Exam Solution

Quiz 1.

Problem 1:

$$E \times F = \begin{vmatrix} a_r & a_\phi & a_z \\ -5 & 10 & 3 \\ 1 & 2 & -6 \end{vmatrix} = (-60-6)\,a_r + (3-30)\,a_\phi + (-10-10)\,a_z$$

$$= (-66, -27, -20)$$

$$|E \times F| = \sqrt{66^2 + 27^2 + 20^2} = 74.06$$

(d) is the right choice

Problem 2:

(a) is the right choice : a_r

Problem 3:

(b) is the right choice:

Problem 4:

$$Q = r\sin\phi\, a_r + r^2 z\, a_\phi + z\cos\phi\, a_z$$

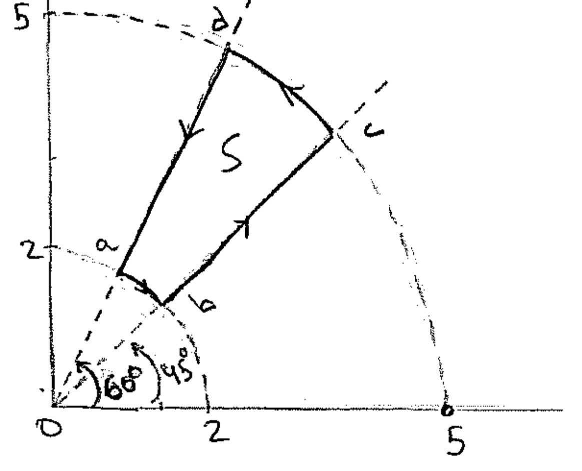
$$\nabla \cdot Q = \frac{1}{r}\frac{\partial}{\partial r}(r Q_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(Q_\phi) + \frac{\partial}{\partial z}(Q_z)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^2 \sin\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(r^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$= 2\sin\phi + \cos\phi$$

(b) is the right choice

Problem 5:



$A = r\cos\phi\, a_r + \sin\phi\, a_\phi$

Let $\oint_L A\cdot dl = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a\right] A\cdot dl$

where path L has been divided into segments ab, bc, cd and da as shown in the figure.

• Along ab, $r = 2$ and $dl = r\,d\phi\, a_\phi$ hence.

$$\int_a^b A\cdot dl = \int_{\phi=60^\circ}^{45^\circ} r\sin\phi\, d\phi = 2(-\cos\phi)\Big|_{60^\circ}^{45^\circ} = -2\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right) = -\sqrt{2}+1$$

• Along bc, $\phi = 45^\circ$ and $dl = dr\, a_r$. Hence.

$$\int_b^c A\cdot dl = \int_{r=2}^{5} r\cos\phi\, dr = \cos 45^\circ \frac{r^2}{2}\Big|_2^5 = \frac{\sqrt{2}}{2}\left(\frac{25}{2} - \frac{4}{2}\right)$$
$$= \frac{\sqrt{2}\times 21}{4}$$

• Along cd, $r = 5$ and $dl = r\,d\phi\, a_\phi$. Hence

$$\int_c^d A\cdot dl = \int_{\phi=45^\circ}^{60^\circ} r\sin\phi\, d\phi = 5(-\cos\phi)\Big|_{45^\circ}^{60^\circ} = -5\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)$$
$$= \frac{-5(1-\sqrt{2})}{2}$$

• Along da, $\phi = 60^\circ$ and $dl = dr\, a_r$. Hence,

$$\int_d^a A\cdot dl = \int_{r=5}^{2} r\cos\phi\, dr = \cos 60^\circ \frac{r^2}{2}\Big|_5^2 = -\frac{21}{4}$$

Putting all these together results in

$$\oint_L A\cdot dl = -\sqrt{2}+1 + \frac{21\sqrt{2}}{4} - \frac{5-5\sqrt{2}}{2} - \frac{21}{4}$$
$$= \frac{-4\sqrt{2}}{4} + \frac{4}{4} + \frac{21\sqrt{2}}{4} - \frac{10-10\sqrt{2}}{4} - \frac{21}{4} = \frac{27\sqrt{2}-27}{4}$$
$$= 2.795$$

Using Stoker's theorem (because L is a closed path)

$$\oint_L A\cdot dl = \int_S (\nabla\times A)\cdot ds$$

But $ds = r\,d\phi\,dr\, a_z$ and

$$\nabla\times A = a_r\left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right] + a_\phi\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + a_z\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right]$$
$$= (0-0)a_r + (0-0)a_\phi + \frac{1}{r}(1+r)\sin\phi\, a_z$$

Hence

$$\int_S (\nabla \times A) \cdot dS = \int_{\phi=45^\circ}^{60^\circ} \int_{r=2}^{5} \frac{1}{r}(1+r)\sin\phi \; r\,dr\,d\phi$$

$$= \int_{45^\circ}^{60^\circ} \sin\phi \, d\phi \int_2^5 (1+r)\,dr$$

$$= -\cos\phi \Big|_{45^\circ}^{60^\circ} \left(r + \frac{r^2}{2}\right)\Big|_2^5$$

$$= -\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)\left(5 + \frac{25}{2} - 2 - \frac{4}{2}\right)$$

$$= \left(\frac{-1+\sqrt{2}}{2}\right)\left(\frac{27}{2}\right) = \frac{-27+27\sqrt{2}}{4} \doteq 2.795$$

$$= \left(\frac{27}{4}\right)(\sqrt{2}-1)$$

(d) is the right choice.

Problem 6:

$$-2 \leqslant x \leqslant 2$$
$$-2 \leqslant y \leqslant 2$$
$$z = 0$$

$$\rho_s = 12\,|y| \;mC/m^2 = 12\times10^{-3}\,|Y|\;C/m^2$$

$$Q = \int_S \rho_s \, ds = \int_{-2}^{2}\int_{-2}^{2} 12\times10^{-3}\,|Y|\,dx\,dy$$

$$= \int_{-2}^{2} 12\times10^{-3}\,dx \left[\int_{-2}^{0} -y\,dy + \int_0^2 y\,dy\right]$$

$$= 12\times10^{-3}\times4\left[-\frac{Y^2}{2}\Big|_{-2}^{0} + \frac{Y^2}{2}\Big|_0^2\right]$$

$$= 12\times4\times10^{-3}\left[\frac{4}{2} + \frac{4}{2} - 0\right]$$

$$= 12\times16\times10^{-3} = 192\,mC$$

(a) is the right choice.

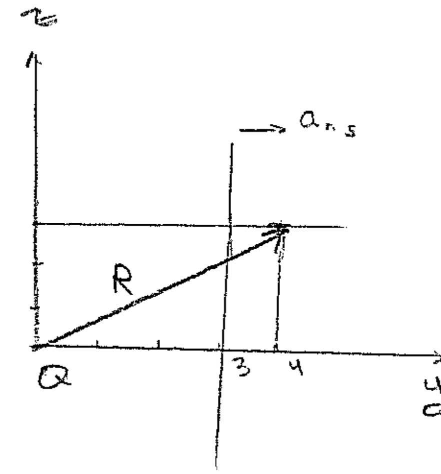
Problem 7:

$Q = 30\,nC$ is at $(0,0,0)$ $\quad Y = 3$ carries $10\,nC/m^2 = \rho_s$

Find D at $(0,4,3)$

at $(0,4,3)$ due to Q: $\vec{R} = 3\vec{a}_z + 4\vec{a}_y$

$$|R| = \sqrt{9+16} = 5$$



so $$\vec{E} = \frac{30\times10^{-9}\,(3\vec{a}_z + 4\vec{a}_y)}{4\pi\times\frac{1}{36\pi}\times10^{-9}\times125}$$

$$D_1 = \varepsilon_0 E = \frac{30\times10^{-9}\,(3\vec{a}_z + 4\vec{a}_y)}{4\pi\times125}$$

due to S_1: $$E = \frac{\rho_s}{2\varepsilon_0}\vec{a}_y = \frac{10\times10^{-9}}{2\varepsilon_0}\vec{a}_y$$

$$D_2 = \varepsilon_0 E = \frac{10}{2} = 5\vec{a}_y$$

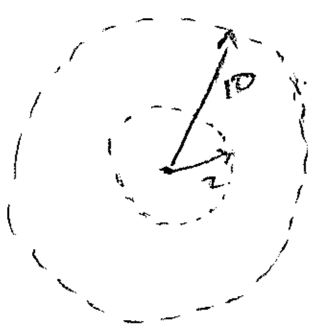
$$D_T = D_1 + D_2 = \frac{90\vec{a}_z + 120\vec{a}_y}{4\pi\times125} + 5\vec{a}_y = 0.0572\,\vec{a}_z + 5.076\,\vec{a}_y\ \frac{nC}{m^2}$$

(c) is the right choice.

Problem 8:

$\rho_v = 2r\ nC/m^3$ $\quad$ for $\quad 0 \leqslant r \leqslant 10\ m$

at $r = 2$; $\quad E = ?$



· By Gauss:

$$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

$$\vec{E} = E\,\vec{a}_x$$

$$\vec{ds} = r^2 \sin\theta\, d\theta\, d\varphi\, \vec{a}_r$$

So $$\int_0^{\pi}\int_0^{2\pi} E\,\vec{a}_r \cdot r^2\sin\theta\, d\theta\, d\varphi = \int_0^{10}\int_0^{\pi}\int_0^{2\pi} \frac{10^{-9}\times 2r\, dV}{\varepsilon_0}$$

$dV = r^2\sin\theta\, dr\, d\theta\, d\varphi$

$$E r^2 \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi = E r^2 (1+1)\cdot 2\pi$$

$$= \frac{1}{\varepsilon_0}\int_0^{10} 10^{-9}\times 2r^3\, dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{2}{\varepsilon_0}\left[\frac{r^4}{4}\Big|_0^{10}\cdot(1+1)\cdot 2\pi\right]$$

hence $$E\times4 = \frac{2}{\varepsilon_0}\left[\frac{10^4}{4}\right] \qquad E = \frac{10^4}{8\varepsilon_0}$$

$$E_r \cdot 4\pi r^2 = \int \frac{\rho_v \, dv}{\varepsilon_0} = \int_0^r \int_0^\pi \int_0^{2\pi} 2r\times10^{-9} \cdot r^3 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \int_0^r 2r^3 dr \times 10^{-9} \int_0^\pi \sin\theta \int_0^{2\pi} d\varphi$$

$$= 2\times10^{-9}\left[\frac{r^4}{4} \times 2 \times 2\pi\right]$$

$$= \frac{2\times10^{-9} \times r^4 \times \pi}{\varepsilon_0}$$

hence $E_r \times 2 = \frac{10^{-9} \times r^2}{\varepsilon_0} \longrightarrow E_r = \frac{r^2\times10^{-9}}{2\times\frac{1}{36\pi}\times10^{-9}} = \frac{r^2}{0.01768}$

$$= \frac{4}{0.01768} = 226.24 \, \vec{a}_r \ V/m.$$

(b) is the right choice.

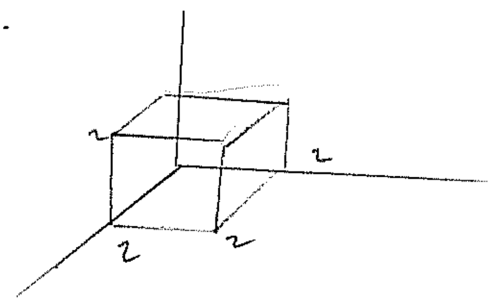
Problem 9:

$V = x - y + xy + 2z$ $\qquad$ E at (1,2,3)??

$E = -\nabla V = -\left[[1+y]\vec{a}_x + (-1+x)\vec{a}_y + 2\vec{a}_z \right]$

E at $(1,2,3) = -(3\vec{a}_x + 0\vec{a}_y + 2\vec{a}_z) = -3\vec{a}_x - 2\vec{a}_z$

(d) is the right choice.



Problem 10:

Since a_z is normal to the boundary plane, we obtain the normal components as:

$E_{1n} = E_1 \cdot \vec{a}_n = E_1 \cdot \vec{a}_z = 3$ $\qquad$ $E_{1n} = 3\vec{a}_z$

$E_{2n} = (E_2 \cdot \vec{a}_z)\vec{a}_z$ $\qquad$ also $\quad E = E_n + E_t$

hence $\quad E_{1t} = E_1 - E_{1n} = 5\vec{a}_x - 2\vec{a}_y$

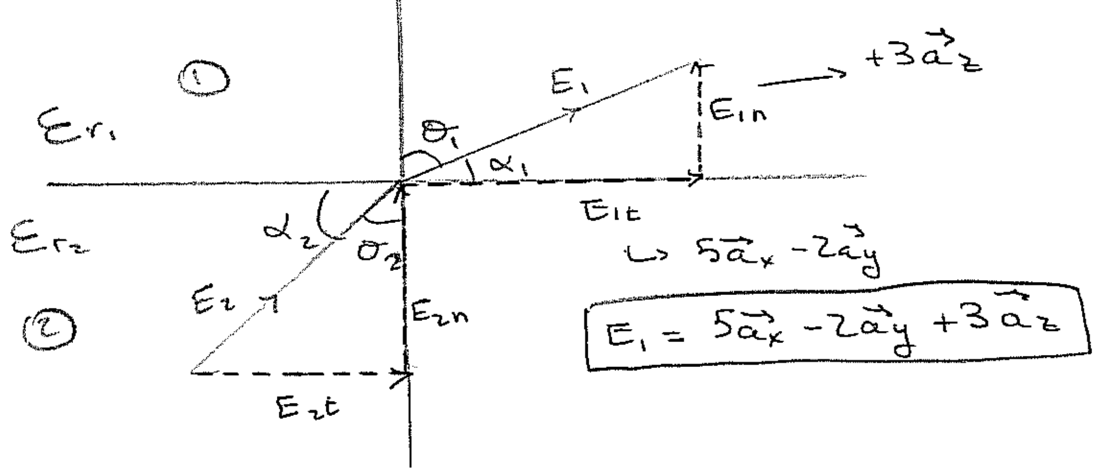
Thus $\quad E_{2t} = E_{1t} = 5\vec{a}_x - 2\vec{a}_y$

similarly $\quad D_{2n} = D_{1n} \longrightarrow \varepsilon_{r2} E_{2n} = \varepsilon_{r1} E_{1n}$

$$E_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{1n} = \frac{6}{3}(3\vec{a}_z) = 6\vec{a}_z$$

Thus $\quad E_2 = E_{2t} + E_{2n} = 5\vec{a}_x - 2\vec{a}_y + 6\vec{a}_z$ K V/m

(d) is the right choice



Problem 11:

$B(-4,1,5)$ is in the dielectric medium since $y=1>0$ at B

$D_n = \rho_s = 2\,nC/m^2$

hence $D = 2a_y \; nC/m^2$

and $E = \dfrac{D}{\varepsilon_0 \varepsilon_r} = 2\times10^{-9}\times\dfrac{36\pi}{4}\times10^{9} a_y = 18\pi\,\vec{a}_y \; V/m = 56.55\; V/m$

ⓐ is the right choice.

Problem 12:

$\rho_v(R) = a + bR$

$\rho_v(0) = a = 0$

$\rho_v(2) = 2b = 10$

hence $b = 5$ $\qquad \rho_v(R) = 5R \;(C/m^3)$

Applying Gauss's law to a spherical surface of radius R

$$\oint_S D \cdot ds = \int_v \rho_v \, dv$$

$$D_R \cdot 4\pi R^2 = \int_0^R 5R \cdot 4\pi R^2 dR = 20\pi \frac{R^4}{4}$$

$$D_R = \frac{5}{4} R^2 \;(C/m^2)$$

$$D = \hat{R} D_R = \hat{R}\,\frac{5}{4} R^2 \;(C/m^2)$$

ⓐ is the right choice