectors $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$, and $\mathbf{C} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$, find

(a) A and $\hat{\mathbf{a}}$,
(b) the component of **B** along **C**,
(c) θ_{AC},
(d) $\mathbf{A} \times \mathbf{C}$,
(e) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$,
(f) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$,
(g) $\hat{\mathbf{x}} \times \mathbf{B}$, and
(h) $(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}}$.

Solution:

(a) From Eq. (3.4),

$$A = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14},$$

and, from Eq. (3.5),

$$\hat{\mathbf{a}}_A = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3}{\sqrt{14}}.$$

(b) The component of **B** along **C** (see Section 3-1.4) is given by

$$B \cos\theta_{BC} = \frac{\mathbf{B} \cdot \mathbf{C}}{C} = \frac{-8}{\sqrt{20}} = -1.8.$$

(c) From Eq. (3.21),

$$\theta_{AC} = \cos^{-1} \frac{\mathbf{A} \cdot \mathbf{C}}{AC} = \cos^{-1} \frac{4+12}{\sqrt{14}\sqrt{20}} = \cos^{-1} \frac{16}{\sqrt{280}} = 17.0^\circ.$$

(d) From Eq. (3.27),

$$\mathbf{A} \times \mathbf{C} = \hat{\mathbf{x}}(2(-4) - (-3)2) + \hat{\mathbf{y}}((-3)0 - 1(-4)) + \hat{\mathbf{z}}(1(2) - 2(0)) = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}2.$$

(e) From Eq. (3.27) and Eq. (3.17),

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = 1(16) + 2(8) + (-3)4 = 20.$$

Eq. (3.30) could also have been used in the solution. Also, Eq. (3.29) could be used in conjunction with the result of part (d).

(f) By repeated application of Eq. (3.27),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = \hat{\mathbf{x}}32 - \hat{\mathbf{y}}52 - \hat{\mathbf{z}}24.$$

Eq. (3.33) could also have been used.

(g) From Eq. (3.27),

$$\hat{\mathbf{x}} \times \mathbf{B} = -\hat{\mathbf{z}}4.$$

(h) From Eq. (3.27) and Eq. (3.17),

$$(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}} = (\hat{\mathbf{x}}3 + \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 1.$$

Eq. (3.29) and Eq. (3.25) could also have been used in the solution.

Problem 3.6 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$ and $\mathbf{B} = \hat{\mathbf{x}}3 - \hat{\mathbf{z}}2$, find a vector $\mathbf{C}$ whose magnitude is 9 and whose direction is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

Solution: The cross product of two vectors produces a new vector which is perpendicular to both of the original vectors. Two vectors exist which have a magnitude of 9 and are orthogonal to both $\mathbf{A}$ and $\mathbf{B}$: one which is 9 units long in the direction of the unit vector parallel to $\mathbf{A} \times \mathbf{B}$, and one in the opposite direction.

$$\begin{aligned}\mathbf{C} = \pm 9 \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} &= \pm 9 \frac{(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \times (\hat{\mathbf{x}}3 - \hat{\mathbf{z}}2)}{|(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \times (\hat{\mathbf{x}}3 - \hat{\mathbf{z}}2)|} \\ &= \pm 9 \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}13 + \hat{\mathbf{z}}3}{\sqrt{2^2 + 13^2 + 3^2}} \approx \pm(\hat{\mathbf{x}}1.34 + \hat{\mathbf{y}}8.67 + \hat{\mathbf{z}}2.0).\end{aligned}$$

Problem 3.7 Given $\mathbf{A} = \hat{\mathbf{x}}(x + 2y) - \hat{\mathbf{y}}(y + 3z) + \hat{\mathbf{z}}(3x - y)$, determine a unit vector parallel to $\mathbf{A}$ at point $P(1, -1, 2)$.

Solution: The unit vector parallel to $\mathbf{A} = \hat{\mathbf{x}}(x + 2y) - \hat{\mathbf{y}}(y + 3z) + \hat{\mathbf{z}}(3x - y)$ at the point $P(1, -1, 2)$ is

$$\frac{\mathbf{A}(1, -1, 2)}{|\mathbf{A}(1, -1, 2)|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 + \hat{\mathbf{z}}4}{\sqrt{(-1)^2 + (-5)^2 + 4^2}} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 + \hat{\mathbf{z}}4}{\sqrt{42}} \approx -\hat{\mathbf{x}}0.15 - \hat{\mathbf{y}}0.77 + \hat{\mathbf{z}}0.62.$$

Problem 3.8 By expansion in Cartesian coordinates, prove:

(a) the relation for the scalar triple product given by (3.29), and

(b) the relation for the vector triple product given by (3.33).

Solution:

(a) Proof of the scalar triple product given by Eq. (3.29): From Eq. (3.27),

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x),$$

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= \hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x),\\ \mathbf{C}\times\mathbf{A} &= \hat{\mathbf{x}}(C_yA_z - C_zA_y) + \hat{\mathbf{y}}(C_zA_x - C_xA_z) + \hat{\mathbf{z}}(C_xA_y - C_yA_x).\end{aligned}$$

Employing Eq. (3.17), it is easily shown that

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x),\\ \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) &= B_x(C_yA_z - C_zA_y) + B_y(C_zA_x - C_xA_z) + B_z(C_xA_y - C_yA_x),\\ \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) &= C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z) + C_z(A_xB_y - A_yB_x),\end{aligned}$$

which are all the same.

(b) Proof of the vector triple product given by Eq. (3.33): The evaluation of the left hand side employs the expression above for $\mathbf{B}\times\mathbf{C}$ with Eq. (3.27):

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times(\hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x))\\ &= \hat{\mathbf{x}}(A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z))\\ &\quad + \hat{\mathbf{y}}(A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x))\\ &\quad + \hat{\mathbf{z}}(A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)),\end{aligned}$$

while the right hand side, evaluated with the aid of Eq. (3.17), is

$$\begin{aligned}\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)\\ &= \hat{\mathbf{x}}(B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z))\\ &\quad + \hat{\mathbf{y}}(B_y(A_xC_x + A_zC_z) - C_y(A_xB_x + A_zB_z))\\ &\quad + \hat{\mathbf{z}}(B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)).\end{aligned}$$

By rearranging the expressions for the components, the left hand side is equal to the right hand side.

Problem 3.9 Find an expression for the unit vector directed toward the origin from an arbitrary point on the line described by $x = 1$ and $z = 2$.

Solution: An arbitrary point on the given line is $(1, y, 2)$. The vector from this point to $(0, 0, 0)$ is:

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}(0-1) + \hat{\mathbf{y}}(0-y) + \hat{\mathbf{z}}(0-2) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}y - 2\hat{\mathbf{z}},\\ |\mathbf{A}| &= \sqrt{1+y^2+4} = \sqrt{5+y^2},\\ \hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}y - \hat{\mathbf{z}}2}{\sqrt{5+y^2}}.\end{aligned}$$

Problem 3.10 Find an expression for the unit vector directed toward the point P located on the z-axis at a height h above the x–y plane from an arbitrary point $Q(x, y, -3)$ in the plane $z = -3$.

Solution: Point P is at $(0, 0, h)$. Vector **A** from $Q(x, y, -3)$ to $P(0, 0, h)$ is:

$$\mathbf{A} = \hat{\mathbf{x}}(0 - x) + \hat{\mathbf{y}}(0 - y) + \hat{\mathbf{z}}(h + 3) = -\hat{\mathbf{x}}x - \hat{\mathbf{y}}y + \hat{\mathbf{z}}(h + 3),$$
$$|\mathbf{A}| = [x^2 + y^2 + (h + 3)^2]^{1/2},$$
$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{-\hat{\mathbf{x}}x - \hat{\mathbf{y}}y + \hat{\mathbf{z}}(h + 3)}{[x^2 + y^2 + (h + 3)^2]^{1/2}}.$$

Problem 3.11 Find a unit vector parallel to either direction of the line described by

$$2x + z = 4.$$

Solution: First, we find any two points on the given line. Since the line equation is not a function of y, the given line is in a plane parallel to the x–z plane. For convenience, we choose the x–z plane with $y = 0$.

For $x = 0$, $z = 4$. Hence, point P is at $(0, 0, 4)$.

For $z = 0$, $x = 2$. Hence, point Q is at $(2, 0, 0)$.

Vector **A** from P to Q is:

$$\mathbf{A} = \hat{\mathbf{x}}(2 - 0) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(0 - 4) = \hat{\mathbf{x}}2 - \hat{\mathbf{z}}4,$$
$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{z}}4}{\sqrt{20}}.$$

Problem 3.12 Two lines in the x–y plane are described by the expressions:

Line 1	$x + 2y = -6,$
Line 2	$3x + 4y = 8.$

Use vector algebra to find the smaller angle between the lines at their intersection point.

Solution: Intersection point is found by solving the two equations simultaneously:

$$-2x - 4y = 12,$$
$$3x + 4y = 8.$$

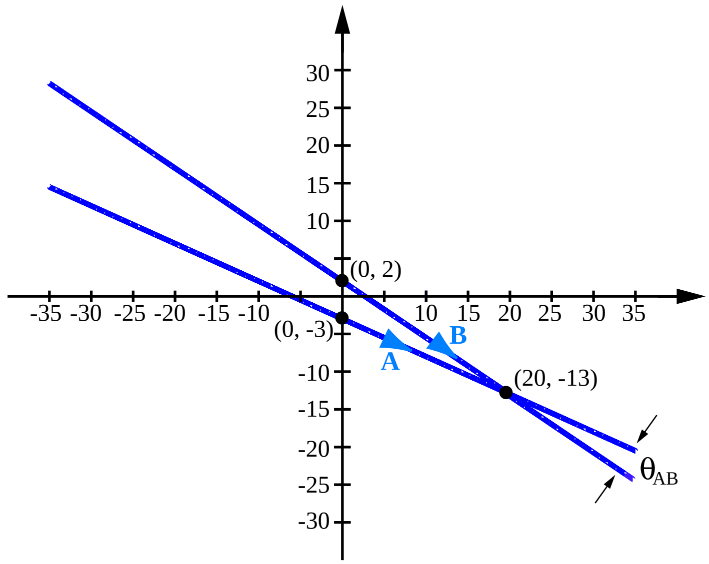


Figure P3.12: Lines 1 and 2.

The sum gives $x = 20$, which, when used in the first equation, gives $y = -13$.

Hence, intersection point is $(20, -13)$.

Another point on line 1 is $x = 0, \ y = -3$. Vector **A** from $(0, -3)$ to $(20, -13)$ is

$$\mathbf{A} = \hat{\mathbf{x}}(20) + \hat{\mathbf{y}}(-13+3) = \hat{\mathbf{x}}20 - \hat{\mathbf{y}}10,$$
$$|\mathbf{A}| = \sqrt{20^2 + 10^2} = \sqrt{500}.$$

A point on line 2 is $x = 0, \ y = 2$. Vector **B** from $(0, 2)$ to $(20, -13)$ is

$$\mathbf{B} = \hat{\mathbf{x}}(20) + \hat{\mathbf{y}}(-13-2) = \hat{\mathbf{x}}20 - \hat{\mathbf{y}}15,$$
$$|\mathbf{B}| = \sqrt{20^2 + 15^2} = \sqrt{625}.$$

Angle between **A** and **B** is

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = \cos^{-1}\left(\frac{400+150}{\sqrt{500}\cdot\sqrt{625}}\right) = 10.3^\circ.$$

Problem 3.13 A given line is described by

$$x + 2y = 4.$$

Vector **A** starts at the origin and ends at point P on the line such that **A** is orthogonal to the line. Find an expression for **A**.

Solution: We first plot the given line. Next we find vector **B** which connects point $P_1(0,2)$ to $P_2(4,0)$, both of which are on the line:

$$\mathbf{B} = \hat{\mathbf{x}}(4-0) + \hat{\mathbf{y}}(0-2) = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}2.$$

Vector **A** starts at the origin and ends on the line at P. If the x-coordinate of P is x,

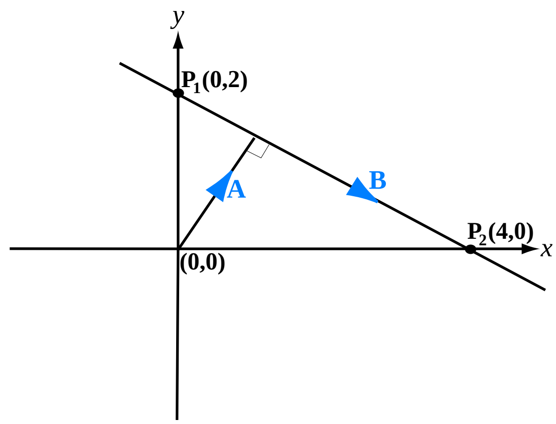


Figure P3.13: Given line and vector **A**.

then its y-coordinate has to be $(4-x)/2$ in order to be on the line. Hence P is at $(x,\, (4-x)/2)$. Vector **A** is

$$\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}\left(\frac{4-x}{2}\right).$$

But **A** is perpendicular to the line. Hence,

$$\mathbf{A}\cdot\mathbf{B} = 0,$$
$$\left[\hat{\mathbf{x}}x + \hat{\mathbf{y}}\left(\frac{4-x}{2}\right)\right]\cdot(\hat{\mathbf{x}}4 - \hat{\mathbf{y}}2) = 0,$$
$$4x - (4-x) = 0, \quad \text{or}$$
$$x = \frac{4}{5} = 0.8.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{x}}0.8 + \hat{\mathbf{y}}\left(\frac{4-0.8}{2}\right) = \hat{\mathbf{x}}0.8 + \hat{\mathbf{y}}1.6.$$

Problem 3.14 Show that, given two vectors **A** and **B**,

(a) the vector **C** defined as the vector component of **B** in the direction of **A** is given by

$$\mathbf{C} = \hat{\mathbf{a}}(\mathbf{B} \cdot \hat{\mathbf{a}}) = \frac{\mathbf{A}(\mathbf{B} \cdot \mathbf{A})}{|\mathbf{A}|^2},$$

where $\hat{\mathbf{a}}$ is the unit vector of **A**, and

(b) the vector **D** defined as the vector component of **B** perpendicular to **A** is given by

$$\mathbf{D} = \mathbf{B} - \frac{\mathbf{A}(\mathbf{B} \cdot \mathbf{A})}{|\mathbf{A}|^2}.$$

Solution:

(a) By definition, $\mathbf{B} \cdot \hat{\mathbf{a}}$ is the component of **B** along $\hat{\mathbf{a}}$. The vector component of $(\mathbf{B} \cdot \hat{\mathbf{a}})$ along **A** is

$$\mathbf{C} = \hat{\mathbf{a}}(\mathbf{B} \cdot \hat{\mathbf{a}}) = \frac{\mathbf{A}}{|\mathbf{A}|}\left(\mathbf{B} \cdot \frac{\mathbf{A}}{|\mathbf{A}|}\right) = \frac{\mathbf{A}(\mathbf{B} \cdot \mathbf{A})}{|\mathbf{A}|^2}.$$

(b) The figure shows vectors **A**, **B**, and **C**, where **C** is the projection of **B** along **A**. It is clear from the triangle that

$$\mathbf{B} = \mathbf{C} + \mathbf{D},$$

or

$$\mathbf{D} = \mathbf{B} - \mathbf{C} = \mathbf{B} - \frac{\mathbf{A}(\mathbf{B} \cdot \mathbf{A})}{|\mathbf{A}|^2}.$$

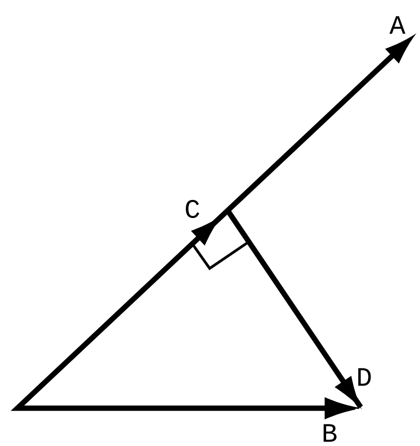


Figure P3.14: Relationships between vectors **A**, **B**, **C**, and **D**.

Problem 3.15 A certain plane is described by

$$2x+3y+4z=16.$$

Find the unit vector normal to the surface in the direction away from the origin.

Solution: Procedure:

1. Use the equation for the given plane to find three points, P_1, P_2 and P_3 on the plane.
2. Find vector **A** from P_1 to P_2 and vector **B** from P_1 to P_3.
3. Cross product of **A** and **B** gives a vector **C** orthogonal to **A** and **B**, and hence to the plane.
4. Check direction of $\hat{\mathbf{c}}$.

Steps:

1. Choose the following three points:

$$P_1 \text{ at } (0,0,4), \\ P_2 \text{ at } (8,0,0), \\ P_3 \text{ at } (0,\tfrac{16}{3},0).$$

2. Vector **A** from P_1 to P_2

$$\mathbf{A}=\hat{\mathbf{x}}(8-0)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}(0-4)=\hat{\mathbf{x}}8-\hat{\mathbf{z}}4$$

Vector **B** from P_1 to P_3

$$\mathbf{B}=\hat{\mathbf{x}}(0-0)+\hat{\mathbf{y}}\left(\frac{16}{3}-0\right)+\hat{\mathbf{z}}(0-4)=\hat{\mathbf{y}}\frac{16}{3}-\hat{\mathbf{z}}4$$

3.

$$\begin{aligned}\mathbf{C}&=\mathbf{A}\times\mathbf{B}\\ &=\hat{\mathbf{x}}(A_yB_z-A_zB_y)+\hat{\mathbf{y}}(A_zB_x-A_xB_z)+\hat{\mathbf{z}}(A_xB_y-A_yB_x)\\ &=\hat{\mathbf{x}}\left(0\cdot(-4)-(-4)\cdot\frac{16}{3}\right)+\hat{\mathbf{y}}\left((-4)\cdot 0-8\cdot(-4)\right)+\hat{\mathbf{z}}\left(8\cdot\frac{16}{3}-0\cdot 0\right)\\ &=\hat{\mathbf{x}}\frac{64}{3}+\hat{\mathbf{y}}32+\hat{\mathbf{z}}\frac{128}{3}\end{aligned}$$

Verify that **C** is orthogonal to **A** and **B**

$$\mathbf{A}\cdot\mathbf{C} = \left(8\cdot\frac{64}{3}\right) + (32\cdot 0) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$
$$\mathbf{B}\cdot\mathbf{C} = \left(0\cdot\frac{64}{3}\right) + \left(32\cdot\frac{16}{3}\right) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

4. $\mathbf{C} = \hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}$

$$\hat{\mathbf{c}} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}}{\sqrt{\left(\frac{64}{3}\right)^2 + 32^2 + \left(\frac{128}{3}\right)^2}} = \hat{\mathbf{x}}\,0.37 + \hat{\mathbf{y}}\,0.56 + \hat{\mathbf{z}}\,0.74.$$

$\hat{\mathbf{c}}$ points away from the origin as desired.

Problem 3.16 Given $\mathbf{B} = \hat{\mathbf{x}}(z-3y) + \hat{\mathbf{y}}(2x-3z) - \hat{\mathbf{z}}(x+y)$, find a unit vector parallel to **B** at point $P(1,0,-1)$.

Solution: At $P(1,0,-1)$,

$$\mathbf{B} = \hat{\mathbf{x}}(-1) + \hat{\mathbf{y}}(2+3) - \hat{\mathbf{z}}(1) = -\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}},$$
$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{1+25+1}} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{27}}.$$

Problem 3.17 When sketching or demonstrating the spatial variation of a vector field, we often use arrows, as in Fig. 3-25 (P3.17), wherein the length of the arrow is made to be proportional to the strength of the field and the direction of the arrow is the same as that of the field's. The sketch shown in Fig. P3.17, which represents the vector field $\mathbf{E} = \hat{\mathbf{r}}r$, consists of arrows pointing radially away from the origin and their lengths increase linearly in proportion to their distance away from the origin. Using this arrow representation, sketch each of the following vector fields:

(a) $\mathbf{E}_1 = -\hat{\mathbf{x}}y$,

(b) $\mathbf{E}_2 = \hat{\mathbf{y}}x$,

(c) $\mathbf{E}_3 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$,

(d) $\mathbf{E}_4 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}2y$,

(e) $\mathbf{E}_5 = \hat{\boldsymbol{\phi}}r$,

(f) $\mathbf{E}_6 = \hat{\mathbf{r}}\sin\phi$.

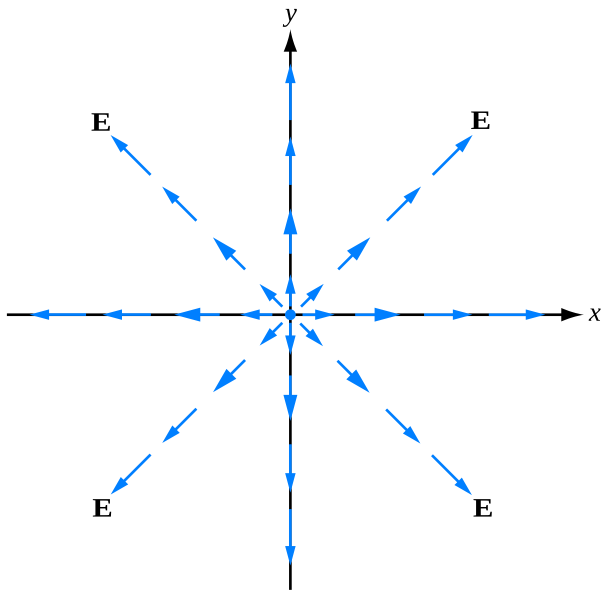
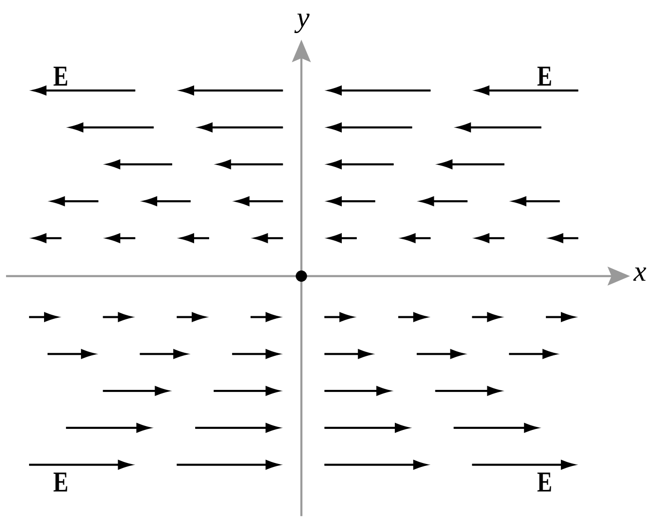


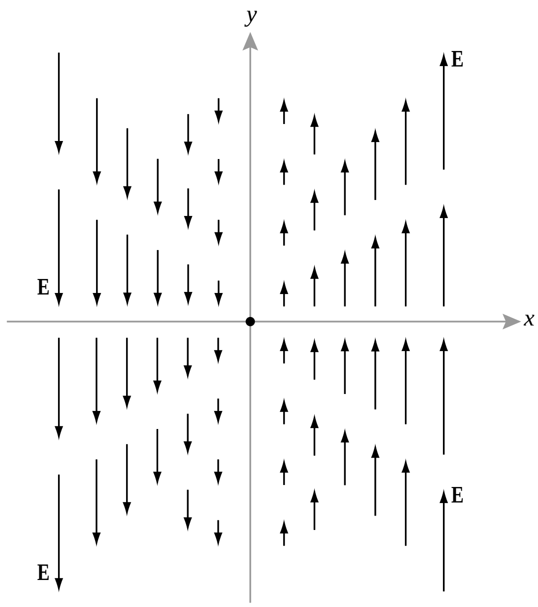
Figure P3.17: Arrow representation for vector field $\mathbf{E} = \hat{\mathbf{r}}r$ (Problem 3.17).

Solution:
(a)



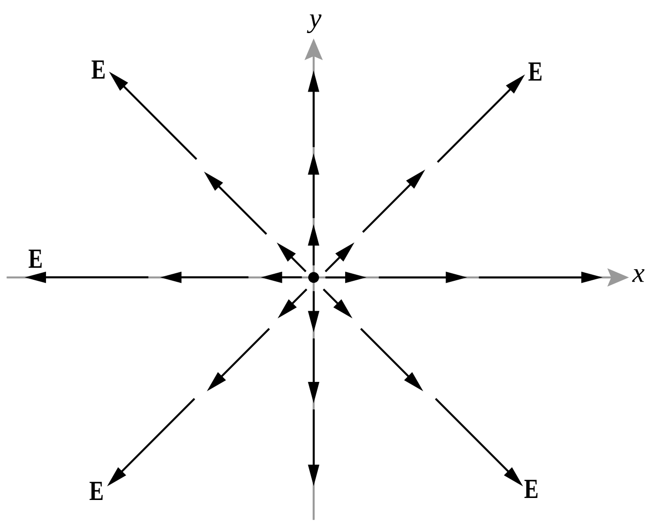
P2.13a: $\mathbf{E}_1 = -\hat{\mathbf{x}}\,y$

(b)



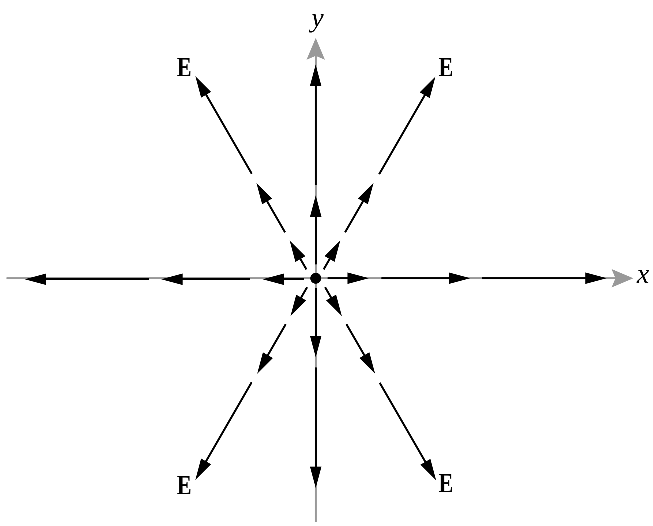
P3.17b: $\mathbf{E}_2 = \hat{\mathbf{y}}x$

(c)



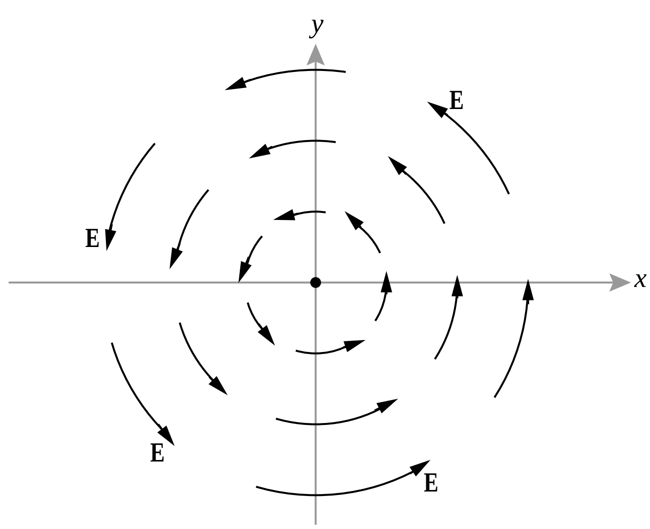
P2.13c: $\mathbf{E}_3 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$

(d)



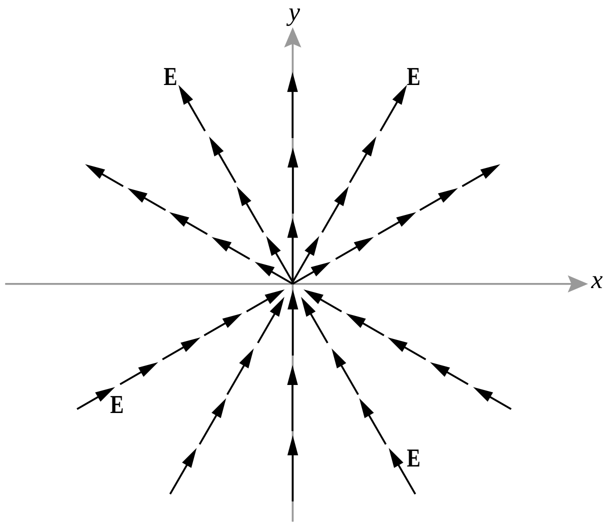
P2.13d: $\mathbf{E}_4 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}2y$

(e)



P2.13e: $\mathbf{E}_5 = \hat{\boldsymbol{\phi}}r$

(f)



P2.13f: $\mathbf{E}_6 = \hat{\mathbf{r}} \sin\phi$

Problem 3.18 Use arrows to sketch each of the following vector fields:

(a) $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$,

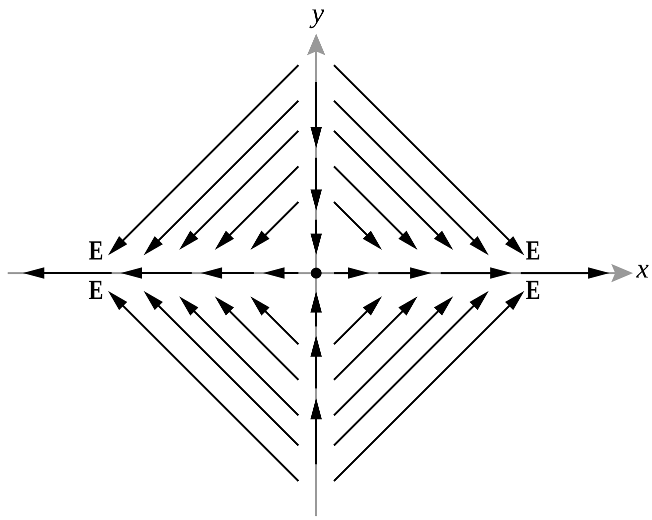
(b) $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$,

(c) $\mathbf{E}_3 = \hat{\mathbf{y}}\frac{1}{x}$,

(d) $\mathbf{E}_4 = \hat{\mathbf{r}} \cos\phi$.

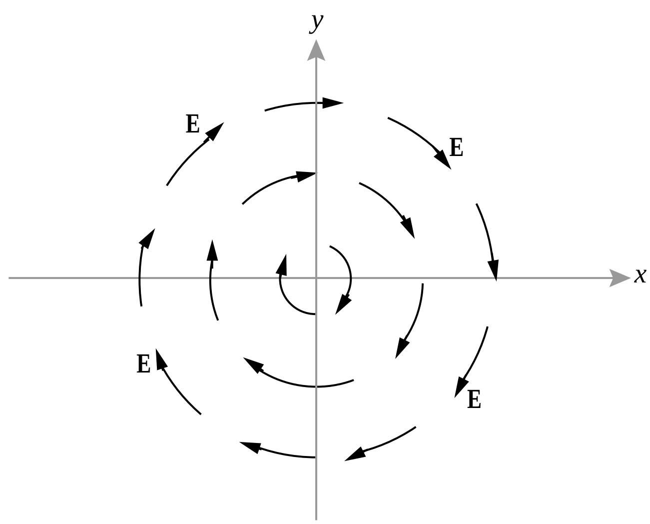
Solution:

(a)



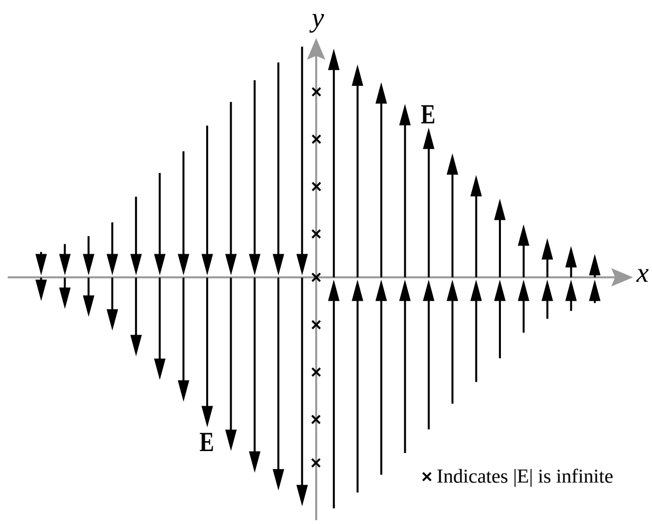
P2.14a: $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$

(b)



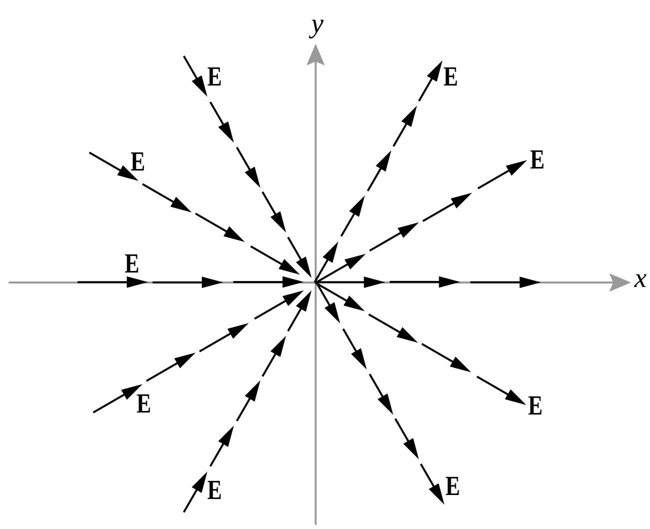
P2.14b: $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$

(c)



P2.14c: $\mathbf{E}_3 = \hat{\mathbf{y}}(1/x)$

(d)



P2.14d: $\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$

Sections 3-2 and 3-3: Coordinate Systems

Problem 3.19 Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates:
(a) $P_1(1,2,0)$,
(b) $P_2(0,0,2)$,
(c) $P_3(1,1,3)$,
(d) $P_4(-2,2,-2)$.

Solution: Use the "coordinate variables" column in Table 3-2.
(a) In the cylindrical coordinate system,

$$P_1 = (\sqrt{1^2+2^2}, \tan^{-1}(2/1), 0) = (\sqrt{5}, 1.107 \text{ rad}, 0) \approx (2.24, 63.4^\circ, 0).$$

In the spherical coordinate system,

$$\begin{aligned} P_1 &= (\sqrt{1^2+2^2+0^2}, \tan^{-1}(\sqrt{1^2+2^2}/0), \tan^{-1}(2/1)) \\ &= (\sqrt{5}, \pi/2 \text{ rad}, 1.107 \text{ rad}) \approx (2.24, 90.0^\circ, 63.4^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.
(b) In the cylindrical coordinate system,

$$P_2 = (\sqrt{0^2+0^2}, \tan^{-1}(0/0), 2) = (0, 0 \text{ rad}, 2) = (0, 0^\circ, 2).$$

In the spherical coordinate system,

$$\begin{aligned} P_2 &= (\sqrt{0^2+0^2+2^2}, \tan^{-1}(\sqrt{0^2+0^2}/2), \tan^{-1}(0/0)) \\ &= (2, 0 \text{ rad}, 0 \text{ rad}) = (2, 0^\circ, 0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is arbitrary and may take any value.
(c) In the cylindrical coordinate system,

$$P_3 = (\sqrt{1^2+1^2}, \tan^{-1}(1/1), 3) = (\sqrt{2}, \pi/4 \text{ rad}, 3) \approx (1.41, 45.0^\circ, 3).$$

In the spherical coordinate system,

$$\begin{aligned} P_3 &= (\sqrt{1^2+1^2+3^2}, \tan^{-1}(\sqrt{1^2+1^2}/3), \tan^{-1}(1/1)) \\ &= (\sqrt{11}, 0.44 \text{ rad}, \pi/4 \text{ rad}) \approx (3.32, 25.2^\circ, 45.0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(d) In the cylindrical coordinate system,

$$\begin{aligned} P_4 &= (\sqrt{(-2)^2+2^2}, \tan^{-1}(2/-2), -2) \\ &= (2\sqrt{2}, 3\pi/4 \text{ rad}, -2) \approx (2.83, 135.0^\circ, -2). \end{aligned}$$

In the spherical coordinate system,

$$\begin{aligned} P_4 &= (\sqrt{(-2)^2+2^2+(-2)^2}, \tan^{-1}(\sqrt{(-2)^2+2^2}/-2), \tan^{-1}(2/-2)) \\ &= (2\sqrt{3}, 2.187 \text{ rad}, 3\pi/4 \text{ rad}) \approx (3.46, 125.3^\circ, 135.0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant II.

Problem 3.20 Convert the coordinates of the following points from cylindrical to Cartesian coordinates:

(a) $P_1(2, \pi/4, -2)$,

(b) $P_2(3, 0, -2)$,

(c) $P_3(4, \pi, 3)$.

Solution:

(a)

$$P_1(x,y,z) = P_1(r\cos\phi, r\sin\phi, z) = P_1\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}, -2\right) = P_1(1.41, 1.41, -2).$$

(b) $P_2(x,y,z) = P_2(3\cos 0, 3\sin 0, -2) = P_2(3, 0, -2)$.

(c) $P_3(x,y,z) = P_3(4\cos\pi, 4\sin\pi, 3) = P_3(-4, 0, 3)$.

Problem 3.21 Convert the coordinates of the following points from spherical to cylindrical coordinates:

(a) $P_1(5, 0, 0)$,

(b) $P_2(5, 0, \pi)$,

(c) $P_3(3, \pi/2, 0)$.

Solution:

(a)

$$\begin{aligned} P_1(r,\phi,z) = P_1(R\sin\theta, \phi, R\cos\theta) &= P_1(5\sin 0, 0, 5\cos 0) \\ &= P_1(0, 0, 5). \end{aligned}$$

(b) $P_2(r,\phi,z) = P_2(5\sin 0, \pi, 5\cos 0) = P_2(0, \pi, 5)$.

(c) $P_3(r,\phi,z) = P_3(3\sin\frac{\pi}{2}, 0, 3\cos\frac{\pi}{2}) = P_3(3,0,0)$.

Problem 3.22 Use the appropriate expression for the differential surface area $d\mathbf{s}$ to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2$,
(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0$,
(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2$,
(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi$,
(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi$.

Also sketch the outlines of each of the surfaces.

Solution:

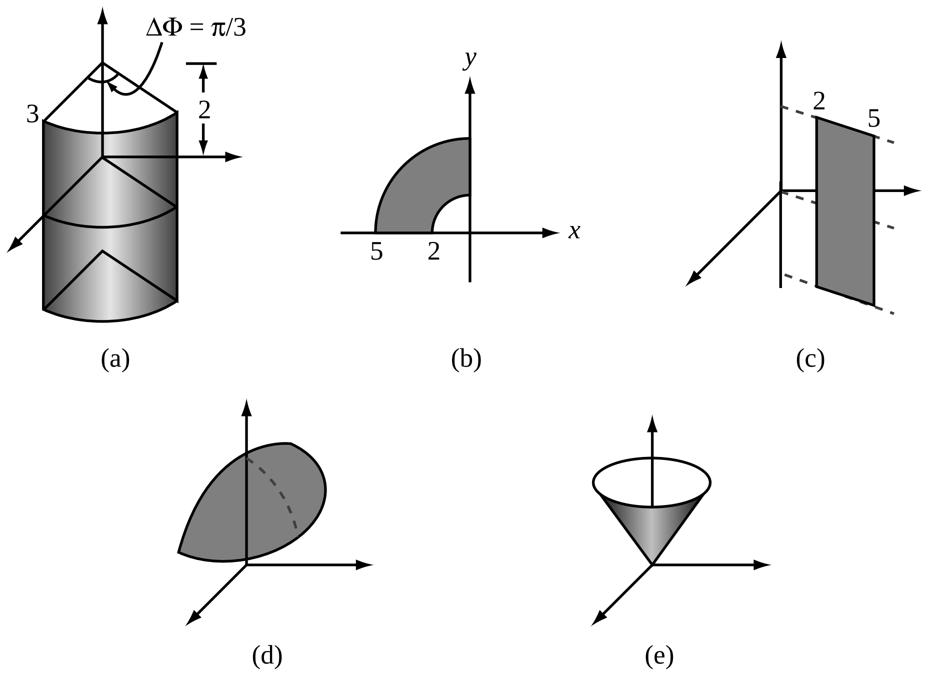


Figure P3.22: Surfaces described by Problem 3.22.

(a) Using Eq. (3.43a),

$$A = \int_{z=-2}^{2}\int_{\phi=0}^{\pi/3} (r)|_{r=3}\, d\phi\, dz = \left((3\phi z)|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq. (3.43c),

$$A = \int_{r=2}^{5} \int_{\phi=\pi/2}^{\pi} (r)|_{z=0} \, d\phi \, dr = \left(\left(\tfrac{1}{2} r^2 \phi \right) \Big|_{r=2}^{5} \right) \Big|_{\phi=\pi/2}^{\pi} = \frac{21\pi}{4}.$$

(c) Using Eq. (3.43b),

$$A = \int_{z=-2}^{2} \int_{r=2}^{5} (1)|_{\phi=\pi/4} \, dr \, dz = \left((rz)|_{z=-2}^{2} \right) \Big|_{r=2}^{5} = 12.$$

(d) Using Eq. (3.50b),

$$A = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{\pi} \left(R^2 \sin\theta \right) \Big|_{R=2} \, d\phi \, d\theta = \left((-4\phi \cos\theta)|_{\theta=0}^{\pi/3} \right) \Big|_{\phi=0}^{\pi} = 2\pi.$$

(e) Using Eq. (3.50c),

$$A = \int_{R=0}^{5} \int_{\phi=0}^{2\pi} (R \sin\theta)|_{\theta=\pi/3} \, d\phi \, dR = \left(\left(\tfrac{1}{2} R^2 \phi \sin \frac{\pi}{3} \right) \Big|_{\phi=0}^{2\pi} \right) \Big|_{R=0}^{5} = \frac{25\sqrt{3}\pi}{2}.$$

Problem 3.23 Find the volumes described by

(a) $2 \le r \le 5$; $\pi/2 \le \phi \le \pi$; $0 \le z \le 2$,

(b) $0 \le R \le 5$; $0 \le \theta \le \pi/3$; $0 \le \phi \le 2\pi$.

Also sketch the outline of each volume.

Solution:

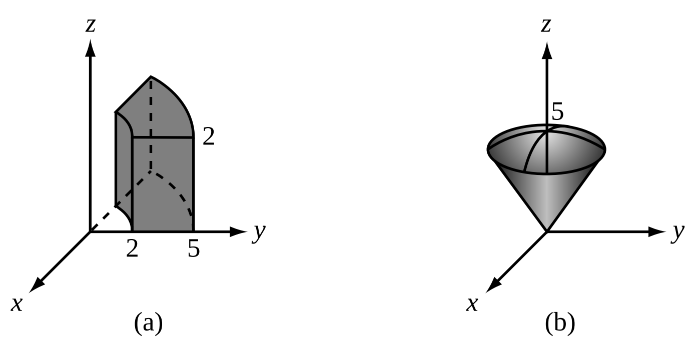


Figure P3.23: Volumes described by Problem 3.23 .

(a) From Eq. (3.44),

$$V = \int_{z=0}^{2} \int_{\phi=\pi/2}^{\pi} \int_{r=2}^{5} r \, dr \, d\phi \, dz = \left(\left(\left(\tfrac{1}{2} r^2 \phi z \right) \Big|_{r=2}^{5} \right) \Big|_{\phi=\pi/2}^{\pi} \right) \Big|_{z=0}^{2} = \frac{21\pi}{2}.$$

(b) From Eq. (3.50e),

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{R=0}^{5} R^2 \sin\theta \, dR \, d\theta \, d\phi$$
$$= \left(\left(\left(-\cos\theta \frac{R^3}{3} \phi \right) \Big|_{R=0}^{5} \right) \Big|_{\theta=0}^{\pi/3} \right) \Big|_{\phi=0}^{2\pi} = \frac{125\pi}{3}.$$

Problem 3.24 A section of a sphere is described by $0 \le R \le 2$, $0 \le \theta \le 90°$, and $30° \le \phi \le 90°$. Find:

(a) the surface area of the spherical section,

(b) the enclosed volume.

Also sketch the outline of the section.

Solution:

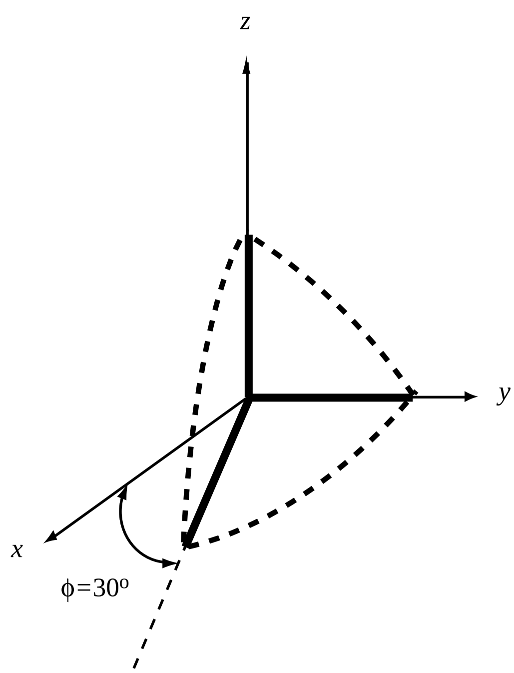


Figure P3.24: Outline of section.

$$S = \int_{\phi=\pi/6}^{\pi/2} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, d\theta \, d\phi |_{R=2}$$
$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \left[-\cos\theta|_0^{\pi/2}\right] = 4 \times \frac{\pi}{3} = \frac{4\pi}{3} \quad (\text{m}^2),$$
$$V = \int_{R=0}^{2} \int_{\phi=\pi/6}^{\pi/2} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, dR \, d\theta \, d\phi$$
$$= \left.\frac{R^3}{3}\right|_0^2 \left(\frac{\pi}{2} - \frac{\pi}{6}\right) [-\cos\theta|_0^{\pi/2}] = \frac{8}{3}\frac{\pi}{3} = \frac{8\pi}{9} \quad (\text{m}^3).$$

Problem 3.25 A vector field is given in cylindrical coordinates by

$$\mathbf{E} = \hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}} r\sin\phi + \hat{\mathbf{z}} z^2.$$

Point $P(2,\pi,3)$ is located on the surface of the cylinder described by $r = 2$. At point P, find:

(a) the vector component of **E** perpendicular to the cylinder,

(b) the vector component of **E** tangential to the cylinder.

Solution:

(a) $\mathbf{E}_\text{n} = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{E}) = \hat{\mathbf{r}}[\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}} r\sin\phi + \hat{\mathbf{z}} z^2)] = \hat{\mathbf{r}} r\cos\phi$.
At $P(2,\pi,3)$, $\mathbf{E}_\text{n} = \hat{\mathbf{r}} 2\cos\pi = -\hat{\mathbf{r}} 2$.
(b) $\mathbf{E}_\text{t} = \mathbf{E} - \mathbf{E}_\text{n} = \hat{\boldsymbol{\phi}} r\sin\phi + \hat{\mathbf{z}} z^2$.
At $P(2,\pi,3)$, $\mathbf{E}_\text{t} = \hat{\boldsymbol{\phi}} 2\sin\pi + \hat{\mathbf{z}} 3^2 = \hat{\mathbf{z}} 9$.

Problem 3.26 At a given point in space, vectors **A** and **B** are given in spherical coordinates by

$$\mathbf{A} = \hat{\mathbf{R}} 4 + \hat{\boldsymbol{\theta}} 2 - \hat{\boldsymbol{\phi}},$$
$$\mathbf{B} = -\hat{\mathbf{R}} 2 + \hat{\boldsymbol{\phi}} 3.$$

Find:

(a) the scalar component, or projection, of **B** in the direction of **A**,

(b) the vector component of **B** in the direction of **A**,

(c) the vector component of **B** perpendicular to **A**.

Solution:

(a) Scalar component of **B** in direction of **A**:

$$C = \mathbf{B}\cdot\hat{\mathbf{a}} = \mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|} = (-\hat{\mathbf{R}}2+\hat{\boldsymbol{\phi}}3)\cdot\frac{(\hat{\mathbf{R}}4+\hat{\boldsymbol{\theta}}2-\hat{\boldsymbol{\phi}})}{\sqrt{16+4+1}}$$
$$= \frac{-8-3}{\sqrt{21}} = -\frac{11}{\sqrt{21}} = -2.4.$$

(b) Vector component of **B** in direction of **A**:

$$\mathbf{C} = \hat{\mathbf{a}}C = \mathbf{A}\frac{C}{|\mathbf{A}|} = (\hat{\mathbf{R}}4+\hat{\boldsymbol{\theta}}2-\hat{\boldsymbol{\phi}})\,\frac{(-2.4)}{\sqrt{21}}$$
$$= -(\hat{\mathbf{R}}\,2.09+\hat{\boldsymbol{\theta}}\,1.05-\hat{\boldsymbol{\phi}}\,0.52).$$

(c) Vector component of **B** perpendicular to **A**:

$$\mathbf{D} = \mathbf{B}-\mathbf{C} = (-\hat{\mathbf{R}}2+\hat{\boldsymbol{\phi}}3)+(\hat{\mathbf{R}}\,2.09+\hat{\boldsymbol{\theta}}\,1.05-\hat{\boldsymbol{\phi}}\,0.52)$$
$$= \hat{\mathbf{R}}\,0.09+\hat{\boldsymbol{\theta}}\,1.05+\hat{\boldsymbol{\phi}}\,2.48.$$

Problem 3.27 Given vectors

$$\mathbf{A} = \hat{\mathbf{r}}(\cos\phi+3z)-\hat{\boldsymbol{\phi}}(2r+4\sin\phi)+\hat{\mathbf{z}}(r-2z),$$
$$\mathbf{B} = -\hat{\mathbf{r}}\sin\phi+\hat{\mathbf{z}}\cos\phi,$$

find

(a) θ_{AB} at $(2,\pi/2,0)$,

(b) a unit vector perpendicular to both **A** and **B** at $(2,\pi/3,1)$.

Solution: It doesn't matter whether the vectors are evaluated before vector products are calculated, or if the vector products are directly calculated and the general results are evaluated at the specific point in question.

(a) At $(2,\pi/2,0)$, $\mathbf{A} = -\hat{\boldsymbol{\phi}}8+\hat{\mathbf{z}}2$ and $\mathbf{B} = -\hat{\mathbf{r}}$. From Eq. (3.21),

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{0}{AB}\right) = 90^\circ.$$

(b) At $(2,\pi/3,1)$, $\mathbf{A} = \hat{\mathbf{r}}\frac{7}{2}-\hat{\boldsymbol{\phi}}4(1+\frac{1}{2}\sqrt{3})$ and $\mathbf{B} = -\hat{\mathbf{r}}\frac{1}{2}\sqrt{3}+\hat{\mathbf{z}}\frac{1}{2}$. Since $\mathbf{A}\times\mathbf{B}$ is perpendicular to both **A** and **B**, a unit vector perpendicular to both **A** and **B** is given by

$$\pm\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \pm\frac{\hat{\mathbf{r}}(-4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2})-\hat{\boldsymbol{\phi}}(\frac{7}{2})(\frac{1}{2})-\hat{\mathbf{z}}(4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2}\sqrt{3})}{\sqrt{(2(1+\frac{1}{2}\sqrt{3}))^2+(\frac{7}{4})^2+(3+2\sqrt{3})^2}}$$
$$\approx \mp(\hat{\mathbf{r}}0.487+\hat{\boldsymbol{\phi}}\,0.228+\hat{\mathbf{z}}\,0.843).$$

Problem 3.28 Find the distance between the following pairs of points:

(a) $P_1(1,2,3)$ and $P_2(-2,-3,-2)$ in Cartesian coordinates,

(b) $P_3(1,\pi/4,3)$ and $P_4(3,\pi/4,4)$ in cylindrical coordinates,

(c) $P_5(4,\pi/2,0)$ and $P_6(3,\pi,0)$ in spherical coordinates.

Solution:

(a)

$$d = [(-2-1)^2 + (-3-2)^2 + (-2-3)^2]^{1/2} = [9+25+25]^{1/2} = \sqrt{59} = 7.68.$$

(b)

$$\begin{aligned} d &= [r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \\ &= \left[9 + 1 - 2\times 3\times 1\times \cos\left(\frac{\pi}{4} - \frac{\pi}{4}\right) + (4-3)^2\right]^{1/2} \\ &= (10 - 6 + 1)^{1/2} = 5^{1/2} = 2.24. \end{aligned}$$

(c)

$$\begin{aligned} d &= \{R_2^2 + R_1^2 - 2R_1R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\}^{1/2} \\ &= \left\{9 + 16 - 2\times 3\times 4\left[\cos\pi \cos\frac{\pi}{2} + \sin\frac{\pi}{2} \sin\pi \cos(0-0)\right]\right\}^{1/2} \\ &= \{9 + 16 - 0\}^{1/2} = \sqrt{25} = 5. \end{aligned}$$

Problem 3.29 Determine the distance between the following pairs of points:

(a) $P_1(1,1,2)$ and $P_2(0,2,3)$,

(b) $P_3(2,\pi/3,1)$ and $P_4(4,\pi/2,3)$,

(c) $P_5(3,\pi,\pi/2)$ and $P_6(4,\pi/2,\pi)$.

Solution:

(a) From Eq. (3.66),

$$d = \sqrt{(0-1)^2 + (2-1)^2 + (3-2)^2} = \sqrt{3}.$$

(b) From Eq. (3.67),

$$d = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + (3-1)^2} = \sqrt{24 - 8\sqrt{3}} \approx 3.18.$$

(c) From Eq. (3.68),

$$d = \sqrt{3^2 + 4^2 - 2(3)(4)\left(\cos\frac{\pi}{2}\cos\pi + \sin\pi\sin\frac{\pi}{2}\cos\left(\pi - \frac{\pi}{2}\right)\right)} = 5.$$

Problem 3.30 Transform the following vectors into cylindrical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}(x+y)$ at $P_1(1,2,3)$,
(b) $\mathbf{B} = \hat{\mathbf{x}}(y-x) + \hat{\mathbf{y}}(x-y)$ at $P_2(1,0,2)$,
(c) $\mathbf{C} = \hat{\mathbf{x}}y^2/(x^2+y^2) - \hat{\mathbf{y}}x^2/(x^2+y^2) + \hat{\mathbf{z}}4$ at $P_3(1,-1,2)$,
(d) $\mathbf{D} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta + \hat{\boldsymbol{\phi}}\cos^2\phi$ at $P_4(2,\pi/2,\pi/4)$,
(e) $\mathbf{E} = \hat{\mathbf{R}}\cos\phi + \hat{\boldsymbol{\theta}}\sin\phi + \hat{\boldsymbol{\phi}}\sin^2\theta$ at $P_5(3,\pi/2,\pi)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned}
\mathbf{A} &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(r\cos\phi + r\sin\phi) \\
&= \hat{\mathbf{r}}r\cos\phi(\cos\phi + \sin\phi) - \hat{\boldsymbol{\phi}}r\sin\phi(\cos\phi + \sin\phi), \\
P_1 &= (\sqrt{1^2+2^2}, \tan^{-1}(2/1), 3) = (\sqrt{5}, 63.4^\circ, 3), \\
\mathbf{A}(P_1) &= (\hat{\mathbf{r}}0.447 - \hat{\boldsymbol{\phi}}0.894)\sqrt{5}(.447 + .894) = \hat{\mathbf{r}}1.34 - \hat{\boldsymbol{\phi}}2.68.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B} &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(r\sin\phi - r\cos\phi) + (\hat{\boldsymbol{\phi}}\cos\phi + \hat{\mathbf{r}}\sin\phi)(r\cos\phi - r\sin\phi) \\
&= \hat{\mathbf{r}}r(2\sin\phi\cos\phi - 1) + \hat{\boldsymbol{\phi}}r(\cos^2\phi - \sin^2\phi) = \hat{\mathbf{r}}r(\sin 2\phi - 1) + \hat{\boldsymbol{\phi}}r\cos 2\phi, \\
P_2 &= (\sqrt{1^2+0^2}, \tan^{-1}(0/1), 2) = (1, 0^\circ, 2), \\
\mathbf{B}(P_2) &= -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{C} &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\frac{r^2\sin^2\phi}{r^2} - (\hat{\boldsymbol{\phi}}\cos\phi + \hat{\mathbf{r}}\sin\phi)\frac{r^2\cos^2\phi}{r^2} + \hat{\mathbf{z}}4 \\
&= \hat{\mathbf{r}}\sin\phi\cos\phi(\sin\phi - \cos\phi) - \hat{\boldsymbol{\phi}}(\sin^3\phi + \cos^3\phi) + \hat{\mathbf{z}}4, \\
P_3 &= (\sqrt{1^2+(-1)^2}, \tan^{-1}(-1/1), 2) = (\sqrt{2}, -45^\circ, 2), \\
\mathbf{C}(P_3) &= \hat{\mathbf{r}}0.707 + \hat{\mathbf{z}}4.
\end{aligned}$$

(d)

$$\mathbf{D} = (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\sin\theta + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\cos\theta + \hat{\boldsymbol{\phi}}\cos^2\phi = \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\cos^2\phi,$$

$$P_4 = (2\sin(\pi/2), \pi/4, 2\cos(\pi/2)) = (2, 45°, 0),$$
$$\mathbf{D}(P_4) = \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\tfrac{1}{2}.$$

(e)

$$\mathbf{E} = (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\cos\phi + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\sin\phi + \hat{\boldsymbol{\phi}}\sin^2\theta,$$
$$P_5 = \left(3, \frac{\pi}{2}, \pi\right),$$
$$\mathbf{E}(P_5) = \left(\hat{\mathbf{r}}\sin\frac{\pi}{2} + \hat{\mathbf{z}}\cos\frac{\pi}{2}\right)\cos\pi + \left(\hat{\mathbf{r}}\cos\frac{\pi}{2} - \hat{\mathbf{z}}\sin\frac{\pi}{2}\right)\sin\pi + \hat{\boldsymbol{\phi}}\sin^2\frac{\pi}{2} = -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}.$$

Problem 3.31 Transform the following vectors into spherical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}xz + \hat{\mathbf{z}}4$ at $P_1(1, -1, 2)$,
(b) $\mathbf{B} = \hat{\mathbf{y}}(x^2 + y^2 + z^2) - \hat{\mathbf{z}}(x^2 + y^2)$ at $P_2(-1, 0, 2)$,
(c) $\mathbf{C} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + \hat{\mathbf{z}}\cos\phi\sin\phi$ at $P_3(2, \pi/4, 2)$, and
(d) $\mathbf{D} = \hat{\mathbf{x}}y^2/(x^2 + y^2) - \hat{\mathbf{y}}x^2/(x^2 + y^2) + \hat{\mathbf{z}}4$ at $P_4(1, -1, 2)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned}
\mathbf{A} &= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(R\sin\theta\sin\phi)^2 \\
&\quad + (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)(R\sin\theta\cos\phi)(R\cos\theta) \\
&\quad + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4 \\
&= \hat{\mathbf{R}}(R^2\sin^2\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) + 4\cos\theta) \\
&\quad + \hat{\boldsymbol{\theta}}(R^2\sin\theta\cos\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) - 4\sin\theta) \\
&\quad + \hat{\boldsymbol{\phi}}R^2\sin\theta(\cos\theta\cos^2\phi - \sin\theta\sin^3\phi), \\
P_1 &= \left(\sqrt{1^2 + (-1)^2 + 2^2}, \tan^{-1}\left(\sqrt{1^2 + (-1)^2}/2\right), \tan^{-1}(-1/1)\right) \\
&= (\sqrt{6}, 35.3°, -45°), \\
\mathbf{A}(P_1) &\approx \hat{\mathbf{R}}2.856 - \hat{\boldsymbol{\theta}}2.888 + \hat{\boldsymbol{\phi}}2.123.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B} &= (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)R^2 - (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)R^2\sin^2\theta \\
&= \hat{\mathbf{R}}R^2\sin\theta(\sin\phi - \sin\theta\cos\theta) + \hat{\boldsymbol{\theta}}R^2(\cos\theta\sin\phi + \sin^3\theta) + \hat{\boldsymbol{\phi}}R^2\cos\phi, \\
P_2 &= \left(\sqrt{(-1)^2 + 0^2 + 2^2}, \tan^{-1}\left(\sqrt{(-1)^2 + 0^2}/2\right), \tan^{-1}(0/(-1))\right) \\
&= (\sqrt{5}, 26.6°, 180°),
\end{aligned}$$

$$\mathbf{B}(P_2) \approx -\hat{\mathbf{R}}0.896 + \hat{\boldsymbol{\theta}}0.449 - \hat{\boldsymbol{\phi}}5.$$

(c)

$$\begin{aligned}
\mathbf{C} &= (\hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\cos\phi\sin\phi \\
&= \hat{\mathbf{R}}\cos\phi(\sin\theta + \cos\theta\sin\phi) + \hat{\boldsymbol{\theta}}\cos\phi(\cos\theta - \sin\theta\sin\phi) - \hat{\boldsymbol{\phi}}\sin\phi, \\
P_3 &= \left(\sqrt{2^2+2^2}, \tan^{-1}(2/2), \pi/4\right) = (2\sqrt{2}, 45^\circ, 45^\circ), \\
\mathbf{C}(P_3) &\approx \hat{\mathbf{R}}0.854 + \hat{\boldsymbol{\theta}}0.146 - \hat{\boldsymbol{\phi}}0.707.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{D} &= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\,\frac{R^2\sin^2\theta\sin^2\phi}{R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi} \\
&\quad - (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\,\frac{R^2\sin^2\theta\cos^2\phi}{R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi} \\
&\quad + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4 \\
&= \hat{\mathbf{R}}(\sin\theta\cos\phi\sin^2\phi - \sin\theta\sin\phi\cos^2\phi + 4\cos\theta) \\
&\quad + \hat{\boldsymbol{\theta}}(\cos\theta\cos\phi\sin^2\phi - \cos\theta\sin\phi\cos^2\phi - 4\sin\theta) \\
&\quad - \hat{\boldsymbol{\phi}}(\cos^3\phi + \sin^3\phi),
\end{aligned}$$

$$\begin{aligned}
P_4(1,-1,2) &= P_4\left[\sqrt{1+1+4}, \tan^{-1}(\sqrt{1+1}/2), \tan^{-1}(-1/1)\right] \\
&= P_4(\sqrt{6}, 35.26^\circ, -45^\circ),
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}(P_4) &= \hat{\mathbf{R}}(\sin 35.26^\circ\cos 45^\circ\sin^2 45^\circ - \sin 35.26^\circ\sin(-45^\circ)\cos^2 45^\circ + 4\cos 35.26^\circ) \\
&\quad + \hat{\boldsymbol{\theta}}(\cos 35.26^\circ\cos 45^\circ\sin^2 45^\circ - \cos 35.26^\circ\sin(-45^\circ)\cos^2 45^\circ - 4\sin 35.26^\circ) \\
&\quad - \hat{\boldsymbol{\phi}}(\cos^3 45^\circ + \sin^3 45^\circ) \\
&= \hat{\mathbf{R}}3.67 - \hat{\boldsymbol{\theta}}1.73 - \hat{\boldsymbol{\phi}}0.707.
\end{aligned}$$

Sections 3-4 to 3-7: Gradient, Divergence, and Curl Operators

Problem 3.32 Find the gradient of the following scalar functions:

(a) $T = 3/(x^2+z^2)$,

(b) $V = xy^2z^4$,

(c) $U = z\cos\phi/(1+r^2)$,
(d) $W = e^{-R}\sin\theta$,
(e) $S = 4x^2e^{-z}+y^3$,
(f) $N = r^2\cos^2\phi$,
(g) $M = R\cos\theta\sin\phi$.

Solution:

(a) From Eq. (3.72),

$$\nabla T = -\hat{\mathbf{x}}\frac{6x}{(x^2+z^2)^2} - \hat{\mathbf{z}}\frac{6z}{(x^2+z^2)^2}\,.$$

(b) From Eq. (3.72),

$$\nabla V = \hat{\mathbf{x}}y^2z^4 + \hat{\mathbf{y}}2xyz^4 + \hat{\mathbf{z}}4xy^2z^3.$$

(c) From Eq. (3.82),

$$\nabla U = -\hat{\mathbf{r}}\frac{2rz\cos\phi}{(1+r^2)^2} - \hat{\boldsymbol{\phi}}\frac{z\sin\phi}{r(1+r^2)} + \hat{\mathbf{z}}\frac{\cos\phi}{1+r^2}\,.$$

(d) From Eq. (3.83),

$$\nabla W = -\hat{\mathbf{R}}e^{-R}\sin\theta + \hat{\boldsymbol{\theta}}(e^{-R}/R)\cos\theta.$$

(e) From Eq. (3.72),

$$S = 4x^2e^{-z}+y^3,$$
$$\nabla S = \hat{\mathbf{x}}\frac{\partial S}{\partial x} + \hat{\mathbf{y}}\frac{\partial S}{\partial y} + \hat{\mathbf{z}}\frac{\partial S}{\partial z} = \hat{\mathbf{x}}8xe^{-z} + \hat{\mathbf{y}}3y^2 - \hat{\mathbf{z}}4x^2e^{-z}.$$

(f) From Eq. (3.82),

$$N = r^2\cos^2\phi,$$
$$\nabla N = \hat{\mathbf{r}}\frac{\partial N}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial N}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial N}{\partial z} = \hat{\mathbf{r}}2r\cos^2\phi - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi.$$

(g) From Eq. (3.83),

$$M = R\cos\theta\sin\phi,$$
$$\nabla M = \hat{\mathbf{R}}\frac{\partial M}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial M}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial M}{\partial \phi} = \hat{\mathbf{R}}\cos\theta\sin\phi - \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{\tan\theta}\,.$$

Problem 3.33 The gradient of a scalar function T is given by

$$\nabla T = \hat{\mathbf{z}} e^{-3z}.$$

If $T = 10$ at $z = 0$, find $T(z)$.

Solution:

$$\nabla T = \hat{\mathbf{z}} e^{-3z}.$$

By choosing P_1 at $z = 0$ and P_2 at any point z, (3.76) becomes

$$\begin{aligned} T(z) - T(0) = \int_0^z \nabla T \cdot d\mathbf{l}' &= \int_0^z \hat{\mathbf{z}} e^{-3z'} \cdot (\hat{\mathbf{x}}\, dx' + \hat{\mathbf{y}}\, dy' + \hat{\mathbf{z}}\, dz') \\ &= \int_0^z e^{-3z'}\, dz' = -\frac{-e^{-3z'}}{3}\Bigg|_0^z = \frac{1}{3}(1 - e^{-3z}). \end{aligned}$$

Hence,

$$T(z) = T(0) + \frac{1}{3}(1 - e^{-3z}) = 10 + \frac{1}{3}(1 - e^{-3z}).$$

Problem 3.34 Follow a procedure similar to that leading to Eq. (3.82) to derive the expression given by Eq. (3.83) for ∇ in spherical coordinates.

Solution: From the chain rule and Table 3-2,

$$\begin{aligned} \nabla T &= \hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \\ &= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} \right) \\ &\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R}\frac{\partial R}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial y} \right) \\ &\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R}\frac{\partial R}{\partial z} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial z} \right) \\ &= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} + \frac{\partial T}{\partial \theta}\frac{\partial}{\partial x}\tan^{-1}\left(\sqrt{x^2+y^2}/z\right) + \frac{\partial T}{\partial \phi}\frac{\partial}{\partial x}\tan^{-1}(y/x) \right) \\ &\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R}\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} + \frac{\partial T}{\partial \theta}\frac{\partial}{\partial y}\tan^{-1}\left(\sqrt{x^2+y^2}/z\right) + \frac{\partial T}{\partial \phi}\frac{\partial}{\partial y}\tan^{-1}(y/x) \right) \\ &\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R}\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2} + \frac{\partial T}{\partial \theta}\frac{\partial}{\partial z}\tan^{-1}\left(\sqrt{x^2+y^2}/z\right) + \frac{\partial T}{\partial \phi}\frac{\partial}{\partial z}\tan^{-1}(y/x) \right) \end{aligned}$$

$$
\begin{aligned}
&= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{\partial T}{\partial \theta}\frac{z}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial T}{\partial \phi}\frac{-y}{x^2+y^2}\right) \\
&\quad + \hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial T}{\partial \theta}\frac{z}{x^2+y^2+z^2}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial T}{\partial \phi}\frac{x}{x^2+y^2}\right) \\
&\quad + \hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\frac{z}{\sqrt{x^2+y^2+z^2}} + \frac{\partial T}{\partial \theta}\frac{-1}{x^2+y^2+z^2}\sqrt{x^2+y^2} + \frac{\partial T}{\partial \phi}0\right) \\
&= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\frac{R\sin\theta\cos\phi}{R} + \frac{\partial T}{\partial \theta}\frac{R\cos\theta}{R^2}\frac{R\sin\theta\cos\phi}{R\sin\theta} + \frac{\partial T}{\partial \phi}\frac{-R\sin\theta\sin\phi}{R^2\sin^2\theta}\right) \\
&\quad + \hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\frac{R\sin\theta\sin\phi}{R} + \frac{\partial T}{\partial \theta}\frac{R\cos\theta}{R^2}\frac{R\sin\theta\sin\phi}{R\sin\theta} + \frac{\partial T}{\partial \phi}\frac{R\sin\theta\cos\phi}{R^2\sin^2\theta}\right) \\
&\quad + \hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\frac{R\cos\theta}{R} + \frac{\partial T}{\partial \theta}\frac{-R\sin\theta}{R^2}\right) \\
&= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\sin\theta\cos\phi + \frac{\partial T}{\partial \theta}\frac{\cos\theta\cos\phi}{R} + \frac{\partial T}{\partial \phi}\frac{-\sin\phi}{R\sin\theta}\right) \\
&\quad + \hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\sin\theta\sin\phi + \frac{\partial T}{\partial \theta}\frac{\cos\theta\sin\phi}{R} + \frac{\partial T}{\partial \phi}\frac{\cos\phi}{R\sin\theta}\right) \\
&\quad + \hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\cos\theta + \frac{\partial T}{\partial \theta}\frac{-\sin\theta}{R}\right) \\
&= (\hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta)\frac{\partial T}{\partial R} \\
&\quad + (\hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta)\frac{1}{R}\frac{\partial T}{\partial \theta} \\
&\quad + (-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi)\frac{1}{R\sin\theta}\frac{\partial T}{\partial \phi} \\
&= \hat{\mathbf{R}}\frac{\partial T}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial T}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial T}{\partial \phi},
\end{aligned}
$$

which is Eq. (3.83).

Problem 3.35 For the scalar function $V = xy^2 - z^2$, determine its directional derivative along the direction of vector $\mathbf{A} = (\hat{\mathbf{x}} - \hat{\mathbf{y}}z)$ and then evaluate it at $P(1,-1,4)$.

Solution: The directional derivative is given by Eq. (3.75) as $dV/dl = \nabla V \cdot \hat{\mathbf{a}}_l$, where the unit vector in the direction of $\mathbf{A}$ is given by Eq. (3.2):

$$\hat{\mathbf{a}}_l = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}z}{\sqrt{1+z^2}},$$

and the gradient of V in Cartesian coordinates is given by Eq. (3.72):

$$\nabla V = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}2xy - \hat{\mathbf{z}}2z.$$

Therefore, by Eq. (3.75),

$$\frac{dV}{dl} = \frac{y^2 - 2xyz}{\sqrt{1+z^2}}.$$

At $P(1,-1,4)$,

$$\left(\frac{dV}{dl}\right)\bigg|_{(1,-1,4)} = \frac{9}{\sqrt{17}} = 2.18.$$

Problem 3.36 For the scalar function $T = \frac{1}{2}e^{-r/5}\cos\phi$, determine its directional derivative along the radial direction $\hat{\mathbf{r}}$ and then evaluate it at $P(2,\pi/4,3)$.

Solution:

$$T = \frac{1}{2}e^{-r/5}\cos\phi,$$

$$\nabla T = \hat{\mathbf{r}}\frac{\partial T}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} = -\hat{\mathbf{r}}\frac{e^{-r/5}\cos\phi}{10} - \hat{\boldsymbol{\phi}}\frac{e^{-r/5}\sin\phi}{2r},$$

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{r}} = -\frac{e^{-r/5}\cos\phi}{10},$$

$$\frac{dT}{dl}\bigg|_{(2,\pi/4,3)} = -\frac{e^{-2/5}\cos\frac{\pi}{4}}{10} = -4.74\times 10^{-2}.$$

Problem 3.37 For the scalar function $U = \frac{1}{R}\sin^2\theta$, determine its directional derivative along the range direction $\hat{\mathbf{R}}$ and then evaluate it at $P(5,\pi/4,\pi/2)$.

Solution:

$$U = \frac{1}{R}\sin^2\theta,$$

$$\nabla U = \hat{\mathbf{R}}\frac{\partial U}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial U}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial U}{\partial \phi} = -\hat{\mathbf{R}}\frac{\sin^2\theta}{R^2} - \hat{\boldsymbol{\theta}}\frac{2\sin\theta\cos\theta}{R},$$

$$\frac{dU}{dl} = \nabla U \cdot \hat{\mathbf{R}} = -\frac{\sin^2\theta}{R^2},$$

$$\frac{dU}{dl}\bigg|_{(5,\pi/4,\pi/2)} = -\frac{\sin^2(\pi/4)}{25} = -0.02.$$

Problem 3.38 Vector field $\mathbf{E}$ is characterized by the following properties: (a) $\mathbf{E}$ points along $\hat{\mathbf{R}}$, (b) the magnitude of $\mathbf{E}$ is a function of only the distance from the origin, (c) $\mathbf{E}$ vanishes at the origin, and (d) $\nabla \cdot \mathbf{E} = 12$, everywhere. Find an expression for $\mathbf{E}$ that satisfies these properties.

Solution: According to properties (a) and (b), $\mathbf{E}$ must have the form

$$\mathbf{E} = \hat{\mathbf{R}} E_R$$

where E_R is a function of R only.

$$\begin{aligned}
\nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) &= 12, \\
\frac{\partial}{\partial R} (R^2 E_R) &= 12R^2, \\
\int_0^R \frac{\partial}{\partial R} (R^2 E_R) \, dR &= \int_0^R 12R^2 \, dR, \\
R^2 E_R \Big|_0^R &= \left. \frac{12R^3}{3} \right|_0^R, \\
R^2 E_R &= 4R^3.
\end{aligned}$$

Hence,

$$E_R = 4R,$$

and

$$\mathbf{E} = \hat{\mathbf{R}} 4R.$$

Problem 3.39 For the vector field $\mathbf{E} = \hat{\mathbf{x}} xz - \hat{\mathbf{y}} yz^2 - \hat{\mathbf{z}} xy$, verify the divergence theorem by computing:

- **(a)** the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and
- **(b)** the integral of $\nabla \cdot \mathbf{E}$ over the cube's volume.

Solution:

(a) For a cube, the closed surface integral has 6 sides:

$$\oint \mathbf{E} \cdot d\mathbf{s} = F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}},$$

$$\begin{aligned}
F_{\text{top}} &= \int_{x=-1}^{1}\int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{z=1} \cdot (\hat{\mathbf{z}}\,dy\,dx) \\
&= -\int_{x=-1}^{1}\int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Bigg|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0, \\
F_{\text{bottom}} &= \int_{x=-1}^{1}\int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{z=-1} \cdot (-\hat{\mathbf{z}}\,dy\,dx) \\
&= \int_{x=-1}^{1}\int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Bigg|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0, \\
F_{\text{right}} &= \int_{x=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{y=1} \cdot (\hat{\mathbf{y}}\,dz\,dx) \\
&= -\int_{x=-1}^{1}\int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3}, \\
F_{\text{left}} &= \int_{x=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{y=-1} \cdot (-\hat{\mathbf{y}}\,dz\,dx) \\
&= -\int_{x=-1}^{1}\int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3}, \\
F_{\text{front}} &= \int_{y=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=1} \cdot (\hat{\mathbf{x}}\,dz\,dy) \\
&= \int_{y=-1}^{1}\int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0, \\
F_{\text{back}} &= \int_{y=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=-1} \cdot (-\hat{\mathbf{x}}\,dz\,dy) \\
&= \int_{y=-1}^{1}\int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0, \\
\oint \mathbf{E}\cdot d\mathbf{s} &= 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}\,.
\end{aligned}$$

(b)

$$\iiint \nabla\cdot\mathbf{E}\,dv = \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} \nabla\cdot(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy)\,dz\,dy\,dx$$
$$= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} (z - z^2)\,dz\,dy\,dx$$
$$= \left(\left(\left(xy\left(\frac{z^2}{2} - \frac{z^3}{3}\right)\right)\Big|_{z=-1}^{1}\right)\Big|_{y=-1}^{1}\right)\Big|_{x=-1}^{1} = \frac{-8}{3}.$$

Problem 3.40 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.

Solution:

$$\oint \mathbf{E}\cdot d\mathbf{s} = \int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(-\hat{\mathbf{z}}r\,dr\,d\phi)\right)\big|_{z=0}$$
$$+ \int_{\phi=0}^{2\pi}\int_{z=0}^{4} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{r}}r\,d\phi\,dz)\right)\big|_{r=2}$$
$$+ \int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\right)\big|_{z=4}$$
$$= 0 + \int_{\phi=0}^{2\pi}\int_{z=0}^{4} 10e^{-2}2\,d\phi\,dz + \int_{r=0}^{2}\int_{\phi=0}^{2\pi} -12r\,dr\,d\phi$$
$$= 160\pi e^{-2} - 48\pi \approx -82.77,$$

$$\iiint \nabla\cdot\mathbf{E}\,d\mathcal{V} = \int_{z=0}^{4}\int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left(\frac{10e^{-r}(1-r)}{r} - 3\right) r\,d\phi\,dr\,dz$$
$$= 8\pi\int_{r=0}^{2} (10e^{-r}(1-r) - 3r)\,dr$$
$$= 8\pi\left(-10e^{-r} + 10e^{-r}(1+r) - \frac{3r^2}{2}\right)\Big|_{r=0}^{2}$$
$$= 160\pi e^{-2} - 48\pi \approx -82.77.$$

Problem 3.41 A vector field $\mathbf{D} = \hat{\mathbf{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating:

(a) $\oint_S \mathbf{D}\cdot d\mathbf{s}$,

(b) $\int_{\mathcal{V}} \nabla\cdot\mathbf{D}\,d\mathcal{V}$.

Solution:

(a)

$$\iint \mathbf{D}\cdot d\mathbf{s} = F_{\text{inner}} + F_{\text{outer}} + F_{\text{bottom}} + F_{\text{top}},$$

$$F_{\text{inner}} = \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3)\cdot(-\hat{\mathbf{r}}r\,dz\,d\phi)\right)\Big|_{r=1}$$

$$= \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left(-r^4\,dz\,d\phi\right)\Big|_{r=1} = -10\pi,$$

$$F_{\text{outer}} = \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3)\cdot(\hat{\mathbf{r}}r\,dz\,d\phi)\right)\Big|_{r=2}$$

$$= \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left(r^4\,dz\,d\phi\right)\Big|_{r=2} = 160\pi,$$

$$F_{\text{bottom}} = \int_{r=1}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3)\cdot(-\hat{\mathbf{z}}r\,d\phi\,dr)\right)\Big|_{z=0} = 0,$$

$$F_{\text{top}} = \int_{r=1}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3)\cdot(\hat{\mathbf{z}}r\,d\phi\,dr)\right)\Big|_{z=5} = 0.$$

Therefore, $\iint \mathbf{D}\cdot d\mathbf{s} = 150\pi$.

(b) From the back cover, $\nabla\cdot\mathbf{D} = (1/r)(\partial/\partial r)(rr^3) = 4r^2$. Therefore,

$$\iiint \nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=1}^{2} 4r^2 r\,dr\,d\phi\,dz = \left(\left(\left(r^4\right)\Big|_{r=1}^{2}\right)\Big|_{\phi=0}^{2\pi}\right)\Big|_{z=0}^{5} = 150\pi.$$

Problem 3.42 For the vector field $\mathbf{D} = \hat{\mathbf{R}}3R^2$, evaluate both sides of the divergence theorem for the region enclosed between the spherical shells defined by $R = 1$ and $R = 2$.

Solution: The divergence theorem is given by Eq. (3.98). Evaluating the left hand side:

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{R=1}^{2} \left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2(3R^2))\right) R^2\sin\theta\,dR\,d\theta\,d\phi$$

$$= 2\pi(-\cos\theta)\big|_{\theta=0}^{\pi}\left(3R^4\right)\Big|_{R=1}^{2} = 180\pi.$$

The right hand side evaluates to

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = \left(\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\hat{\mathbf{R}}3R^2)\cdot(-\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi)\right)\Bigg|_{R=1} + \left(\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\hat{\mathbf{R}}3R^2)\cdot(\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi)\right)\Bigg|_{R=2}$$

$$= -2\pi\int_{\theta=0}^{\pi} 3\sin\theta\,d\theta + 2\pi\int_{\theta=0}^{\pi} 48\sin\theta\,d\theta = 180\pi.$$

Problem 3.43 For the vector field $\mathbf{E} = \hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)$, calculate

(a) $\oint_C \mathbf{E}\cdot d\mathbf{l}$ around the triangular contour shown in Fig. P3.43(a), and

(b) $\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{s}$ over the area of the triangle.

Solution: In addition to the independent condition that $z=0$, the three lines of the triangle are represented by the equations $y=0$, $x=1$, and $y=x$, respectively.

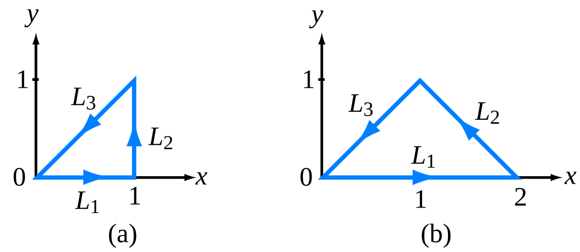


Figure P3.43: Contours for (a) Problem 3.43 and (b) Problem 3.44.

(a)

$$\oint \mathbf{E}\cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$L_1 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2))\cdot(\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$

$$= \int_{x=0}^{1} (xy)\big|_{y=0,z=0}\,dx - \int_{y=0}^{0} \left(x^2+2y^2\right)\big|_{z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=0}\,dz = 0,$$

$$L_2 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{1} (xy)\big|_{z=0}\,dx - \int_{y=0}^{1} \left(x^2+2y^2\right)\big|_{x=1,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{x=1}\,dz$$
$$= 0 - \left(y + \frac{2y^3}{3}\right)\bigg|_{y=0}^{1} + 0 = \frac{-5}{3},$$
$$L_3 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{0} (xy)\big|_{y=x,\;\; z=0}\,dx - \int_{y=1}^{0} \left(x^2+2y^2\right)\big|_{x=y,\;\; z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=x}\,dz$$
$$= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}.$$

Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

(b) From Eq. (3.105), $\nabla\times\mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\iint \nabla\times\mathbf{E} \cdot d\mathbf{s} = \int_{x=0}^{1}\int_{y=0}^{x} \left((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0}$$
$$= -\int_{x=0}^{1}\int_{y=0}^{x} 3x\,dy\,dx = -\int_{x=0}^{1} 3x(x-0)\,dx = -\left(x^3\right)\big|_0^1 = -1.$$

Problem 3.44 Repeat Problem 3.43 for the contour shown in Fig. P3.43(b).

Solution: In addition to the independent condition that $z=0$, the three lines of the triangle are represented by the equations $y=0$, $y=2-x$, and $y=x$, respectively.

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$
$$L_1 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=0}^{2} (xy)\big|_{y=0,z=0}\,dx - \int_{y=0}^{0} \left(x^2+2y^2\right)\big|_{z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=0}\,dz = 0,$$
$$L_2 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=2}^{1} (xy)\big|_{z=0,y=2-x}\,dx - \int_{y=0}^{1} \left(x^2+2y^2\right)\big|_{x=2-y,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=2-x}\,dz$$
$$= \left(x^2 - \frac{x^3}{3}\right)\bigg|_{x=2}^{1} - \left(4y - 2y^2 + y^3\right)\big|_{y=0}^{1} + 0 = \frac{-11}{3},$$

$$L_3 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{0} (xy)\big|_{y=x,z=0}\,dx - \int_{y=1}^{0} \left(x^2+2y^2\right)\big|_{x=y,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=x}\,dz$$
$$= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}\,.$$

Therefore,

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) From Eq. (3.105), $\nabla\times\mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\iint \nabla\times\mathbf{E}\cdot d\mathbf{s} = \int_{x=0}^{1}\int_{y=0}^{x} \left((-\hat{\mathbf{z}}3x)\cdot(\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0}$$
$$+ \int_{x=1}^{2}\int_{y=0}^{2-x} \left((-\hat{\mathbf{z}}3x)\cdot(\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0}$$
$$= -\int_{x=0}^{1}\int_{y=0}^{x} 3x\,dy\,dx - \int_{x=1}^{2}\int_{y=0}^{2-x} 3x\,dy\,dx$$
$$= -\int_{x=0}^{1} 3x(x-0)\,dx - \int_{x=1}^{2} 3x((2-x)-0)\,dx$$
$$= -\left(x^3\right)\big|_{0}^{1} - \left(3x^2 - x^3\right)\big|_{x=1}^{2} = -3.$$

Problem 3.45 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.46(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l},$$
$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$
$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$
$$= \left(\tfrac{1}{2}r^2\right)\big|_{r=0}^{2} + 0 = 2,$$

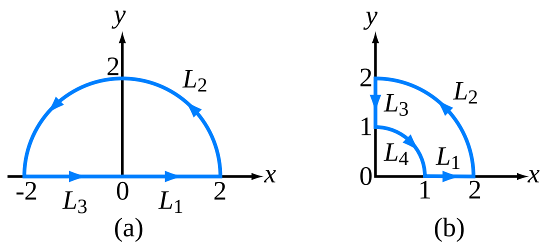


Figure P3.46: Contour paths for (a) Problem 3.45 and (b) Problem 3.46.

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\Big|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\Big|_{r=2,\ z=0}$$
$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$
$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\Big|_{\phi=\pi,z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\Big|_{z=0}$$
$$= \left(-\tfrac{1}{2}r^2\right)\big|_{r=2}^{0} + 0 = 2,$$
$$\oint \mathbf{B}\cdot d\mathbf{l} = 2+4+2 = 8.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$
$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$
$$+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$
$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),$$
$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)$$
$$= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\,dr\,d\phi = \left(\left(-\cos\phi\left(\tfrac{1}{2}r^2+r\right)\right)\Big|_{r=0}^{2}\right)\Big|_{\phi=0}^{\pi} = 8.$$

Problem 3.46 Repeat Problem 3.45 for the contour shown in Fig. P3.46(b).

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l} + \int_{L_4} \mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=1}^{2} + 0 = \frac{3}{2},$$

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi/} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi/2} = 2,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{1} r\cos\phi\,dr\right)\bigg|_{\phi=\pi/2,\ z=0} + \left(\int_{\phi=\pi/2}^{\pi/2} r\sin\phi\,d\phi\right)\bigg|_{z=0} = 0,$$

$$\int_{L_4} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{1} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=\pi/2}^{0} r\sin\phi\,d\phi\right)\bigg|_{r=1,\ z=0}$$

$$= 0 + (-\cos\phi)\big|_{\phi=\pi/2}^{0} = -1,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+ \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),$$

$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}} r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2} \sin\phi(r+1)\,dr\,d\phi$$

$$= \left(\left(-\cos\phi(\tfrac{1}{2}r^2 + r)\right)\Big|_{r=1}^{2}\right)\bigg|_{\phi=0}^{\pi/2} = \frac{5}{2}.$$

Problem 3.47 Verify Stokes's Theorem for the vector field $\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta$ by evaluating it on the hemisphere of unit radius.

Solution:

$$\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi.$$

Hence, $A_R = \cos\theta$, $A_\theta = 0$, $A_\phi = \sin\theta$.

$$\begin{aligned}
\nabla \times \mathbf{A} &= \hat{\mathbf{R}}\,\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\right) - \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial}{\partial R}(RA_\phi) - \hat{\boldsymbol{\phi}}\,\frac{1}{R}\,\frac{\partial A_R}{\partial\theta} \\
&= \hat{\mathbf{R}}\,\frac{1}{R\sin\theta}\,\frac{\partial}{\partial\theta}(\sin^2\theta) - \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial}{\partial R}(R\sin\theta) - \hat{\boldsymbol{\phi}}\,\frac{1}{R}\,\frac{\partial}{\partial\theta}(\cos\theta) \\
&= \hat{\mathbf{R}}\,\frac{2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\,\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\,\frac{\sin\theta}{R}\,.
\end{aligned}$$

For the hemispherical surface, $d\mathbf{s} = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$.

$$\begin{aligned}
&\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}(\nabla\times\mathbf{A})\cdot d\mathbf{s} \\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\frac{\hat{\mathbf{R}}2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\,\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\,\frac{\sin\theta}{R}\right)\cdot\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\,\bigg|_{R=1} \\
&= 4\pi R\,\left.\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right|_{R=1} = 2\pi.
\end{aligned}$$

The contour C is the circle in the x–y plane bounding the hemispherical surface.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi}(\hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta)\cdot\hat{\boldsymbol{\phi}}R\,d\phi\,\bigg|_{\substack{\theta=\pi/2\\R=1}} = R\sin\theta\int_0^{2\pi}d\phi\,\bigg|_{\substack{\theta=\pi/2\\R=1}} = 2\pi.$$

Problem 3.48 Determine if each of the following vector fields is solenoidal, conservative, or both:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y2xy$,
(b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z$,
(c) $\mathbf{C} = \hat{\mathbf{r}}(\sin\phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos\phi)/r^2$,
(d) $\mathbf{D} = \hat{\mathbf{R}}/R$,
(e) $\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z$,
(f) $\mathbf{F} = (\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)/(x^2 + y^2)$,
(g) $\mathbf{G} = \hat{\mathbf{x}}(x^2 + z^2) - \hat{\mathbf{y}}(y^2 + x^2) - \hat{\mathbf{z}}(y^2 + z^2)$,
(h) $\mathbf{H} = \hat{\mathbf{R}}(Re^{-R})$.

Solution:

(a)

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \nabla\cdot(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2xy = 2x - 2x = 0,\\
\nabla\times\mathbf{A} &= \nabla\times(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy)\\
&= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-2xy)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}0\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right)\\
&= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 - \hat{\mathbf{z}}(2y) \neq 0\,.
\end{aligned}$$

The field **A** is solenoidal but not conservative.

(b)

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \nabla\cdot(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}2z = 2x - 2y + 2 \neq 0,\\
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z)\\
&= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(-y^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(2z)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x^2)\right)\\
&= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 + \hat{\mathbf{z}}0.
\end{aligned}$$

The field **B** is conservative but not solenoidal.

(c)

$$\begin{aligned}
\nabla\cdot\mathbf{C} &= \nabla\cdot\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\sin\phi}{r^2}\right)\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r^2}\right) + \frac{\partial}{\partial z}0\\
&= \frac{-\sin\phi}{r^3} + \frac{-\sin\phi}{r^3} + 0 = \frac{-2\sin\phi}{r^3}\,,\\
\nabla\times\mathbf{C} &= \nabla\times\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}\left(\frac{\cos\phi}{r^2}\right)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}\left(\frac{\sin\phi}{r^2}\right) - \frac{\partial}{\partial r}0\right)\\
&\quad + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\left(\frac{\cos\phi}{r^2}\right)\right) - \frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r^2}\right)\right)\\
&= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}\left(-\left(\frac{\cos\phi}{r^2}\right) - \left(\frac{\cos\phi}{r^2}\right)\right) = \hat{\mathbf{z}}\frac{-2\cos\phi}{r^3}\,.
\end{aligned}$$

The field **C** is neither solenoidal nor conservative.

(d)

$$\nabla\cdot\mathbf{D} = \nabla\cdot\left(\frac{\hat{\mathbf{R}}}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(\frac{1}{R}\right)\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(0\sin\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}0 = \frac{1}{R^2},$$

$$\begin{aligned}\nabla\times\mathbf{D} &= \nabla\times\left(\frac{\hat{\mathbf{R}}}{R}\right)\\ &= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(0\sin\theta) - \frac{\partial}{\partial\phi}0\right) + \hat{\boldsymbol{\theta}}\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right) - \frac{\partial}{\partial R}(R(0))\right)\\ &\quad + \hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}(R(0)) - \frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right) = \hat{\mathbf{r}}0 + \hat{\boldsymbol{\theta}}0 + \hat{\boldsymbol{\phi}}0.\end{aligned}$$

The field **D** is conservative but not solenoidal.

(e)

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z,\\ \nabla\cdot\mathbf{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE_{\text{r}}) + \frac{1}{r}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(3r - \frac{r^2}{1+r}\right) + 1\\ &= \frac{1}{r}\left[3 - \frac{2r}{1+r} + \frac{r^2}{(1+r)^2}\right] + 1\\ &= \frac{1}{r}\left[\frac{3+3r^2+6r-2r-2r^2+r^2}{(1+r)^2}\right] + 1 = \frac{2r^2+4r+3}{r(1+r)^2} + 1 \neq 0,\\ \nabla\times\mathbf{E} &= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) + \hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) - \frac{1}{r}\frac{\partial E_r}{\partial\phi}\right) = 0.\end{aligned}$$

Hence, **E** is conservative, but not solenoidal.

(f)

$$\begin{aligned}\mathbf{F} &= \frac{\hat{\mathbf{x}}y + \hat{\mathbf{y}}x}{x^2+y^2} = \hat{\mathbf{x}}\frac{y}{x^2+y^2} + \hat{\mathbf{y}}\frac{x}{x^2+y^2},\\ \nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\\ &= \frac{-2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} \neq 0,\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{F} &= \hat{\mathbf{x}}(0-0) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] \\
&= \hat{\mathbf{z}}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right) \\
&= \hat{\mathbf{z}}\,\frac{2(y^2-x^2)}{(x^2+y^2)^2} \neq 0.
\end{aligned}$$

Hence, **F** is neither solenoidal nor conservative.

(g)

$$\begin{aligned}
\mathbf{G} &= \hat{\mathbf{x}}(x^2+z^2) - \hat{\mathbf{y}}(y^2+x^2) - \hat{\mathbf{z}}(y^2+z^2), \\
\nabla \cdot \mathbf{G} &= \frac{\partial}{\partial x}(x^2+z^2) - \frac{\partial}{\partial y}(y^2+x^2) - \frac{\partial}{\partial z}(y^2+z^2) \\
&= 2x - 2y - 2z \neq 0, \\
\nabla \times \mathbf{G} &= \hat{\mathbf{x}}\left(-\frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(y^2+x^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2+z^2) + \frac{\partial}{\partial x}(y^2+z^2)\right) \\
&\quad + \hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}(y^2+x^2) - \frac{\partial}{\partial y}(x^2+z^2)\right) \\
&= -\hat{\mathbf{x}}\,2y + \hat{\mathbf{y}}\,2z - \hat{\mathbf{z}}\,2x \neq 0.
\end{aligned}$$

Hence, **G** is neither solenoidal nor conservative.

(h)

$$\begin{aligned}
\mathbf{H} &= \hat{\mathbf{R}}(Re^{-R}), \\
\nabla \cdot \mathbf{H} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^3 e^{-R}) = \frac{1}{R^2}(3R^2 e^{-R} - R^3 e^{-R}) = e^{-R}(3-R) \neq 0, \\
\nabla \times \mathbf{H} &= 0.
\end{aligned}$$

Hence, **H** is conservative, but not solenoidal.

Problem 3.49 Find the Laplacian of the following scalar functions:

(a) $V = 4xy^2z^3$,
(b) $V = xy + yz + zx$,
(c) $V = 3/(x^2+y^2)$,
(d) $V = 5e^{-r}\cos\phi$,
(e) $V = 10e^{-R}\sin\theta$.

Solution:

(a) From Eq. (3.110), $\nabla^2(4xy^2z^3) = 8xz^3 + 24xy^2z$.

(b) $\nabla^2(xy+yz+zx)=0$.
(c) From the inside back cover of the book,

$$\nabla^2\left(\frac{3}{x^2+y^2}\right)=\nabla^2(3r^{-2})=12r^{-4}=\frac{12}{(x^2+y^2)^2}\,.$$

(d)

$$\nabla^2(5e^{-r}\cos\phi)=5e^{-r}\cos\phi\left(1-\frac{1}{r}-\frac{1}{r^2}\right).$$

(e)

$$\nabla^2(10e^{-R}\sin\theta)=10e^{-R}\left(\sin\theta\left(1-\frac{2}{R}\right)+\frac{\cos^2\theta-\sin^2\theta}{R^2\sin\theta}\right).$$

Problem 3.50 Find a vector **G** whose magnitude is 4 and whose direction is perpendicular to both vectors **E** and **F**, where $\mathbf{E}=\hat{\mathbf{x}}+\hat{\mathbf{y}}\,2-\hat{\mathbf{z}}\,2$ and $\mathbf{F}=\hat{\mathbf{y}}\,3-\hat{\mathbf{z}}\,6$.

Solution: The cross product of two vectors produces a third vector which is perpendicular to both of the original vectors. Two vectors exist that satisfy the stated conditions, one along $\mathbf{E}\times\mathbf{F}$ and another along the opposite direction. Hence,

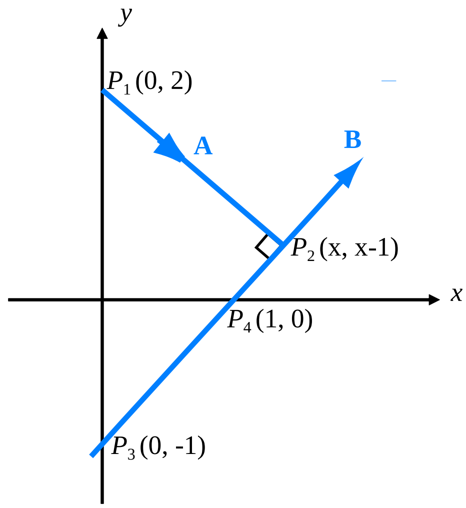
$$\begin{aligned}\mathbf{G}=\pm4\frac{\mathbf{E}\times\mathbf{F}}{|\mathbf{E}\times\mathbf{F}|}&=\pm4\frac{(\hat{\mathbf{x}}+\hat{\mathbf{y}}\,2-\hat{\mathbf{z}}\,2)\times(\hat{\mathbf{y}}\,3-\hat{\mathbf{z}}\,6)}{|(\hat{\mathbf{x}}+\hat{\mathbf{y}}\,2-\hat{\mathbf{z}}\,2)\times(\hat{\mathbf{y}}\,3-\hat{\mathbf{z}}\,6)|}\\&=\pm4\frac{(-\hat{\mathbf{x}}\,6+\hat{\mathbf{y}}\,6+\hat{\mathbf{z}}\,3)}{\sqrt{36+36+9}}\\&=\pm\frac{4}{9}(-\hat{\mathbf{x}}\,6+\hat{\mathbf{y}}\,6+\hat{\mathbf{z}}\,3)=\pm\left(-\hat{\mathbf{x}}\,\frac{8}{3}+\hat{\mathbf{y}}\,\frac{8}{3}+\hat{\mathbf{z}}\,\frac{4}{3}\right).\end{aligned}$$

Problem 3.51 A given line is described by the equation:

$$y=x-1.$$

Vector **A** starts at point $P_1(0,2)$ and ends at point P_2 on the line such that **A** is orthogonal to the line. Find an expression for **A**.

Solution: We first plot the given line.



Next we find a vector **B** which connects point $P_3(0,1)$ to point $P_4(1,0)$, both of which are on the line. Hence,

$$\mathbf{B} = \hat{\mathbf{x}}(1-0) + \hat{\mathbf{y}}(0+1) = \hat{\mathbf{x}} + \hat{\mathbf{y}}.$$

Vector **A** starts at $P_1(0,2)$ and ends on the line at P_2. If the x-coordinate of P_2 is x, then its y-coordinate has to be $y = x-1$, per the equation for the line. Thus, P_2 is at $(x, x-1)$, and vector **A** is

$$\mathbf{A} = \hat{\mathbf{x}}(x-0) + \hat{\mathbf{y}}(x-1-2) = \hat{\mathbf{x}}x + \hat{\mathbf{y}}(x-3).$$

Since **A** is orthogonal to **B**,

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= 0, \\ [\hat{\mathbf{x}}x + \hat{\mathbf{y}}(x-3)]\cdot(\hat{\mathbf{x}}+\hat{\mathbf{y}}) &= 0 \\ x + x - 3 &= 0 \\ x &= \frac{3}{2}\,. \end{aligned}$$

Finally,

$$\begin{aligned} \mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}(x-3) &= \hat{\mathbf{x}}\frac{3}{2} + \hat{\mathbf{y}}\left(\frac{3}{2}-3\right) \\ &= \hat{\mathbf{x}}\frac{3}{2} - \hat{\mathbf{y}}\frac{3}{2}\,. \end{aligned}$$

Problem 3.52 Vector field **E** is given by

$$\mathbf{E} = \hat{\mathbf{R}}\, 5R\cos\theta - \hat{\boldsymbol{\theta}}\frac{12}{R}\sin\theta\cos\phi + \hat{\boldsymbol{\phi}}3\sin\phi.$$

Determine the component of **E** tangential to the spherical surface $R = 2$ at point $P(2, 30°, 60°)$.

Solution: At P, **E** is given by

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{R}}\, 5\times 2\cos 30° - \hat{\boldsymbol{\theta}}\frac{12}{2}\sin 30°\cos 60° + \hat{\boldsymbol{\phi}}3\sin 60° \\ &= \hat{\mathbf{R}}\, 8.67 - \hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.\end{aligned}$$

The $\hat{\mathbf{R}}$ component is normal to the spherical surface while the other two are tangential. Hence,

$$\mathbf{E}_t = -\hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.$$

Problem 3.53 Transform the vector

$$\mathbf{A} = \hat{\mathbf{R}}\sin^2\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos^2\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

into cylindrical coordinates and then evaluate it at $P(2, \pi/2, \pi/2)$.

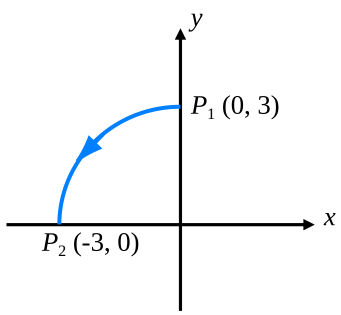
Solution: From Table 3-2,

$$\begin{aligned}\mathbf{A} &= (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\sin^2\theta\cos\phi + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\cos^2\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ &= \hat{\mathbf{r}}(\sin^3\theta\cos\phi + \cos\theta\cos^2\phi) - \hat{\boldsymbol{\phi}}\sin\phi + \hat{\mathbf{z}}(\cos\theta\sin^2\theta\cos\phi - \sin\theta\cos^2\phi)\end{aligned}$$

At $P(2, \pi/2, \pi/2)$,

$$\mathbf{A} = -\hat{\boldsymbol{\phi}}.$$

Problem 3.54 Evaluate the line integral of $\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$ along the segment P_1 to P_2 of the circular path shown in the figure.



Solution: We need to calculate:

$$\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

Since the path is along the perimeter of a circle, it is best to use cylindrical coordinates, which requires expressing both $\mathbf{E}$ and $d\boldsymbol{\ell}$ in cylindrical coordinates. Using Table 3-2,

$$\begin{aligned}\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)r\cos\phi - (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)r\sin\phi \\ &= \hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi\end{aligned}$$

The designated path is along the ϕ-direction at a constant $r = 3$. From Table 3-1, the applicable component of $d\boldsymbol{\ell}$ is:

$$d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}} r\, d\phi.$$

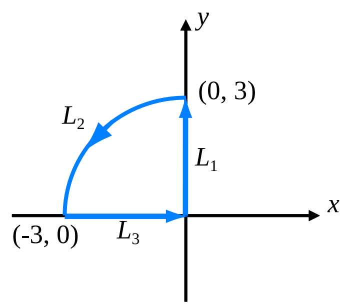
Hence,

$$\begin{aligned}\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{\phi=180^\circ} \left[\hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}\, 2r\sin\phi\cos\phi\right] \cdot \hat{\boldsymbol{\phi}} r\, d\phi\Big|_{r=3} \\ &= \int_{90^\circ}^{180^\circ} -2r^2\sin\phi\cos\phi\, d\phi\Big|_{r=3} \\ &= -2r^2 \left.\left.\frac{\sin^2\phi}{2}\right|_{\phi=90^\circ}^{180^\circ}\right|_{r=3} = 9.\end{aligned}$$

Problem 3.55 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell}$ over the path comprising a quarter section of a circle, as shown in the figure, and

(b) $\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$ over the surface of the quarter section.



Solution:

(a)

$$\oint_C \mathbf{B}\cdot d\boldsymbol{\ell} = \int_{L_1} \mathbf{B}\cdot d\boldsymbol{\ell} + \int_{L_2} \mathbf{B}\cdot d\boldsymbol{\ell} + \int_{L_3} \mathbf{B}\cdot d\boldsymbol{\ell}$$

Given the shape of the path, it is best to use cylindrical coordinates. **B** is already expressed in cylindrical coordinates, and we need to choose $d\boldsymbol{\ell}$ in cylindrical coordinates:

$$d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr + \hat{\boldsymbol{\phi}} r\ d\phi + \hat{\mathbf{z}}\ dz.$$

Along path L_1, $d\phi = 0$ and $dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr$ and

$$\begin{aligned}\int_{L_1} \mathbf{B}\cdot d\boldsymbol{\ell} &= \int_{r=0}^{r=3} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\mathbf{r}}\ dr\bigg|_{\phi=90^\circ}\\ &= \int_{r=0}^{3} \cos\phi\ dr\bigg|_{\phi=90^\circ} = r\cos\phi\big|_{r=0}^{3}\Big|_{\phi=90^\circ} = 0.\end{aligned}$$

Along L_2, $dr = dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}} r\ d\phi$ and

$$\begin{aligned}\int_{L_2} \mathbf{B}\cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{180^\circ} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\boldsymbol{\phi}} r\ d\phi\Big|_{r=3}\\ &= -3\cos\phi\big|_{90^\circ}^{180^\circ} = 3.\end{aligned}$$

Along L_3, $dz = 0$ and $d\phi = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr$ and

$$\begin{aligned}\int_{L_3} \mathbf{B}\cdot d\boldsymbol{\ell} &= \int_{r=3}^{0} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\mathbf{r}}\ dr\Big|_{\phi=180^\circ}\\ &= \int_{r=3}^{0} \cos\phi\ dr\big|_{\phi=180^\circ} = -r\big|_{3}^{0} = 3.\end{aligned}$$

Hence,

$$\oint_C \mathbf{B}\cdot d\boldsymbol{\ell} = 0 + 3 + 3 = 6.$$

(b)

$$\begin{aligned}\nabla\times\mathbf{B} &= \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rB_\phi - \frac{\partial B_r}{\partial\phi}\right)\right)\\ &= \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r\sin\phi) - \frac{\partial}{\partial\phi}(\cos\phi)\right)\\ &= \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + \sin\phi) = \hat{\mathbf{z}}\frac{2}{r}\sin\phi.\end{aligned}$$

$$\begin{aligned}\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s} &= \int_{r=0}^{3}\int_{\phi=90^\circ}^{180^\circ}\left(\hat{\mathbf{z}}\frac{2}{r}\sin\phi\right)\cdot\hat{\mathbf{z}} r\ dr\ d\phi\\ &= -2r\big|_{r=0}^{3}\cos\phi\Big|_{\phi=90^\circ}^{180^\circ} = 6.\end{aligned}$$

Hence, Stokes's theorem is verified.

Problem 3.56 Find the Laplacian of the following scalar functions:

(a) $V_1 = 10r^3 \sin 2\phi$

(b) $V_2 = (2/R^2)\cos\theta \sin\phi$

Solution:
(a)

$$\begin{aligned}
\nabla^2 V_1 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_1}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(10r^3 \sin 2\phi)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}(10r^3 \sin 2\phi) + 0 \\
&= \frac{1}{r}\frac{\partial}{\partial r}(30r^3 \sin 2\phi) - \frac{1}{r^2}(10r^3)4\sin 2\phi \\
&= 90r\sin 2\phi - 40r\sin 2\phi = 50r\sin 2\phi.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla^2 V_2 &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_2}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V_2}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V_2}{\partial \phi^2} \\
&= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right) \\
&\quad + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right) \\
&\quad + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left(\frac{2}{R^2}\cos\theta\sin\phi\right) \\
&= \frac{4}{R^4}\cos\theta\sin\phi - \frac{4}{R^4}\cos\theta\sin\phi - \frac{2}{R^4}\frac{\cos\theta}{\sin^2\theta}\sin\phi \\
&= -\frac{2}{R^4}\frac{\cos\theta\sin\phi}{\sin^2\theta}\,.
\end{aligned}$$