AMERICAN UNIVERSITY OF BEIRUT

MATH 101 — Calculus & Analytic Geometry I

Fall Semester 2013–14

Mid-term Exam 1

Saturday 5 October, 2013

Time: 1 hour

Name: ____ ID: ____________________

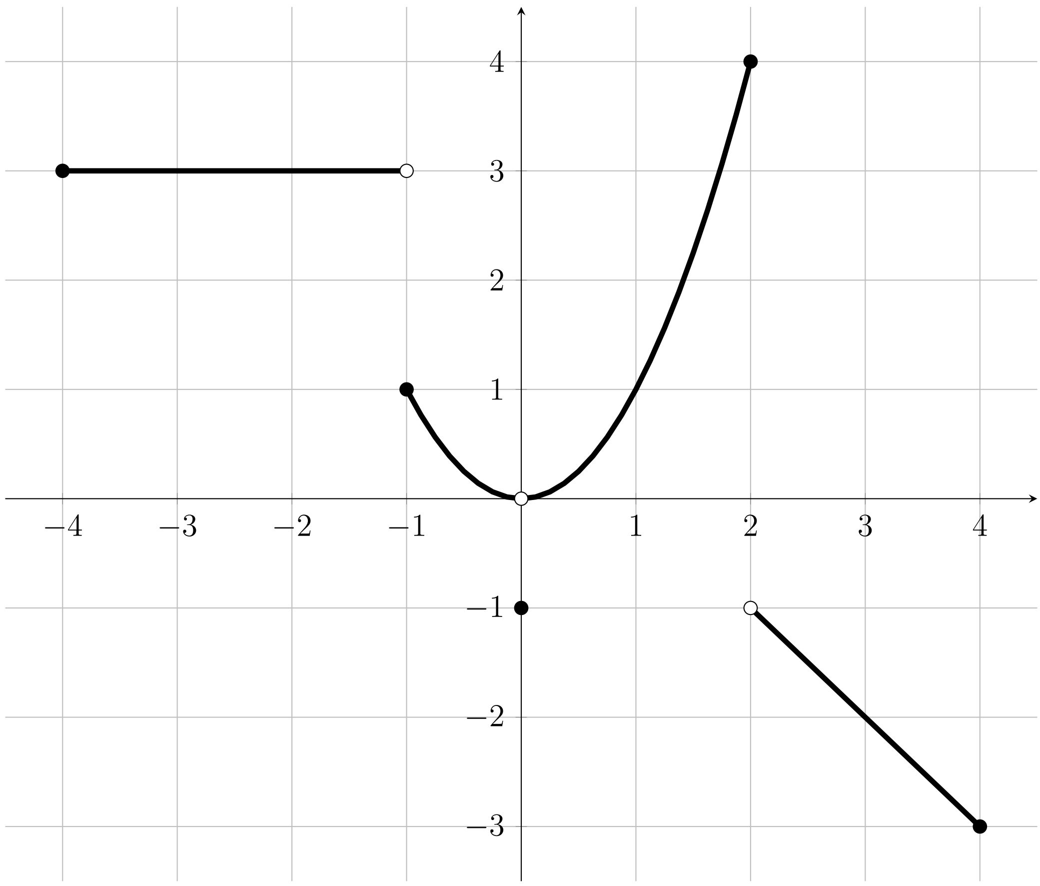
Circle your recitation section:

1 Fri 9am	2 Fri 10am	3 Fri 4pm	4 Fri 11am	5 Thurs 2pm
6 Thurs 11am	7 Thurs 12:30pm	8 Mon 8am	9 Mon 9am	10 Mon 2pm

The backs of the pages may be used as scratch paper.
Please **do not open this paper** until you are instructed to do so.

Question:	1	2	3	4	5	6	Total
Points:	6	15	10	6	9	14	60
Score:							

1. Here is the graph of a piecewise-defined function $f(x)$.



(a) (4 points) Fill in the blanks:

$$\lim_{x \to -1^-} f(x) = 3 \qquad \lim_{x \to -1^+} f(x) = 1$$

$$\lim_{x \to 4^-} f(x) = -3 \qquad \lim_{x \to 0} f(x) = 0$$

(b) (2 points) Is f continuous at $x = 0$? Justify your answer.

Solution: We have $\lim_{x \to 0} f(x) = 0$, but $f(0) = -1$. So f is not continuous at $x = 0$.

2. (15 points) Calculate the following limits. Show your working.

(a) $\lim\limits_{x\to 0} \dfrac{\cos x + 1}{5x - 2}$

Solution:

$$\lim_{x\to 0} \frac{\cos x + 1}{5x - 2} = \frac{\lim_{x\to 0} \cos x + \lim_{x\to 0} 1}{\lim_{x\to 0} 5x - \lim_{x\to 0} 2} = \frac{1+1}{0-2} = -1.$$

(b) $\lim\limits_{x\to 2} \sqrt{\dfrac{x^2 - 3x + 2}{x^2 - 4}}$

Solution:

$$\lim_{x\to 2} \sqrt{\frac{x^2 - 3x + 2}{x^2 - 4}} = \lim_{x\to 2} \sqrt{\frac{(x-2)(x-1)}{(x-2)(x+2)}} = \lim_{x\to 2} \sqrt{\frac{x-1}{x+2}} = \frac{1}{2}.$$

(c) $\lim\limits_{x\to \infty} \dfrac{2x + 1}{x^2 - 9}$

Solution:

$$\lim_{x\to \infty} \frac{2x+1}{x^2 - 9} = \lim_{x\to \infty} \frac{2/x^2 + 1/x}{1 - 9/x^2} = \frac{0}{1} = 0.$$

(d) $\lim\limits_{x\to 7} \dfrac{\sqrt{x+2} - 3}{x - 7}$

Solution:

$$\lim_{x\to 7} \frac{\sqrt{x+2} - 3}{x - 7} = \lim_{x\to 7} \frac{(\sqrt{x+2} - 3)(\sqrt{x+2} + 3)}{(x-7)(\sqrt{x+2} + 3)}$$
$$= \lim_{x\to 7} \frac{x-7}{(x-7)(\sqrt{x+2} + 3)} = \lim_{x\to 7} \frac{1}{\sqrt{x+2} + 3} = \frac{1}{6}.$$

(e) $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{x^4 + x}$

Solution:

$$\lim_{x\to 0} \frac{\sin(3x)}{x^4 + x} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \frac{3x}{x^4 + x} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \lim_{x\to 0} \frac{3}{x^3 + 1} = 1 \times 3 = 3.$$

3. (a) (8 points) Complete the following table by putting a tick (check mark) ✓ or a cross × in each empty box to indicate whether each function is: even; odd; increasing; decreasing.

Function	even	odd	increasing	decreasing
$4\sin x$	×	✓	×	×
7	✓	×	×	×
$x + \lfloor x \rfloor$	×	×	✓	×
$\lvert 2x+1 \rvert$	×	×	×	×

(b) (2 points) If f is an increasing function and g is a decreasing function, does the composition $f \circ g$ have to be increasing or decreasing? Justify your answer.

Solution: For any numbers x_1 and x_2 with $x_1 > x_2$, we have $g(x_1) < g(x_2)$ because g is decreasing, and then $f(g(x_1)) < f(g(x_2))$ because f is increasing. So $f \circ g$ is decreasing.

4. (6 points) Use the Intermediate Value Theorem to show that the equation

$$\cos x = x$$

has a solution between 0.7 and 0.8.

Solution: The equation can be re-arranged to $f(x) = \cos x - x = 0$. Now

$$f(0.7) = \cos(0.7) - 0.7 = 0.065\ldots \quad \text{and} \quad f(0.8) = \cos(0.8) - 0.8 = -0.103\ldots.$$

Since f is a continuous function, and 0 lies between -0.103 and 0.065, the Intermediate Value Theorem implies that there is some $c \in [0.7, 0.8]$ satisfying $f(c) = 0$, that is, $\cos c = c$.

5. (a) (5 points) Consider the function f defined as follows.

$$f(x) = \begin{cases} \lfloor x \rfloor + 1 & \text{if } x < 2; \\ x^2 - x & \text{if } x \geq 2. \end{cases}$$

Is the function $f(x)$ continuous at $x = 2$? Justify your answer.

Solution: If x is slightly smaller than 2, then we have

$$f(x) = \lfloor x \rfloor + 1 = 1 + 1 = 2.$$

Therefore $\lim_{x\to 2^-} f(x) = 2$. On the other hand,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 - x) = 2$$

because $x^2 - x$ is continuous. So we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 2$$

and therefore the function is continuous at $x = 2$.

(b) (4 points) Suppose that f and g are continuous functions with $f(2) = 5$ and

$$\lim_{x\to 2} \big(4f(x) - g(x)\big) = 11.$$

Find $g(2)$, and justify your answer.

Solution: Because f and g are continuous, we have

$$\lim_{x\to 2} f(x) = f(2) = 5 \qquad \text{and} \qquad \lim_{x\to 2} g(x) = g(2).$$

The limit laws give

$$\lim_{x\to 2} \big(4f(x) - g(x)\big) = 4\lim_{x\to 2} f(x) - \lim_{x\to 2} g(x) = 20 - \lim_{x\to 2} g(x).$$

So $20 - \lim_{x\to 2} g(x) = 11$, and therefore $\lim_{x\to 2} g(x) = 9$.

6. (14 points) Indicate whether each of the following statements is true or false. You receive two points for each correct answer.

T **[F]** The line $y = 3$ is a horizontal asymptote to the graph $y = \dfrac{6x^2 - x^4 - 5}{2x^4 + 3}$.

[T] **F** The function $f(x) = \lfloor |x - 2| \rfloor$ is continuous at $x = 2$.

[T] **F** The function $f(x) = \sin(x - \pi/2)$ is an even function.

T **[F]** If $f(3) = 5$ and $g(5) = 2$, then $(f \circ g)(3) = 2$.

[T] **F** If f is a function satisfying $|f(x)| \leq 2$ for all x, then

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0.$$

T **[F]** The formula

$$\sin(3\theta) = \sin^3\theta - 3\sin\theta\cos^2\theta$$

is true for all θ.

T **[F]** The domain of the function $f(x) = \sqrt{\cos x}$ is $[-\pi/2, \pi/2]$.