Chapter 1: Introduction: Waves and Phasors

Lesson #1

Chapter — Section: Chapter 1

Topics: EM history and how it relates to other fields

Highlights:

- EM in Classical era: 1000 BC to 1900
- Examples of Modern Era Technology timelines
- Concept of "fields" (gravitational, electric, magnetic)
- Static vs. dynamic fields
- The EM Spectrum

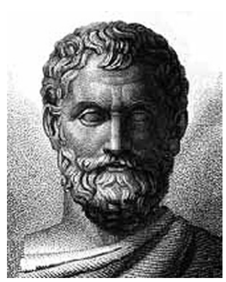
Special Illustrations:

- Timelines from CD-ROM

Timeline for Electromagnetics in the Classical Era

ca. 900 BC — Legend has it that while walking across a field in northern Greece, a shepherd named Magnus experiences a pull on the iron nails in his sandals by the black rock he was standing on. The region was later named Magnesia and the rock became known as magnetite [a form of iron with permanent magnetism].

ca. 600 BC — Greek philosopher **Thales** describes how amber, after being rubbed with cat fur, can pick up feathers [static electricity].



ca. 1000 — Magnetic compass used as a navigational device.

1752 — **Benjamin Franklin** (American) invents the lightning rod and demonstrates that lightning is electricity.

1785 — **Charles-Augustin de Coulomb** (French) demonstrates that the electrical force between charges is proportional to the inverse of the square of the distance between them.

1800 — **Alessandro Volta** (Italian) develops the first electric battery.

1820 — **Hans Christian Oersted** (Danish) demonstrates the interconnection between electricity and magnetism through his discovery that an electric current in a wire causes a compass needle to orient itself perpendicular to the wire.

Lessons #2 and 3

Chapter — Sections: 1-1 to 1-6

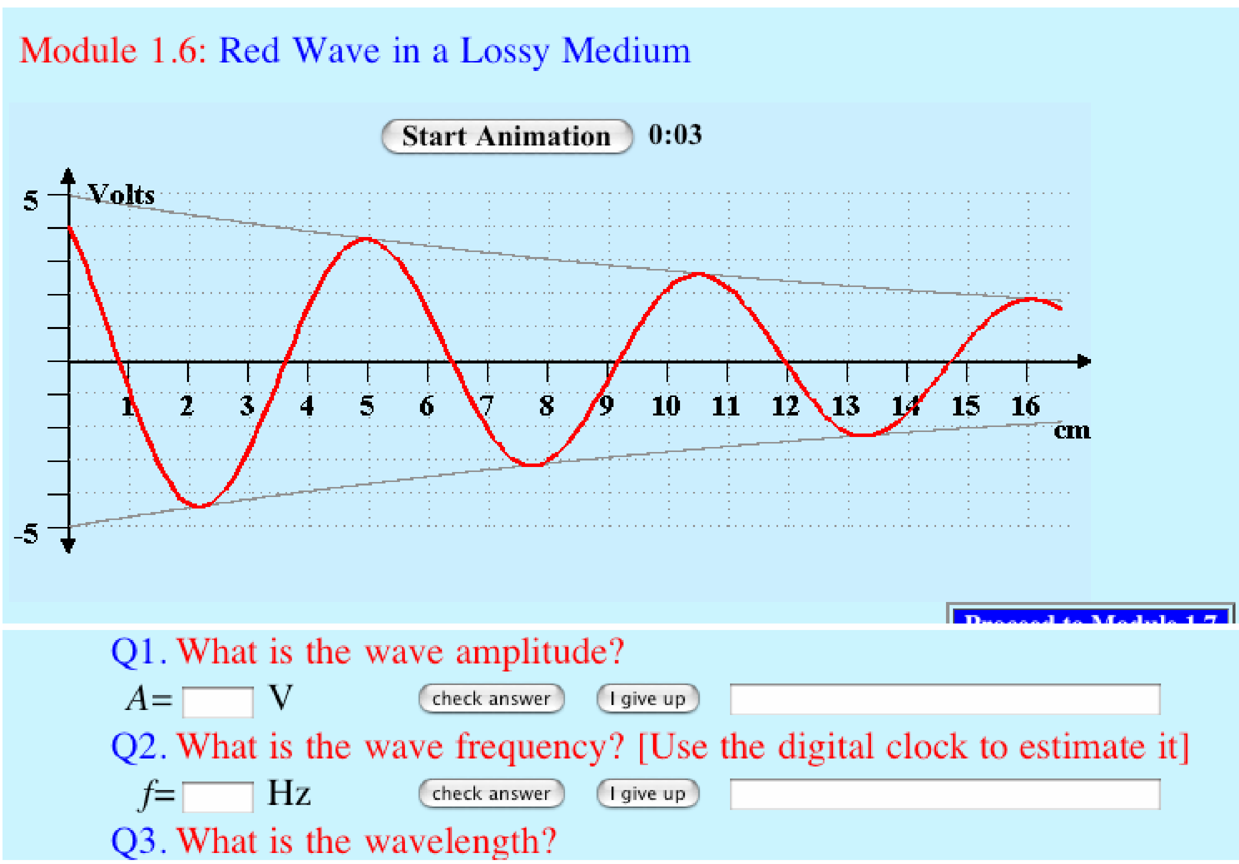
Topics: Waves

Highlights:

- Wave properties
- Complex numbers
- Phasors

Special Illustrations:

- CD-ROM Modules 1.1-1.9
- CD-ROM Demos 1.1-1.3



Chapter 1

Section 1-3: Traveling Waves

Problem 1.1 A 2-kHz sound wave traveling in the x-direction in air was observed to have a differential pressure $p(x,t) = 10$ N/m^2 at $x = 0$ and $t = 50$ μs. If the reference phase of $p(x,t)$ is 36°, find a complete expression for $p(x,t)$. The velocity of sound in air is 330 m/s.

Solution: The general form is given by Eq. (1.17),

$$p(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right),$$

where it is given that $\phi_0 = 36°$. From Eq. (1.26), $T = 1/f = 1/(2\times 10^3) = 0.5$ ms. From Eq. (1.27),

$$\lambda = \frac{u_\text{p}}{f} = \frac{330}{2\times 10^3} = 0.165 \text{ m.}$$

Also, since

$$\begin{aligned} p(x = 0,\ t = 50\ \mu\text{s}) = 10 \text{ (N/m}^2) &= A\cos\left(\frac{2\pi\times 50\times 10^{-6}}{5\times 10^{-4}} + 36°\frac{\pi \text{ rad}}{180°}\right) \\ &= A\cos(1.26 \text{ rad}) = 0.31A, \end{aligned}$$

it follows that $A = 10/0.31 = 32.36$ N/m^2. So, with t in (s) and x in (m),

$$\begin{aligned} p(x,t) &= 32.36\cos\left(2\pi\times 10^6\frac{t}{500} - 2\pi\times 10^3\frac{x}{165} + 36°\right) \quad \text{(N/m}^2) \\ &= 32.36\cos(4\pi\times 10^3 t - 12.12\pi x + 36°) \quad \text{(N/m}^2). \end{aligned}$$

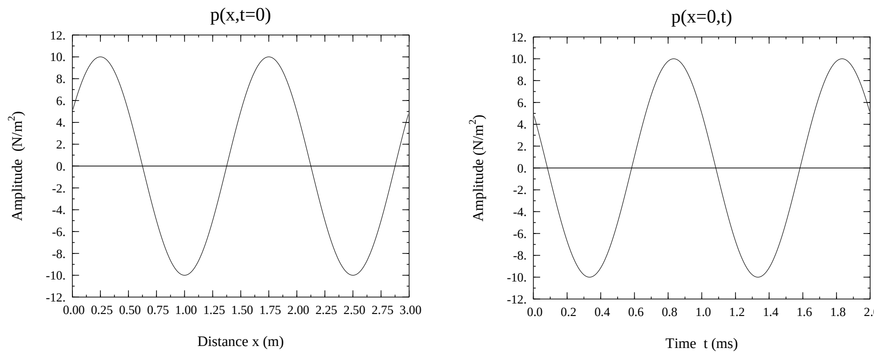
Problem 1.2 For the pressure wave described in Example 1-1, plot

(a) $p(x,t)$ versus x at $t = 0$,

(b) $p(x,t)$ versus t at $x = 0$.

Be sure to use appropriate scales for x and t so that each of your plots covers at least two cycles.

Solution: Refer to Fig. P1.2(a) and Fig. P1.2(b).



(a) (b)

Figure P1.2: (a) Pressure wave as a function of distance at $t = 0$ and (b) pressure wave as a function of time at $x = 0$.

Problem 1.3 A harmonic wave traveling along a string is generated by an oscillator that completes 180 vibrations per minute. If it is observed that a given crest, or maximum, travels 300 cm in 10 s, what is the wavelength?

Solution:

$$f = \frac{180}{60} = 3 \text{ Hz}.$$

$$u_{\text{p}} = \frac{300 \text{ cm}}{10 \text{ s}} = 0.3 \text{ m/s}.$$

$$\lambda = \frac{u_{\text{p}}}{f} = \frac{0.3}{3} = 0.1 \text{ m} = 10 \text{ cm}.$$

Problem 1.4 Two waves, $y_1(t)$ and $y_2(t)$, have identical amplitudes and oscillate at the same frequency, but $y_2(t)$ leads $y_1(t)$ by a phase angle of 60°. If

$$y_1(t) = 4\cos(2\pi \times 10^3 t),$$

write down the expression appropriate for $y_2(t)$ and plot both functions over the time span from 0 to 2 ms.

Solution:

$$y_2(t) = 4\cos(2\pi \times 10^3 t + 60°).$$

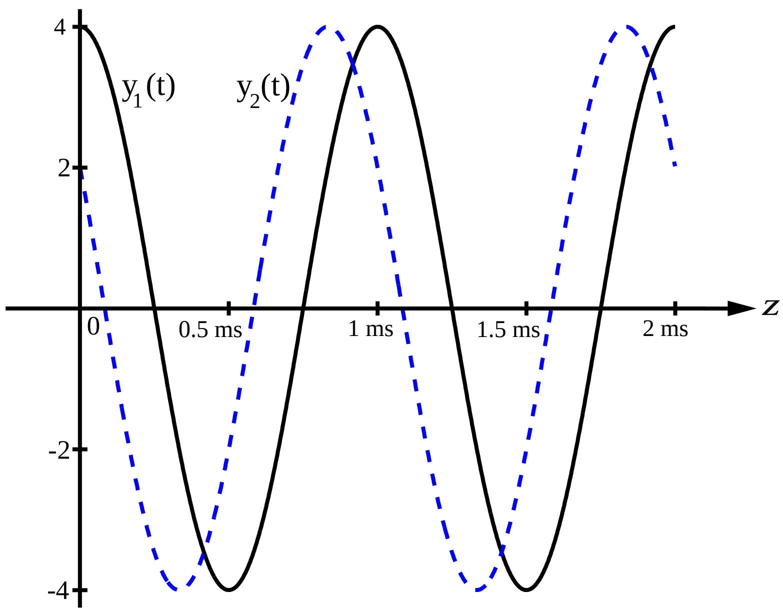


Figure P1.4: Plots of $y_1(t)$ and $y_2(t)$.

Problem 1.5 The height of an ocean wave is described by the function

$$y(x,t) = 1.5\sin(0.5t - 0.6x) \quad \text{(m)}.$$

Determine the phase velocity and the wavelength and then sketch $y(x,t)$ at $t = 2$ s over the range from $x = 0$ to $x = 2\lambda$.

Solution: The given wave may be rewritten as a cosine function:

$$y(x,t) = 1.5\cos(0.5t - 0.6x - \pi/2).$$

By comparison of this wave with Eq. (1.32),

$$y(x,t) = A\cos(\omega t - \beta x + \phi_0),$$

we deduce that

$$\omega = 2\pi f = 0.5 \text{ rad/s}, \qquad \beta = \frac{2\pi}{\lambda} = 0.6 \text{ rad/m},$$

$$u_{\mathrm{p}} = \frac{\omega}{\beta} = \frac{0.5}{0.6} = 0.83 \text{ m/s}, \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.6} = 10.47 \text{ m}.$$

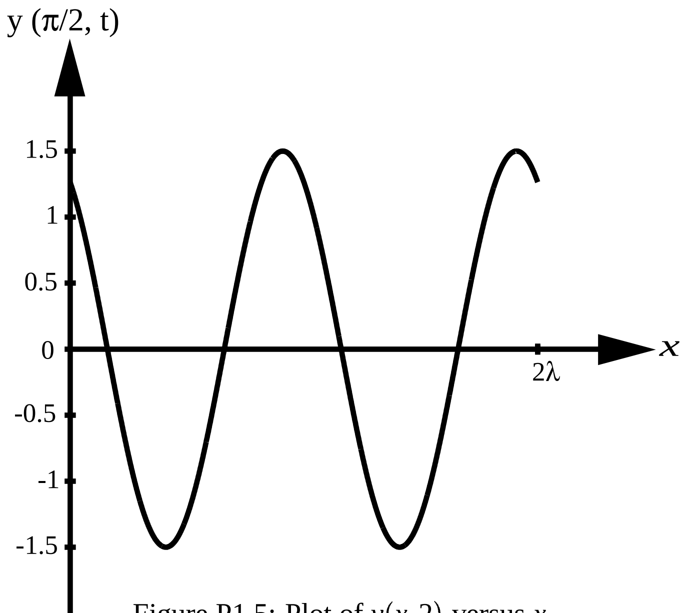


Figure P1.5: Plot of $y(x,2)$ versus x.

At $t = 2$ s, $y(x,2) = 1.5\sin(1 - 0.6x)$ (m), with the argument of the cosine function given in radians. Plot is shown in Fig. P1.5.

Problem 1.6 A wave traveling along a string in the $+x$-direction is given by

$$y_1(x,t) = A\cos(\omega t - \beta x),$$

where $x = 0$ is the end of the string, which is tied rigidly to a wall, as shown in Fig. 1-21 (P1.6). When wave $y_1(x,t)$ arrives at the wall, a reflected wave $y_2(x,t)$ is generated. Hence, at any location on the string, the vertical displacement y_s will be the sum of the incident and reflected waves:

$$y_s(x,t) = y_1(x,t) + y_2(x,t).$$

(a) Write down an expression for $y_2(x,t)$, keeping in mind its direction of travel and the fact that the end of the string cannot move.

(b) Generate plots of $y_1(x,t)$, $y_2(x,t)$ and $y_s(x,t)$ versus x over the range $-2\lambda \leq x \leq 0$ at $\omega t = \pi/4$ and at $\omega t = \pi/2$.

Solution:

(a) Since wave $y_2(x,t)$ was caused by wave $y_1(x,t)$, the two waves must have the same angular frequency ω, and since $y_2(x,t)$ is traveling on the same string as $y_1(x,t)$,

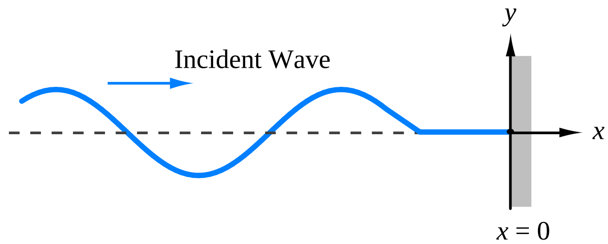


Figure P1.6: Wave on a string tied to a wall at $x = 0$ (Problem 1.6).

the two waves must have the same phase constant β. Hence, with its direction being in the negative x-direction, $y_2(x,t)$ is given by the general form

$$y_2(x,t) = B\cos(\omega t + \beta x + \phi_0), \tag{1}$$

where B and ϕ_0 are yet-to-be-determined constants. The total displacement is

$$y_s(x,t) = y_1(x,t) + y_2(x,t) = A\cos(\omega t - \beta x) + B\cos(\omega t + \beta x + \phi_0).$$

Since the string cannot move at $x = 0$, the point at which it is attached to the wall, $y_s(0,t) = 0$ for all t. Thus,

$$y_s(0,t) = A\cos\omega t + B\cos(\omega t + \phi_0) = 0. \tag{2}$$

(i) Easy Solution: The physics of the problem suggests that a possible solution for (2) is $B = -A$ and $\phi_0 = 0$, in which case we have

$$y_2(x,t) = -A\cos(\omega t + \beta x). \tag{3}$$

(ii) Rigorous Solution: By expanding the second term in (2), we have

$$A\cos\omega t + B(\cos\omega t\cos\phi_0 - \sin\omega t\sin\phi_0) = 0,$$

or

$$(A + B\cos\phi_0)\cos\omega t - (B\sin\phi_0)\sin\omega t = 0. \tag{4}$$

This equation has to be satisfied for all values of t. At $t = 0$, it gives

$$A + B\cos\phi_0 = 0, \tag{5}$$

and at $\omega t = \pi/2$, (4) gives

$$B \sin \phi_0 = 0. \tag{6}$$

Equations (5) and (6) can be satisfied simultaneously only if

$$A = B = 0 \tag{7}$$

or

$$A = -B \quad \text{and} \quad \phi_0 = 0. \tag{8}$$

Clearly (7) is not an acceptable solution because it means that $y_1(x,t) = 0$, which is contrary to the statement of the problem. The solution given by (8) leads to (3).

(b) At $\omega t = \pi/4$,

$$y_1(x,t) = A\cos(\pi/4 - \beta x) = A\cos\left(\frac{\pi}{4} - \frac{2\pi x}{\lambda}\right),$$
$$y_2(x,t) = -A\cos(\omega t + \beta x) = -A\cos\left(\frac{\pi}{4} + \frac{2\pi x}{\lambda}\right).$$

Plots of y_1, y_2, and y_3 are shown in Fig. P1.6(b).

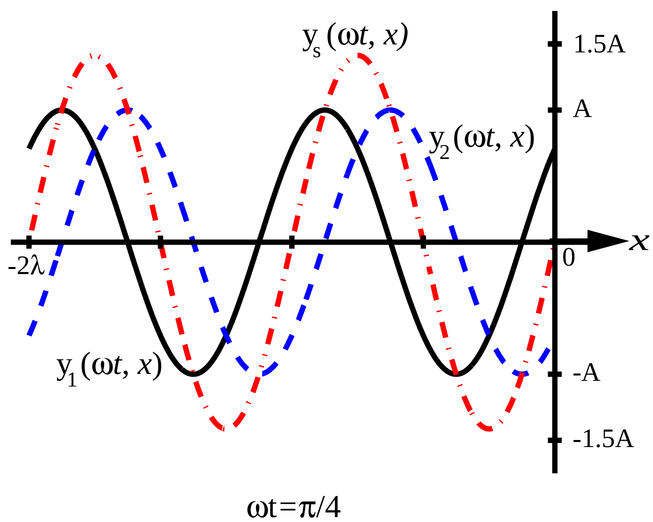


Figure P1.6: (b) Plots of y_1, y_2, and y_s versus x at $\omega t = \pi/4$.

At $\omega t = \pi/2$,

$$y_1(x,t) = A\cos(\pi/2 - \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda},$$

$$y_2(x,t) = -A\cos(\pi/2 + \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda}.$$

Plots of y_1, y_2, and y_3 are shown in Fig. P1.6(c).

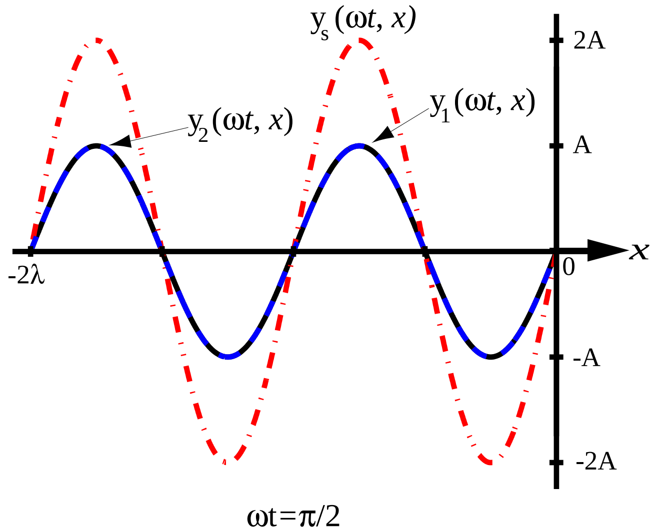


Figure P1.6: (c) Plots of y_1, y_2, and y_s versus x at $\omega t = \pi/2$.

Problem 1.7 Two waves on a string are given by the following functions:

$$y_1(x,t) = 4\cos(20t - 30x) \quad \text{(cm)},$$
$$y_2(x,t) = -4\cos(20t + 30x) \quad \text{(cm)},$$

where x is in centimeters. The waves are said to interfere constructively when their superposition $|y_s| = |y_1 + y_2|$ is a maximum and they interfere destructively when $|y_s|$ is a minimum.

(a) What are the directions of propagation of waves $y_1(x,t)$ and $y_2(x,t)$?

(b) At $t = (\pi/50)$ s, at what location x do the two waves interfere constructively, and what is the corresponding value of $|y_s|$?

(c) At $t = (\pi/50)$ s, at what location x do the two waves interfere destructively, and what is the corresponding value of $|y_s|$?

Solution:

(a) $y_1(x,t)$ is traveling in positive x-direction. $y_2(x,t)$ is traveling in negative x-direction.

(b) At $t = (\pi/50)$ s, $y_s = y_1 + y_2 = 4[\cos(0.4\pi - 30x) - \cos(0.4\pi + 3x)]$. Using the formulas from Appendix C,

$$2\sin x \sin y = \cos(x-y) - (\cos x + y),$$

we have

$$y_s = 8\sin(0.4\pi)\sin 30x = 7.61\sin 30x.$$

Hence,

$$|y_{\rm s}|_{\max} = 7.61$$

and it occurs when $\sin 30x = 1$, or $30x = \dfrac{\pi}{2} + 2n\pi$, or $x = \left(\dfrac{\pi}{60} + \dfrac{2n\pi}{30}\right)$ cm, where $n = 0, 1, 2, \ldots$.

(c) $|y_{\rm s}|_{\min} = 0$ and it occurs when $30x = n\pi$, or $x = \dfrac{n\pi}{30}$ cm.

Problem 1.8 Give expressions for $y(x,t)$ for a sinusoidal wave traveling along a string in the negative x-direction, given that $y_{\max} = 40$ cm, $\lambda = 30$ cm, $f = 10$ Hz, and

(a) $y(x,0) = 0$ at $x = 0$,

(b) $y(x,0) = 0$ at $x = 7.5$ cm.

Solution: For a wave traveling in the negative x-direction, we use Eq. (1.17) with $\omega = 2\pi f = 20\pi$ (rad/s), $\beta = 2\pi/\lambda = 2\pi/0.3 = 20\pi/3$ (rad/s), $A = 40$ cm, and x assigned a positive sign:

$$y(x,t) = 40\cos\left(20\pi t + \frac{20\pi}{3}x + \phi_0\right) \quad \text{(cm)},$$

with x in meters.

(a) $y(0,0) = 0 = 40\cos\phi_0$. Hence, $\phi_0 = \pm\pi/2$, and

$$\begin{aligned} y(x,t) &= 40\cos\left(20\pi t + \frac{20\pi}{3}x \pm \frac{\pi}{2}\right) \\ &= \begin{cases} -40\sin\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = \pi/2, \\ 40\sin\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = -\pi/2. \end{cases} \end{aligned}$$

(b) At $x = 7.5$ cm = 7.5×10^{-2} m, $y = 0 = 40\cos(\pi/2 + \phi_0)$. Hence, $\phi_0 = 0$ or π, and

$$y(x,t) = \begin{cases} 40\cos\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = 0, \\ -40\cos\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = \pi. \end{cases}$$

Problem 1.9 An oscillator that generates a sinusoidal wave on a string completes 20 vibrations in 50 s. The wave peak is observed to travel a distance of 2.8 m along the string in 50 s. What is the wavelength?

Solution:

$$T = \frac{50}{20} = 2.5 \text{ s}, \qquad u_{\text{p}} = \frac{2.8}{5} = 0.56 \text{ m/s},$$
$$\lambda = u_{\text{p}} T = 0.56 \times 2.5 = 1.4 \text{ m}.$$

Problem 1.10 The vertical displacement of a string is given by the harmonic function:

$$y(x,t) = 6\cos(16\pi t - 20\pi x) \quad \text{(m)},$$

where x is the horizontal distance along the string in meters. Suppose a tiny particle were to be attached to the string at $x = 5$ cm, obtain an expression for the vertical velocity of the particle as a function of time.

Solution:

$$y(x,t) = 6\cos(16\pi t - 20\pi x) \quad \text{(m)}.$$

$$\begin{aligned} u(0.05,t) &= \left.\frac{dy(x,t)}{dt}\right|_{x=0.05} \\ &= 96\pi \sin(16\pi t - 20\pi x)|_{x=0.05} \\ &= 96\pi \sin(16\pi t - \pi) \\ &= -96\pi \sin(16\pi t) \quad \text{(m/s)}. \end{aligned}$$

Problem 1.11 Given two waves characterized by

$$\begin{aligned} y_1(t) &= 3\cos\omega t, \\ y_2(t) &= 3\sin(\omega t + 36^\circ), \end{aligned}$$

does $y_2(t)$ lead or lag $y_1(t)$, and by what phase angle?

Solution: We need to express $y_2(t)$ in terms of a cosine function:

$$\begin{aligned} y_2(t) &= 3\sin(\omega t + 36^\circ) \\ &= 3\cos\left(\frac{\pi}{2} - \omega t - 36^\circ\right) = 3\cos(54^\circ - \omega t) = 3\cos(\omega t - 54^\circ). \end{aligned}$$

Hence, $y_2(t)$ lags $y_1(t)$ by 54°.

Problem 1.12 The voltage of an electromagnetic wave traveling on a transmission line is given by $v(z,t) = 5e^{-\alpha z}\sin(4\pi \times 10^9 t - 20\pi z)$ (V), where z is the distance in meters from the generator.

(a) Find the frequency, wavelength, and phase velocity of the wave.

(b) At $z = 2$ m, the amplitude of the wave was measured to be 1 V. Find α.

Solution:

(a) This equation is similar to that of Eq. (1.28) with $\omega = 4\pi \times 10^9$ rad/s and $\beta = 20\pi$ rad/m. From Eq. (1.29a), $f = \omega/2\pi = 2 \times 10^9$ Hz $= 2$ GHz; from Eq. (1.29b), $\lambda = 2\pi/\beta = 0.1$ m. From Eq. (1.30),

$$u_p = \omega/\beta = 2 \times 10^8 \text{ m/s}.$$

(b) Using just the amplitude of the wave,

$$1 = 5e^{-\alpha 2}, \qquad \alpha = \frac{-1}{2\text{ m}} \ln\left(\frac{1}{5}\right) = 0.81 \text{ Np/m}.$$

Problem 1.13 A certain electromagnetic wave traveling in sea water was observed to have an amplitude of 98.02 (V/m) at a depth of 10 m and an amplitude of 81.87 (V/m) at a depth of 100 m. What is the attenuation constant of sea water?

Solution: The amplitude has the form $Ae^{\alpha z}$. At $z = 10$ m,

$$Ae^{-10\alpha} = 98.02$$

and at $z = 100$ m,

$$Ae^{-100\alpha} = 81.87$$

The ratio gives

$$\frac{e^{-10\alpha}}{e^{-100\alpha}} = \frac{98.02}{81.87} = 1.20$$

or

$$e^{-10\alpha} = 1.2e^{-100\alpha}.$$

Taking the natural log of both sides gives

$$\ln(e^{-10\alpha}) = \ln(1.2e^{-100\alpha}),$$
$$-10\alpha = \ln(1.2) - 100\alpha,$$
$$90\alpha = \ln(1.2) = 0.18.$$

Hence,

$$\alpha = \frac{0.18}{90} = 2 \times 10^{-3} \quad \text{(Np/m)}.$$

Section 1-5: Complex Numbers

Problem 1.14 Evaluate each of the following complex numbers and express the result in rectangular form:

(a) $z_1 = 4e^{j\pi/3}$,
(b) $z_2 = \sqrt{3}\, e^{j3\pi/4}$,
(c) $z_3 = 6e^{-j\pi/2}$,
(d) $z_4 = j^3$,
(e) $z_5 = j^{-4}$,
(f) $z_6 = (1 - j)^3$,
(g) $z_7 = (1 - j)^{1/2}$.

Solution: (Note: In the following solutions, numbers are expressed to only two decimal places, but the final answers are found using a calculator with 10 decimal places.)

(a) $z_1 = 4e^{j\pi/3} = 4(\cos\pi/3 + j\sin\pi/3) = 2.0 + j3.46.$

(b)

$$z_2 = \sqrt{3}e^{j3\pi/4} = \sqrt{3}\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right] = -1.22 + j1.22 = 1.22(-1 + j).$$

(c) $z_3 = 6e^{-j\pi/2} = 6[\cos(-\pi/2) + j\sin(-\pi/2)] = -j6.$

(d) $z_4 = j^3 = j \cdot j^2 = -j$, or

$$z_4 = j^3 = (e^{j\pi/2})^3 = e^{j3\pi/2} = \cos(3\pi/2) + j\sin(3\pi/2) = -j.$$

(e) $z_5 = j^{-4} = (e^{j\pi/2})^{-4} = e^{-j2\pi} = 1.$

(f)

$$\begin{aligned} z_6 = (1 - j)^3 = (\sqrt{2}e^{-j\pi/4})^3 &= (\sqrt{2})^3 e^{-j3\pi/4} \\ &= (\sqrt{2})^3[\cos(3\pi/4) - j\sin(3\pi/4)] \\ &= -2 - j2 = -2(1 + j). \end{aligned}$$

(g)

$$\begin{aligned} z_7 = (1 - j)^{1/2} = (\sqrt{2}e^{-j\pi/4})^{1/2} = \pm 2^{1/4}e^{-j\pi/8} &= \pm 1.19(0.92 - j0.38) \\ &= \pm(1.10 - j0.45). \end{aligned}$$

Problem 1.15 Complex numbers z_1 and z_2 are given by

$$z_1 = 3 - j2,$$
$$z_2 = -4 + j3.$$

(a) Express z_1 and z_2 in polar form.
(b) Find $|z_1|$ by applying Eq. (1.41) and again by applying Eq. (1.43).
(c) Determine the product $z_1 z_2$ in polar form.
(d) Determine the ratio z_1/z_2 in polar form.
(e) Determine z_1^3 in polar form.

Solution:
(a) Using Eq. (1.41),

$$z_1 = 3 - j2 = 3.6e^{-j33.7^\circ},$$
$$z_2 = -4 + j3 = 5e^{j143.1^\circ}.$$

(b) By Eq. (1.41) and Eq. (1.43), respectively,

$$|z_1| = |3 - j2| = \sqrt{3^2 + (-2)^2} = \sqrt{13} = 3.60,$$
$$|z_1| = \sqrt{(3 - j2)(3 + j2)} = \sqrt{13} = 3.60.$$

(c) By applying Eq. (1.47b) to the results of part (a),

$$z_1 z_2 = 3.6e^{-j33.7^\circ} \times 5e^{j143.1^\circ} = 18e^{j109.4^\circ}.$$

(d) By applying Eq. (1.48b) to the results of part (a),

$$\frac{z_1}{z_2} = \frac{3.6e^{-j33.7^\circ}}{5e^{j143.1^\circ}} = 0.72e^{-j176.8^\circ}.$$

(e) By applying Eq. (1.49) to the results of part (a),

$$z_1^3 = \left(3.6e^{-j33.7^\circ}\right)^3 = (3.6)^3 e^{-j3\times 33.7^\circ} = 46.66e^{-j101.1^\circ}.$$

Problem 1.16 If $z = -2 + j4$, determine the following quantities in polar form:
(a) $1/z$,
(b) z^3,
(c) $|z|^2$,
(d) $\mathfrak{Im}\{z\}$,
(e) $\mathfrak{Im}\{z^*\}$.

Solution: (Note: In the following solutions, numbers are expressed to only two decimal places, but the final answers are found using a calculator with 10 decimal places.)

(a)

$$\frac{1}{z} = \frac{1}{-2+j4} = (-2+j4)^{-1} = (4.47\,e^{j116.6^\circ})^{-1} = (4.47)^{-1}e^{-j116.6^\circ} = 0.22\,e^{-j116.6^\circ}.$$

(b) $z^3 = (-2+j4)^3 = (4.47\,e^{j116.6^\circ})^3 = (4.47)^3 e^{j350.0^\circ} = 89.44\,e^{-j10^\circ}$.
(c) $|z|^2 = z \cdot z^* = (-2+j4)(-2-j4) = 4+16 = 20$.
(d) $\mathfrak{Im}\{z\} = \mathfrak{Im}\{-2+j4\} = 4$.
(e) $\mathfrak{Im}\{z^*\} = \mathfrak{Im}\{-2-j4\} = -4 = 4e^{j\pi}$.

Problem 1.17 Find complex numbers $t = z_1 + z_2$ and $s = z_1 - z_2$, both in polar form, for each of the following pairs:
(a) $z_1 = 2+j3,\ z_2 = 1-j3$,
(b) $z_1 = 3,\ z_2 = -j3$,
(c) $z_1 = 3\angle 30^\circ,\ z_2 = 3\angle -30^\circ$,
(d) $z_1 = 3\angle 30^\circ,\ z_2 = 3\angle -150^\circ$.

Solution:
(a)

$$\begin{aligned} t &= z_1 + z_2 = (2+j3) + (1-j3) = 3, \\ s &= z_1 - z_2 = (2+j3) - (1-j3) = 1+j6 = 6.08\,e^{j80.5^\circ}. \end{aligned}$$

(b)

$$\begin{aligned} t &= z_1 + z_2 = 3 - j3 = 4.24\,e^{-j45^\circ}, \\ s &= z_1 - z_2 = 3 + j3 = 4.24\,e^{j45^\circ}. \end{aligned}$$

(c)

$$\begin{aligned} t &= z_1 + z_2 = 3\angle 30^\circ + 3\angle -30^\circ \\ &= 3e^{j30^\circ} + 3e^{-j30^\circ} = (2.6+j1.5) + (2.6-j1.5) = 5.2, \\ s &= z_1 - z_2 = 3e^{j30^\circ} - 3e^{-j30^\circ} = (2.6+j1.5) - (2.6-j1.5) = j3 = 3e^{j90^\circ}. \end{aligned}$$

(d)

$$\begin{aligned} t &= z_1 + z_2 = 3\angle 30^\circ + 3\angle -150^\circ = (2.6+j1.5) + (-2.6-j1.5) = 0, \\ s &= z_1 - z_2 = (2.6+j1.5) - (-2.6-j1.5) = 5.2+j3 = 6e^{j30^\circ}. \end{aligned}$$

Problem 1.18 Complex numbers z_1 and z_2 are given by

$$\begin{aligned} z_1 &= 5\angle -60^\circ, \\ z_2 &= 2\angle 45^\circ. \end{aligned}$$

(a) Determine the product $z_1 z_2$ in polar form.
(b) Determine the product $z_1 z_2^*$ in polar form.
(c) Determine the ratio z_1/z_2 in polar form.
(d) Determine the ratio z_1^*/z_2^* in polar form.
(e) Determine $\sqrt{z_1}$ in polar form.

Solution:

(a) $z_1 z_2 = 5e^{-j60^\circ} \times 2e^{j45^\circ} = 10e^{-j15^\circ}$.

(b) $z_1 z_2^* = 5e^{-j60^\circ} \times 2e^{-j45^\circ} = 10e^{-j105^\circ}$.

(c) $\dfrac{z_1}{z_2} = \dfrac{5e^{-j60^\circ}}{2e^{j45^\circ}} = 2.5^{-j105^\circ}$.

(d) $\dfrac{z_1^*}{z_2^*} = \left(\dfrac{z_1}{z_2}\right)^* = 2.5^{j105^\circ}$.

(e) $\sqrt{z_1} = \sqrt{5e^{-j60^\circ}} = \pm\sqrt{5}\,e^{-j30^\circ}$.

Problem 1.19 If $z = 3 - j5$, find the value of $\ln(z)$.

Solution:

$$|z| = +\sqrt{3^2 + 5^2} = 5.83, \quad \theta = \tan^{-1}\left(\frac{-5}{3}\right) = -59^\circ,$$

$$z = |z|e^{j\theta} = 5.83e^{-j59^\circ},$$

$$\begin{aligned}\ln(z) &= \ln(5.83e^{-j59^\circ}) \\ &= \ln(5.83) + \ln(e^{-j59^\circ}) \\ &= 1.76 - j59^\circ = 1.76 - j\frac{59^\circ \pi}{180^\circ} = 1.76 - j1.03.\end{aligned}$$

Problem 1.20 If $z = 3 - j4$, find the value of e^z.

Solution:

$$e^z = e^{3-j4} = e^3 \cdot e^{-j4} = e^3(\cos 4 - j\sin 4),$$

$$e^3 = 20.09, \qquad \text{and} \qquad 4 \text{ rad} = \frac{4}{\pi} \times 180^\circ = 229.18^\circ.$$

Hence, $e^z = 20.08(\cos 229.18^\circ - j\sin 229.18^\circ) = -13.13 + j15.20$.

Section 1-6: Phasors

Problem 1.21 A voltage source given by $v_s(t) = 25\cos(2\pi \times 10^3 t - 30°)$ (V) is connected to a series RC load as shown in Fig. 1-19. If $R = 1$ MΩ and $C = 200$ pF, obtain an expression for $v_c(t)$, the voltage across the capacitor.

Solution: In the phasor domain, the circuit is a voltage divider, and

$$\widetilde{V}_c = \widetilde{V}_s \frac{1/j\omega C}{R + 1/j\omega C} = \frac{\widetilde{V}_s}{(1 + j\omega RC)}.$$

Now $\widetilde{V}_s = 25e^{-j30°}$ V with $\omega = 2\pi \times 10^3$ rad/s, so

$$\begin{aligned}\widetilde{V}_c &= \frac{25e^{-j30°} \text{ V}}{1 + j((2\pi \times 10^3 \text{ rad/s}) \times (10^6\ \Omega) \times (200 \times 10^{-12} \text{ F}))} \\ &= \frac{25e^{-j30°} \text{ V}}{1 + j2\pi/5} = 15.57e^{-j81.5°} \text{ V}.\end{aligned}$$

Converting back to an instantaneous value,

$$v_c(t) = \mathfrak{Re}\,\widetilde{V}_c e^{j\omega t} = \mathfrak{Re}\,15.57e^{j(\omega t - 81.5°)} \text{ V} = 15.57\cos(2\pi \times 10^3 t - 81.5°) \text{ V},$$

where t is expressed in seconds.

Problem 1.22 Find the phasors of the following time functions:

(a) $v(t) = 3\cos(\omega t - \pi/3)$ (V),
(b) $v(t) = 12\sin(\omega t + \pi/4)$ (V),
(c) $i(x,t) = 2e^{-3x}\sin(\omega t + \pi/6)$ (A),
(d) $i(t) = -2\cos(\omega t + 3\pi/4)$ (A),
(e) $i(t) = 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6)$ (A).

Solution:

(a) $\widetilde{V} = 3e^{-j\pi/3}$ V.

(b) $v(t) = 12\sin(\omega t + \pi/4) = 12\cos(\pi/2 - (\omega t + \pi/4)) = 12\cos(\omega t - \pi/4)$ V, $\widetilde{V} = 12e^{-j\pi/4}$ V.

(c)

$$\begin{aligned}i(t) &= 2e^{-3x}\sin(\omega t + \pi/6) \text{ A} = 2e^{-3x}\cos(\pi/2 - (\omega t + \pi/6)) \text{ A} \\ &= 2e^{-3x}\cos(\omega t - \pi/3) \text{ A}, \\ \widetilde{I} &= 2e^{-3x}e^{-j\pi/3} \text{ A}.\end{aligned}$$

(d)

$$i(t) = -2\cos(\omega t + 3\pi/4),$$
$$\widetilde{I} = -2e^{j3\pi/4} = 2e^{-j\pi}e^{j3\pi/4} = 2e^{-j\pi/4} \text{ A}.$$

(e)

$$\begin{aligned} i(t) &= 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6) \\ &= 4\cos[\pi/2 - (\omega t + \pi/3)] + 3\cos(\omega t - \pi/6) \\ &= 4\cos(-\omega t + \pi/6) + 3\cos(\omega t - \pi/6) \\ &= 4\cos(\omega t - \pi/6) + 3\cos(\omega t - \pi/6) = 7\cos(\omega t - \pi/6), \\ \widetilde{I} &= 7e^{-j\pi/6} \text{ A}. \end{aligned}$$

Problem 1.23 Find the instantaneous time sinusoidal functions corresponding to the following phasors:

(a) $\widetilde{V} = -5e^{j\pi/3}$ (V),
(b) $\widetilde{V} = j6e^{-j\pi/4}$ (V),
(c) $\widetilde{I} = (6 + j8)$ (A),
(d) $\tilde{I} = -3 + j2$ (A),
(e) $\tilde{I} = j$ (A),
(f) $\tilde{I} = 2e^{j\pi/6}$ (A).

Solution:

(a)

$$\widetilde{V} = -5e^{j\pi/3} \text{ V} = 5e^{j(\pi/3-\pi)} \text{ V} = 5e^{-j2\pi/3} \text{ V},$$
$$v(t) = 5\cos(\omega t - 2\pi/3) \text{ V}.$$

(b)

$$\widetilde{V} = j6e^{-j\pi/4} \text{ V} = 6e^{j(-\pi/4+\pi/2)} \text{ V} = 6e^{j\pi/4} \text{ V},$$
$$v(t) = 6\cos(\omega t + \pi/4) \text{ V}.$$

(c)

$$\widetilde{I} = (6 + j8) \text{ A} = 10e^{j53.1^\circ} \text{ A},$$
$$i(t) = 10\cos(\omega t + 53.1^\circ) \text{ A}.$$

(d)

$$\widetilde{I} = -3 + j2 = 3.61\,e^{j146.31^\circ},$$
$$i(t) = \mathfrak{Re}\{3.61\,e^{j146.31^\circ}\,e^{j\omega t}\} = 3.61\,\cos(\omega t + 146.31^\circ) \text{ A}.$$

(e)

$$\widetilde{I} = j = e^{j\pi/2},$$
$$i(t) = \mathfrak{Re}\{e^{j\pi/2}e^{j\omega t}\} = \cos(\omega t + \pi/2) = -\sin\omega t \text{ A}.$$

(f)

$$\widetilde{I} = 2e^{j\pi/6},$$
$$i(t) = \mathfrak{Re}\{2e^{j\pi/6}e^{j\omega t}\} = 2\cos(\omega t + \pi/6) \text{ A}.$$

Problem 1.24 A series RLC circuit is connected to a generator with a voltage $v_s(t) = V_0\cos(\omega t + \pi/3)$ (V).

(a) Write down the voltage loop equation in terms of the current $i(t)$, R, L, C, and $v_s(t)$.

(b) Obtain the corresponding phasor-domain equation.

(c) Solve the equation to obtain an expression for the phasor current $\widetilde{I}$.

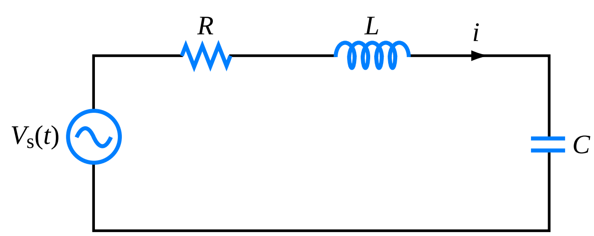


Figure P1.24: RLC circuit.

Solution:

(a) $v_s(t) = Ri + L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\,dt.$

(b) In phasor domain: $\widetilde{V}_s = R\tilde{I} + j\omega L\tilde{I} + \dfrac{\tilde{I}}{j\omega C}$.

(c) $\tilde{I} = \dfrac{\widetilde{V}_s}{R + j(\omega L - 1/\omega C)} = \dfrac{V_0 e^{j\pi/3}}{R + j(\omega L - 1/\omega C)} = \dfrac{\omega C V_0 e^{j\pi/3}}{\omega RC + j(\omega^2 LC - 1)}$.

Problem 1.25 A wave traveling along a string is given by

$$y(x,t) = 2\sin(4\pi t + 10\pi x) \quad \text{(cm)}$$

where x is the distance along the string in meters and y is the vertical displacement. Determine: **(a)** the direction of wave travel, **(b)** the reference phase ϕ_0, **(c)** the frequency, **(d)** the wavelength, and **(e)** the phase velocity.

Solution:

(a) We start by converting the given expression into a cosine function of the form given by (1.17):

$$y(x,t) = 2\cos\left(4\pi t + 10\pi x - \frac{\pi}{2}\right) \quad \text{(cm)}.$$

Since the coefficients of t and x both have the same sign, the wave is traveling in the negative x-direction.

(b) From the cosine expression, $\phi_0 = -\pi/2$.

(c) $\omega = 2\pi f = 4\pi$,

$$f = 4\pi/2\pi = 2 \text{ Hz}.$$

(d) $2\pi/\lambda = 10\pi$,

$$\lambda = 2\pi/10\pi = 0.2 \text{ m}.$$

(e) $u_p = f\lambda = 2 \times 0.2 = 0.4$ (m/s).

Problem 1.26 A laser beam traveling through fog was observed to have an intensity of 1 (μW/m^2) at a distance of 2 m from the laser gun and an intensity of 0.2 (μW/m^2) at a distance of 3 m. Given that the intensity of an electromagnetic wave is proportional to the square of its electric-field amplitude, find the attenuation constant α of fog.

Solution: If the electric field is of the form

$$E(x,t) = E_0 e^{-\alpha x} \cos(\omega t - \beta x),$$

then the intensity must have a form

$$\begin{aligned} I(x,t) &\approx [E_0 e^{-\alpha x} \cos(\omega t - \beta x)]^2 \\ &\approx E_0^2 e^{-2\alpha x} \cos^2(\omega t - \beta x) \end{aligned}$$

or

$$I(x,t) = I_0 e^{-2\alpha x} \cos^2(\omega t - \beta x)$$

where we define $I_0 \approx E_0^2$. We observe that the magnitude of the intensity varies as $I_0 e^{-2\alpha x}$. Hence,

$$\begin{aligned} &\text{at } x = 2 \text{ m}, \qquad I_0 e^{-4\alpha} = 1 \times 10^{-6} \quad \text{(W/m}^2\text{)}, \\ &\text{at } x = 3 \text{ m}, \qquad I_0 e^{-6\alpha} = 0.2 \times 10^{-6} \quad \text{(W/m}^2\text{)}. \end{aligned}$$

$$\frac{I_0 e^{-4\alpha}}{I_0 e^{-6\alpha}} = \frac{10^{-6}}{0.2\times 10^{-6}} = 5$$
$$e^{-4\alpha}\cdot e^{6\alpha} = e^{2\alpha} = 5$$
$$\alpha = 0.8 \quad \text{(NP/m)}.$$

Problem 1.27 Complex numbers z_1 and z_2 are given by

$$z_1 = -3 + j2$$
$$z_2 = 1 - j2$$

Determine **(a)** $z_1 z_2$, **(b)** z_1/z_2^*, **(c)** z_1^2, and **(d)** $z_1 z_1^*$, all all in polar form.

Solution:

(a) We first convert z_1 and z_2 to polar form:

$$\begin{aligned} z_1 = -(3 - j2) &= -\left(\sqrt{3^2+2^2}\, e^{-j\tan^{-1} 2/3}\right) \\ &= -\sqrt{13}\, e^{-j33.7^\circ} \\ &= \sqrt{13}\, e^{j(180^\circ - 33.7^\circ)} \\ &= \sqrt{13}\, e^{j146.3^\circ}. \end{aligned}$$

$$\begin{aligned} z_2 = 1 - j2 &= \sqrt{1+4}\, e^{-j\tan^{-1} 2} \\ &= \sqrt{5}\, e^{-j63.4^\circ}. \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= \sqrt{13}\, e^{j146.3^\circ} \times \sqrt{5}\, e^{-j63.4^\circ} \\ &= \sqrt{65}\, e^{j82.9^\circ}. \end{aligned}$$

(b)

$$\frac{z_1}{z_2^*} = \frac{\sqrt{13}\, e^{j146.3^\circ}}{\sqrt{5}\, e^{j63.4^\circ}} = \sqrt{\frac{13}{5}}\, e^{j82.9^\circ}.$$

(c)

$$\begin{aligned} z_1^2 = (\sqrt{13})^2 (e^{j146.3^\circ})^2 &= 13 e^{j292.6^\circ} \\ &= 13 e^{-j360^\circ} e^{j292.6^\circ} \\ &= 13 e^{-j67.4^\circ}. \end{aligned}$$

(d)

$$z_1 z_1^* = \sqrt{13}\, e^{j146.3^\circ} \times \sqrt{13}\, e^{-j146.3^\circ}$$
$$= 13.$$

Problem 1.28 If $z = 3e^{j\pi/6}$, find the value of e^z.

Solution:

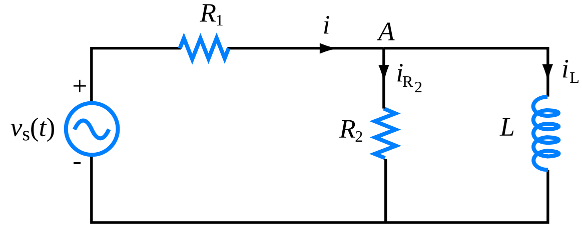
$$z = 3e^{j\pi/6} = 3\cos\pi/6 + j3\sin\pi/6$$
$$= 2.6 + j1.5$$

$$e^z = e^{2.6+j1.5} = e^{2.6} \times e^{j1.5}$$
$$= e^{2.6}(\cos 1.5 + j\sin 1.5)$$
$$= 13.46(0.07 + j0.98)$$
$$= 0.95 + j13.43.$$

Problem 1.29 The voltage source of the circuit shown in the figure is given by

$$v_s(t) = 25\cos(4\times 10^4 t - 45^\circ) \quad \text{(V)}.$$

Obtain an expression for $i_L(t)$, the current flowing through the inductor.



$R_1 = 20\ \Omega$, $R_2 = 30\ \Omega$, $L = 0.4$ mH

Solution: Based on the given voltage expression, the phasor source voltage is

$$\widetilde{V}_s = 25e^{-j45^\circ} \quad \text{(V)}. \tag{9}$$

The voltage equation for the left-hand side loop is

$$R_1 i + R_2 i_{R_2} = v_s \tag{10}$$

For the right-hand loop,

$$R_2 i_{R_2} = L\frac{di_{\mathrm{L}}}{dt} \,, \tag{11}$$

and at node A,

$$i = i_{R_2} + i_{\mathrm{L}}. \tag{12}$$

Next, we convert Eqs. (2)–(4) into phasor form:

$$R_1\widetilde{I} + R_2\widetilde{I}_{R_2} = \widetilde{V}_{\mathrm{s}} \tag{13}$$

$$R_2\widetilde{I}_{R_2} = j\omega L\widetilde{I}_{\mathrm{L}} \tag{14}$$

$$\widetilde{I} = \widetilde{I}_{R_2} + \widetilde{I}_{\mathrm{L}} \tag{15}$$

Upon combining (6) and (7) to solve for $\widetilde{I}_{R_2}$ in terms of $\widetilde{I}$, we have:

$$\widetilde{I}_{R_2} = \frac{j\omega L}{R_2 + j\omega L} I \,. \tag{16}$$

Substituting (8) in (5) and then solving for $\widetilde{I}$ leads to:

$$\begin{aligned}
R_1\widetilde{I} + \frac{jR_2\omega L}{R_2 + j\omega L}\widetilde{I} &= \widetilde{V}_{\mathrm{s}} \\
\widetilde{I}\left(R_1 + \frac{jR_2\omega L}{R_2 + j\omega L}\right) &= \widetilde{V}_{\mathrm{s}} \\
\widetilde{I}\left(\frac{R_1R_2 + jR_1\omega L + jR_2\omega L}{R_2 + j\omega L}\right) &= \widetilde{V}_{\mathrm{s}} \\
\widetilde{I} &= \left(\frac{R_2 + j\omega L}{R_1R_2 + j\omega L(R_1 + R_2)}\right)\widetilde{V}_{\mathrm{s}}.
\end{aligned} \tag{17}$$

Combining (6) and (7) to solve for $\widetilde{I}_{\mathrm{L}}$ in terms of $\widetilde{I}$ gives

$$\widetilde{I}_{\mathrm{L}} = \frac{R_2}{R_2 + j\omega L}\widetilde{I}. \tag{18}$$

Combining (9) and (10) leads to

$$\begin{aligned}
\widetilde{I}_{\mathrm{L}} &= \left(\frac{R_2}{R_2 + j\omega L}\right)\left(\frac{R_2 + j\omega L}{R_1R_2 + j\omega L(R_1 + R_2)}\right)\widetilde{V}_{\mathrm{s}} \\
&= \frac{R_2}{R_1R_2 + + j\omega L(R_1 + R_2)}\widetilde{V}_{\mathrm{s}}.
\end{aligned}$$

Using (1) for $\widetilde{V}_{\text{s}}$ and replacing R_1, R_2, L and ω with their numerical values, we have

$$\begin{aligned}
\widetilde{I}_{\text{L}} &= \frac{30}{20\times 30 + j4\times 10^4 \times 0.4\times 10^{-3}(20+30)}\ 25e^{-j45^\circ} \\
&= \frac{30\times 25}{600+j800}\ e^{-j45^\circ} \\
&= \frac{7.5}{6+j8}\ e^{-j45^\circ} = \frac{7.5e^{-j45^\circ}}{10e^{j53.1^\circ}} = 0.75e^{-j98.1^\circ} \quad \text{(A)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
i_{\text{L}}(t) &= \mathfrak{Re}[\widetilde{I}_{\text{L}} e^{j\omega t}] \\
&= 0.75\cos(4\times 10^4 t - 98.1^\circ) \quad \text{(A)}.
\end{aligned}$$