1. Maxwell's equations

Modern electromagnetism is based on a set of four fundamental relations known as Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho_v,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

where $\mathbf{E}$ and $\mathbf{D}$ are the electric quantities interrelated by $\mathbf{D} = \varepsilon \mathbf{E}$, with ε being the electric permittivity of the material; $\mathbf{B}$ and $\mathbf{H}$ are magnetic field quantities interrelated by $\mathbf{B} = \mu \mathbf{H}$, with μ being the magnetic permeability of the material; ρ_v is the electric charge density per unit volume; and $\mathbf{J}$ is the current density per unit area.

In the static case, none of the quantities appearing in Maxwell's equations are a function of time (i.e., $\partial/\partial t = 0$). This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that ρ_v and $\mathbf{J}$ are constant in time. Under these circumstances, the time derivatives of $\mathbf{B}$ and $\mathbf{D}$ in the above equations are zero, and the Maxwell's equations reduce to:

Electrostatics:

$$\nabla \cdot \mathbf{D} = \rho_v,$$

$$\nabla \times \mathbf{E} = 0,$$

Magnetostatics:

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Maxwell's four equations separate into two uncoupled pairs, with the first involving only the electric field quantities $\mathbf{E}$ and $\mathbf{D}$ and the second pair involving only the magnetic field quantities $\mathbf{B}$ and $\mathbf{H}$. The electric and magnetic fields are no longer interconnected in the static case. This allows us to study electricity and magnetism as two distinct and separate phenomena, as long as the spatial distributions of a charge and current flow remain constant in time.

We define the volume charge density ρ_v as

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad (\mathrm{C}/m^3),$$

where Δq is the charge contained in Δv. In general ρ_v is defined at a given point in space, specified by (x, y, z) in a Cartesian coordinate system, and at a given time t. Thus, $\rho_v = \rho_v(x, y, z, t)$. Physically, ρ_v represents the average charge per unit volume for a volume Δv centered at (x, y, z). The total charge contained in a given volume v is given by

$$Q = \int_v \rho_v dv \ \ \text{(C)}.$$

In some cases, the surface charge density ρ_s is defined as

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \ \ (\text{C}/m^2),$$

where Δq is the charge present across an elemental surface area Δs. Similarly, if the charge is distributed along a line, which need not to be straight, we characterize the distribution in terms of the line charge density ρ_l, defined as

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \ \ (\text{C}/m).$$

For electrostatics in free space, only the electric field intensity E is needed. E is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists:

$E=\lim_{q\to 0} (F/q)$ (V/m).

E-is in the direction of F.

Alternatively: F=qE (N)

The two fundamental postulates of electrostatics in free space specify the divergence and the curl of E:

$$\nabla.E=\rho_v/\varepsilon_o \quad \text{(in free space) \& } \nabla\times E=0 \text{ where } \rho_v \text{ in c/m}^3.$$

The curl of E =0 means that a static electric fields in irrotational, & $\nabla.E=\rho_v/\varepsilon_o$ implies that this field is not solenoidal unless $\rho_v=0$.

The above equation is point relations; i.e. they hold at every point in space taking the volume ρ:

$$\oint_v \nabla .E\, dv=1/\varepsilon_o \oint_v \rho_v dv$$

This can be written as $\oint_s$ E.ds =Q/ε_o (gauss's law) where Q-total charge in V integrating ∇×E over an open surface &applying Stokes 's theorem:

$\oint_c$ E.dl=0 where C is an arbitrary closed contour. This means that the line integral of a static electric field around any closed path vanishes, which is equivalent to KVL in circuit theory.

Summary: the two fundamental postulates of electrostatics in free space are

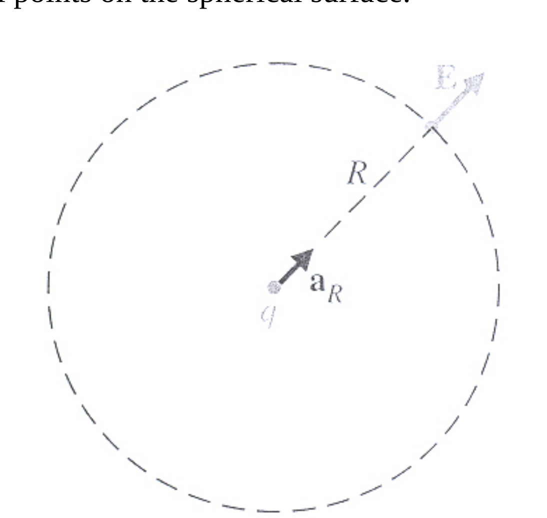
Differential form	Integral form
$\nabla.E=\rho_v/\varepsilon_o$ $\nabla\times E=0$	$\oint_s$ E.ds= Q/ε_o $\oint_s$ E.dl=0

2. Coulomb's law

Consider a single point charge q, at rest in a boundless free space. Find the electric field intensity due to q.

For this, we draw a spherical surface of an arbitrary radius R centered at q(a hypothetical
Enclosed surface –Gaussian surface)

Since a point charge has no directions, its electric field must be everywhere radial as the same intensity at all points on the spherical surface.



$\oint_s \vec{E}.\vec{ds} = \oint_s (a_R E_R).a_R \, ds = q/\varepsilon_o$.

Or $E_R \oint_s ds = q/\varepsilon_o \rightarrow E_R \, (4\pi R^2) = q/\varepsilon_o$.

Therefore: $\vec{E}._R = a_R.E_R = a_R \, q/(4\pi R^2 \varepsilon_o)$ V/m

Conclusion: the electric field intensity of a positive point charge is in the outward radial direction. It has a magnitude proportional to q and inv. proportional to R^2.

For a material with electric permittivity ε, the electrical field quantities **E** and **D** are related by

$$\mathbf{D} = \varepsilon \mathbf{E}$$

with

$$\varepsilon = \varepsilon_r \varepsilon_0$$

where

$$\varepsilon_0 = 8.85 \times 10^{-12} \approx (1/36\pi) \times 10^{-9}$$

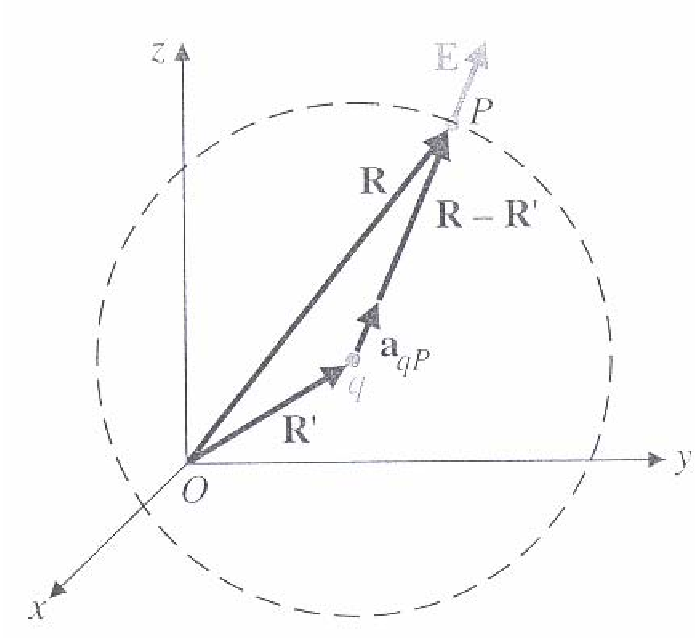
is the electric permittivity of free space, and $\varepsilon_r = \varepsilon / \varepsilon_0$ is called the relative permittivity (or dielectric constant) of the material. For most materials and under most conditions, ε of the material has a constant value independent of both the magnitude and direction of **E**, then the material is said to be linear because **D** and **E** are related linearly, and if it is independent of the direction of **E**, the material is said to be isotopic.

If the charge q is not located at the origin of a chosen coordinate system, suitable changes should be made to $\vec{a}_R$ and R. Let the position vector of q be R' and that of a field point p be R. From the above derived equation

$$\overrightarrow{E_p} = \overrightarrow{a_{pq}} \frac{q}{4\pi\varepsilon_0 \left| \vec{R} - \vec{R}' \right|^2},$$

where $\overrightarrow{a_{pq}}$ is the unit vector drawn from q to p. Since $a_{qp} = (\vec{R} - \vec{R}')/\left| \vec{R} - \vec{R}' \right|$. We have:

$$\overrightarrow{E_p} = \frac{q(\vec{R} - \vec{R}')}{4\pi\varepsilon_0 \left| \vec{R} - \vec{R}' \right|^3} \;\; (V/m) \qquad (1)$$



Example: Determine the electric field intensity at p(-0.2,0,-2.3) due to a point charge of +5(nC) at Q(0.2,0.1,-2.5) in air. All dimensions are in meters.

The position vector for p is:

$\vec{R} = \overrightarrow{op}$ =-a_x0.2-a_z2.3

and $\vec{R}$' is the position vector for Q:

$\vec{R}$'= $\overrightarrow{OQ}$ =a_x0.2+a_y0.1-a_z2.5

$\vec{R}$ - $\vec{R}$'=-a_x0.4-a_y0.1+a_z0.2

$$\left|\vec{R}-\vec{R}'\right| = \sqrt{\left[(-0.4)^2 + (-0.1)^2 + (0.2)^2\right]} = 0.458\ (m).$$

Substituting in (1) above:

$$\overrightarrow{E_p} = \frac{q(\vec{R}-\vec{R}')}{4\pi\varepsilon_0\left|\vec{R}-\vec{R}'\right|^3} = 214.5(-a_x 0.873 - a_y 0.218 + a_z 0.437)\ (\mathrm{V/m})$$

The quantity in within the parentheses is ($\vec{R}$ - $\vec{R}$')/| $\vec{R}$ - $\vec{R}$' | = $\overline{a_{pq}}$

The magnitude of $\vec{E}_p$ is 214.5

When a point charge q_2 is placed in the field of another point charge q_1,a force F_{12} is experienced by q_2 due to electric field intensity $\vec{E}_{12}$ of q_1 at q_2.

We have:

$$F_{12} = q_2 E_{12} = q_{12} \frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2} \ (N)$$

Coulomb's law: the force between 2 points charges is proportional to $q_1 q_2$ and inversely proportional to R^2.

Example: point charges 1 mC and – 2 mC are located at (3,2,-1) and (-1,-1,4) respectively. Calculate the electric force on a 10nC located at (0,3,1) and E at that point.

$\vec{F} = \sum_{k=1,2} Q\, Q_k\, a_R / 4\pi\varepsilon_o R^2 = \sum_{k=1,2} Q\, Q_k (\vec{r} - \vec{r}_k) / 4\pi\varepsilon_o |r - r_k|^3$

$\vec{F} = Q/4\pi\varepsilon_o \{10^{-3}[(0,3,1)-(3,2,1)]/|(0,3,1)-(3,2,-1)|^3$
$-2.10^{-3}[(0,3,1)-(-1,-1,4)]/|(0,3,1)-(-1,-1,4)|^3 \}$

$\vec{F} = 9.10^{-2}[(-3,1,2)/14\sqrt{14} + (-2,8,6)/26\sqrt{26}]$.

$\vec{F} = -6.507\vec{a}_x - 3.81\vec{a}_y + 7.506\vec{a}_z$ mN.

At that point:
$\vec{E} = \vec{F}/Q = \sum_{k=1,2} Q_k(\vec{r} - \vec{r}\,') / 4\pi\varepsilon_o |r - r_k|^3 = (-6.507,-3.81,7.506)10^{-3}/10\times10^{-9}$
$\vec{E} = -650,7\vec{a}_x - 381.7\vec{a}_y + 750.6\vec{a}_z$ kv/m.

3. Electric field due to a system of discrete charges.

Suppose the electrostatic field is created by n discrete point charges. Since the E is a linear function of $a_R\, q/R^2$, the principle of superposition applies and the total E field is vector sum of E_s of individual q_s.

Let the positions of q_1,q_2, q_3, ------q_n be noted by positions vectors R_1',R_2',------R_n' and let the position of the field point at which E is to be calculated be denoted by R.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1(\vec{R} - \vec{R_1}')}{\left|\vec{R} - \vec{R_1}'\right|^3} + \frac{q_2(\vec{R} - \vec{R_2}')}{\left|\vec{R} - \vec{R_2}'\right|^3} + \dots\dots\dots + \frac{q_n(\vec{R} - \vec{R_n}')}{\left|\vec{R} - \vec{R_n}'\right|^3}\right]$$

or

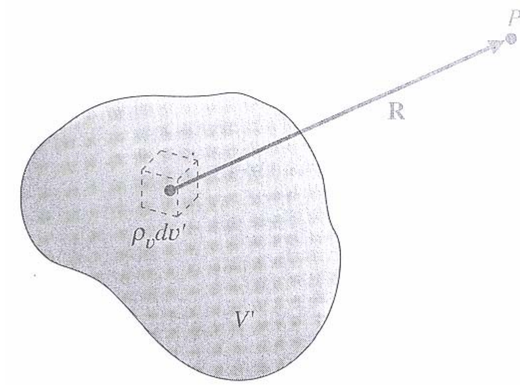
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k(\vec{R} - \vec{R_k}')}{\left|\vec{R} - \vec{R_k}'\right|^3} \quad (V/m).$$

A better approach will be discussed later.

4. Electric field due to a continuous distribution of charge

Here the electric field can be obtained by integrating (superposing) the contribution of an element of charge over the charge distribution.

The contribution of the volume charge ρ_v dV' in a differential volume element dv' to the electric field intensity at the field point P is:



$$dE = a_R \frac{\rho_v dv'}{4\pi\varepsilon_0 R^2}$$, so

$$E = \frac{1}{4\pi\varepsilon_0} \int_{v'} a_R \frac{\rho_v}{R^2} dv' \ \ (V/m).$$

If the charge is distributed on a surface with ρ_s (C/m^2),

$$E = \frac{1}{4\pi\varepsilon_0} \int_{s'} a_R \frac{\rho_s}{R^2} ds' \ \ (V/m).$$ where Q= $\int \rho_s \, ds$

and for a line charge:

$$E = \frac{1}{4\pi\varepsilon_0} \int_{L'} a_R \frac{\rho_l}{R^2} dl' \ \ (V/m).$$

ρ_l (C/m) is the charge density and Q= $\int \rho_L \, dL$.