1. Overview:

The magnetic fields in a medium with magnetic permeability μ are governed by the second pair of Maxwell's equations,

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0,$$

$$\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}.$$

where $\vec{\mathbf{J}}$ is the current density. The magnetic flux density $\vec{\mathbf{B}}$ and the magnetic field intensity $\vec{\mathbf{H}}$ are related by

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$$

With the exception of ferromagnetic materials, for which the relation ship between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ in nonlinear, most materials are characterized by constant magnetic permeability. Furthermore, $\mu = \mu_o$ for most dielectrics and metals (excluding ferromagnetic materials)

Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges	Steady currents
Fields	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

2. Magnetic Forces and Torques:

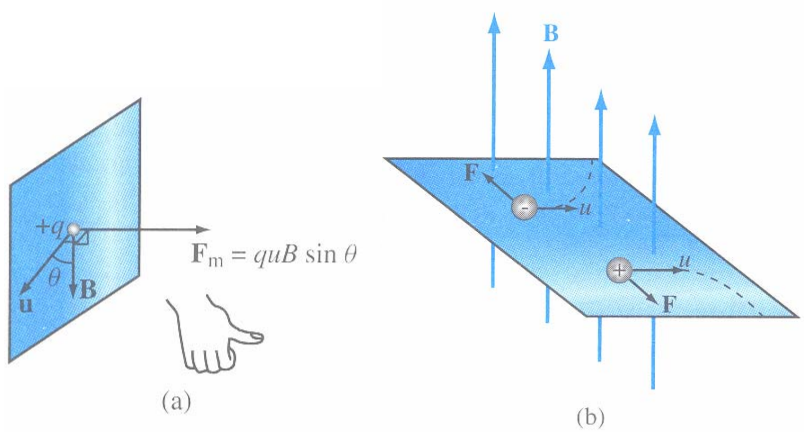
The magnetic flux density $\vec{\mathbf{B}}$ at a point in space in terms of the magnetic force $\overrightarrow{\mathbf{F}_{\mathrm{m}}}$ that would be exerted on a charged particle moving with a velocity $\vec{\mathbf{u}}$ were it to be passing through that point. The magnetic force $\overrightarrow{\mathbf{F}_{\mathrm{m}}}$ acting on a particle of charge q can be cast in the form

$$\overrightarrow{\mathbf{F}_{\mathrm{m}}} = q\vec{\mathbf{u}} \times \vec{\mathbf{B}} \qquad (N)$$

For a positively charged particle, the direction of $\overrightarrow{\mathbf{F}_{\mathrm{m}}}$ is in the direction of the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{B}}$, which is perpendicular to the plane containing $\vec{\mathbf{u}}$ and $\vec{\mathbf{B}}$ and governed by the right-hand rule. The magnitude of $\overrightarrow{\mathbf{F}_{\mathrm{m}}}$ is given by

$$F_m = q\,uB\,\sin\theta, \qquad (N)$$

where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{B}}$. We note that F_{m} is maximum when $\vec{\mathbf{u}}$ is perpendicular to $\vec{\mathbf{B}}$ ($\theta = 90^{\circ}$), and it is zero when $\vec{\mathbf{u}}$ is parallel to $\vec{\mathbf{B}}$ ($\theta = 0$ or 180°).



The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both $\vec{\mathbf{B}}$ and $\vec{\mathbf{u}}$ and (b) depends on the charge polarity (positive or negative).

If a charged particle is in the presence of both electric field $\vec{\mathbf{E}}$ and a magnetic field $\vec{\mathbf{B}}$, the total electromagnetic force acting on it is

$$\overrightarrow{\mathbf{F}_{\mathbf{e}}} + \overrightarrow{\mathbf{F}_{\mathbf{m}}} = q\vec{\mathbf{E}} + q\vec{\mathbf{u}} \times \vec{\mathbf{B}} = q(\vec{\mathbf{E}} + \vec{\mathbf{u}} \times \vec{\mathbf{B}})$$

The force expressed in this equation is known as the Lorentz force. Electric and Magnetic forces exhibit a number of important differences:

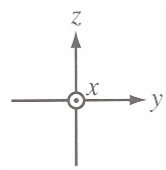
1. Whereas the electric force is always in the direction of the electric field, the magnetic force is always perpendicular to the magnetic filed.
2. Whereas the electric force acts on a charged particle whether or not it is moving, the magnetic force acts on it only when it is in motion
3. Whereas the electric force expends energy in displacing a charged particle, the magnetic force does no work when a particle is displaced.

Because the magnetic force $\overrightarrow{\mathbf{F}_\mathrm{m}}$ is always perpendicular to $\vec{\mathbf{u}}$, $\overrightarrow{\mathbf{F}_\mathbf{m}} \cdot \vec{\mathbf{u}} = 0$. Hence, the work performed when a particle is displaced by a differential distance $\overrightarrow{d\,\mathbf{l}} = \vec{\mathbf{u}}\,dt$ is

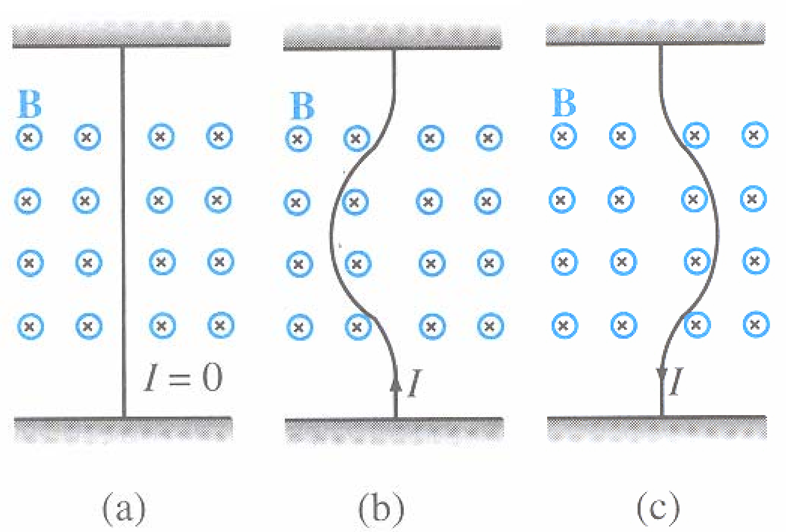
$$d\,W = \overrightarrow{\mathbf{F}_\mathbf{m}} \cdot \overrightarrow{d\,\mathbf{l}} = \left(\overrightarrow{\mathbf{F}_\mathbf{m}} \cdot \vec{\mathbf{u}}\right)dt = 0$$

Since no work is done, a magnetic field cannot change the kinetic energy of a charged particle; the magnetic field can change the direction of motion of a charged particle, but it cannot change its speed.

2.1. Magnetic Force on a current-Carrying Conductor



Consider the arrangement shown in figure in which a vertical wire oriented along the *z*-direction is placed in a magnetic field $\vec{\mathbf{B}}$ (produced by a magnet) oriented along the $-\hat{\mathbf{x}}$-direction (into the page).



When a slightly flexible vertical is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when ***I*** is upward, and (c) deflected to the right when ***I*** is downward.

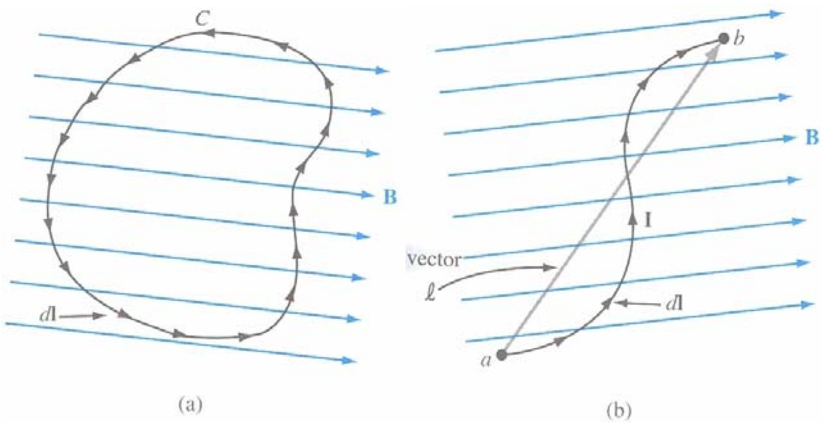
When a current is introduced in the wire, the wire deflects to the left ($-\hat{\mathbf{y}}$-direction) if the current's direction is upward ($+\hat{\mathbf{z}}$-direction), and it deflects to the right ($+\hat{\mathbf{y}}$-direction) if the current's direction is downward ($-\hat{\mathbf{z}}$-direction).
For a closed circuit of contour ***C*** carrying a current ***I***, the total magnetic force is

$$\overrightarrow{\mathbf{F_m}} = I\oint_C \overrightarrow{d\,\mathbf{l}} \times \overrightarrow{\mathbf{B}} \qquad (N) \qquad (*)$$

Closed Circuit in a Uniform $\vec{\mathbf{B}}$ Field

Consider a closed wire carrying a current ***I*** and placed in a uniform external magnetic field B, as shown in figure. Since $\vec{\mathbf{B}}$ is constant, it can be taken outside the integral of equation $(*)$, in which case we have

$$\overrightarrow{\mathbf{F_m}} = I\left(\oint_C \overrightarrow{d\,\mathbf{l}}\right) \times \vec{\mathbf{B}} = 0$$



In a uniform magnetic field, (a) the net force on a closed current loop is zero because the integral of the displacement vector $d\mathbf{l}$ over a closed contour is zero, and (b) the force on a line segment is proportional to the vector between the end point ($\overrightarrow{\mathbf{F_m}} = I\vec{\ell} \times \overrightarrow{\mathbf{B}}$).

So, the total magnetic force on any closed current loop in a uniform magnetic field is zero.

Curved Wire in a Uniform B Field

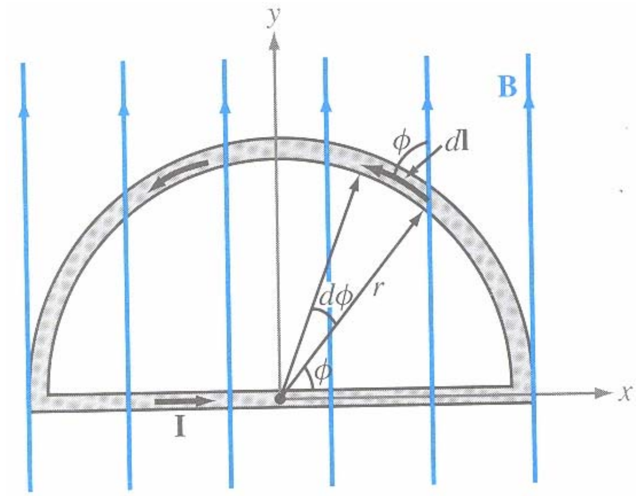
If we are interested in the magnetic force exerted on wire segment, such shown in the figure (b), when placed in a uniform field B, then equation $(*)$ becomes

$$\overrightarrow{\mathbf{F_m}} = I\left(\int_a^b \overrightarrow{d\,\mathbf{l}}\right) \times \vec{\mathbf{B}} = I\vec{\ell} \times \vec{\mathbf{B}},$$

where $\vec{\ell}$ is the vector directed from *a* to *b*, as shown in figure (b). The integral of $d\vec{\mathbf{l}}$ from *a* to *b* has the same value irrespective of the path taken between *a* and *b*.

Example: Force on a Semicircular Conductor

The semicircular conductor shown in the figure lies in the *x-y* plane and caries a current $\boldsymbol{I}$. The closed circuit is exposed to a uniform magnetic field $\vec{\mathbf{B}} = \hat{\mathbf{y}} B_o$.



Semicircular conductor in a uniform field

Determine,

a) The magnetic force $\vec{\mathbf{F}}_1$ on the straight section of the wire.
b) The force $\vec{\mathbf{F}}_2$ on the curved section.

Solution:

a) The straight section of the circuit is of length 2*r*, and flowing through it is along the $+\hat{\mathbf{x}}$-direction. With $\vec{\ell} = \hat{\mathbf{x}}\ 2r$ we will have,

$$\vec{\mathbf{F}}_1 = \hat{\mathbf{x}}\left(2\boldsymbol{I}\, r\right) \times \hat{\mathbf{y}} B_o = \hat{\mathbf{z}}\, 2\boldsymbol{I}\, r B_o \qquad (N)$$

b) Let us consider a segment of differential length $d\vec{\mathbf{l}}$ on the curved part of the circle. The direction of $d\vec{\mathbf{l}}$ is chosen to coincide with the direction of the current. Since $d\vec{\mathbf{l}}$ and $\vec{\mathbf{B}}$ are both in the *x-y* plane, their cross product $d\vec{\mathbf{l}} \times \vec{\mathbf{B}}$ points in the negative *z*-direction, and the magnitude of $d\vec{\mathbf{l}} \times \vec{\mathbf{B}}$ is proportional to $\sin\varphi$, where φ is the angle between $d\vec{\mathbf{l}}$ and $\vec{\mathbf{B}}$. Moreover, the magnitude of $d\vec{\mathbf{l}}$ is $dl = r\, d\varphi$.

Hence,

$$\overrightarrow{\mathbf{F}_2} = I\int_{\varphi=0}^{\pi} \overrightarrow{d\,\mathbf{l}} \times \overrightarrow{\mathbf{B}}$$

$$= -\hat{\mathbf{z}}\,\boldsymbol{I}\int_{\varphi=0}^{\pi} r\,B_o \sin\phi\, d\phi = -\,\hat{\mathbf{z}}\,2\boldsymbol{I}rB_o \qquad (N)$$

We note that $\overrightarrow{\mathbf{F}_2} = -\overrightarrow{\mathbf{F}_1}$, and consequently the net force on the closed loop is zero.