Solution

Version 2

American University of Beirut
Mathematics Department
Spring Semester 2011-2012
Math 101 Quiz I

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Name : ______________________ **ID#:** ________________________

Circle your problem solving section:

Section 1 : M @8:00 **Section 2 :** M @ 9:00 **Section 3 :** M @ 11:00

Answer table for Part I

1	b		6	b
2	d		7	c
3	a		8	a
4	a		9	b
5	b		10	a

# of correct answers : ----------------------- # of wrong answers : -------------------------					**Grade of Part I** 30%
1.	2.	3.	4.	5.	**Grade of Part II** 70%

Final Grade

Part I **(30%) 10 multiple choices with 3 points for each correct answer and -0.5 penalty on each wrong answer.**

Circle the correct answer then, copy your answers as a, b, c or d on the table provided on page 1:

1. The graph of the function is the graph of $F(x)=\sqrt{x+1}-3$ is the graph of $f(x)=\sqrt{x}$ Shifted :

   a. 1 unit to the right and 3 units down b. 1 unit to the left and 3 units down

   c. 1 unit to the right and 3 units up d. 1 unit to the left and 3 units up

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2. Let $f(x)=3\sqrt{x}+1$ and $g(x)=\frac{x^2-1}{3}$ then domain of $\left(\frac{f}{g}\right)(x)=$

   a. $(1;+\infty)$ b. $[0;1]\cup[1;+\infty)$

   c. $[1;+\infty)$ d. $[0;1)\cup(1;+\infty)$

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3. The range of $f(x)=3+\frac{x^2}{x^2+5}$ is

   a. $[3,4)$ b. $(-\infty,3)$ c. $(3,+\infty)$ d. $(3,4)$

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4. $\lim\limits_{x\to 0}\frac{5x^3-2x^2+3x+7}{-2x^3-6x-5}=$

   a. $\frac{-7}{5}$ b. ∞ c. 0 d. $\frac{-5}{2}$

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5. The function $f(x)=x\sin x$ is :

   a. a polynomial b. an even function c. a cubic function d. an odd function

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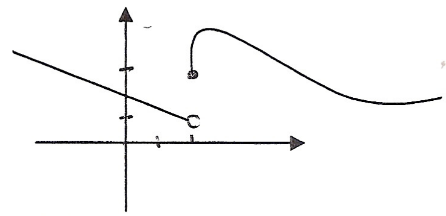
6. The function $f(x)$ is represented graphically as:



$f(x)$ is not continuous at $x = 2$ because :

a $\lim\limits_{x \to 2} f(x) = f(2)$

b. $\lim\limits_{x \to 2} f(x)$ doesn't exist

c. it is not defined at 2

d. $f(2) = 1$

If $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x-1}$ then: (Answer questions 7and 8)

7. $f \circ g(10) =$

a. 3

b. 9

c. $\dfrac{1}{3}$

d. $\dfrac{1}{9}$

8. the domain of the function $g \circ f(x)$ is:

a. $[0,1) \cup (1;+\infty)$

b. $(0,1] \cup [1;+\infty)$

c. $[1;+\infty)$

d. $(1;+\infty)$

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9. If $\tan x = \dfrac{1}{2}$, $x \in \left[\pi , \dfrac{3\pi}{2}\right]$ then $\sec x =$

a. $\dfrac{2}{\sqrt{5}}$

b. $-\dfrac{\sqrt{5}}{2}$

c. $\dfrac{\sqrt{5}}{2}$

d. $-\dfrac{2}{\sqrt{5}}$

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10. The equation of the axis of symmetry of the parabola $y = x^2 - 6x - 7$ is

a. $x = 3$

b. $x = -3$

c. $x = 7$

d. $x = -1$

Part II (70%) Answer each of the following questions. **Explain and show your work.**

1. a) Express the given in terms $\cos x$ and $\sin x$

(4 %) $\cos(\frac{3\pi}{2} - x) + \sin(x + \frac{\pi}{2}) =$

$= \underbrace{\cos\frac{3\pi}{2}}_{0} \cos x + \underbrace{\sin\frac{3\pi}{2}}_{-1} \sin x + \sin x \underbrace{\cos\frac{\pi}{2}}_{0} + \cos x \underbrace{\sin\frac{\pi}{2}}_{1}$

$= -\sin x + \cos x$

b) Show that : $\frac{1-\cos x}{\sin x} - \frac{\sin x}{1+\cos x} = 0$

(4 %) $\frac{1-\cos x}{\sin x} - \frac{\sin x}{1+\cos x} = \frac{(1-\cos x)(1+\cos x) - \sin^2 x}{\sin x\,(1+\cos x)}$

$= \frac{(1-\cos^2 x) - \sin^2 x}{\sin x\,(1+\cos x)}$

$= \frac{\sin^2 x - \sin^2 x}{\sin x\,(1+\cos x)} = 0$

c) Show that $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and then find the value of $\cos^2 \frac{\pi}{12}$

(4 %)

$\cos^2 x + \sin^2 x = 1$

$+ \quad \cos^2 x - \sin^2 x = \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

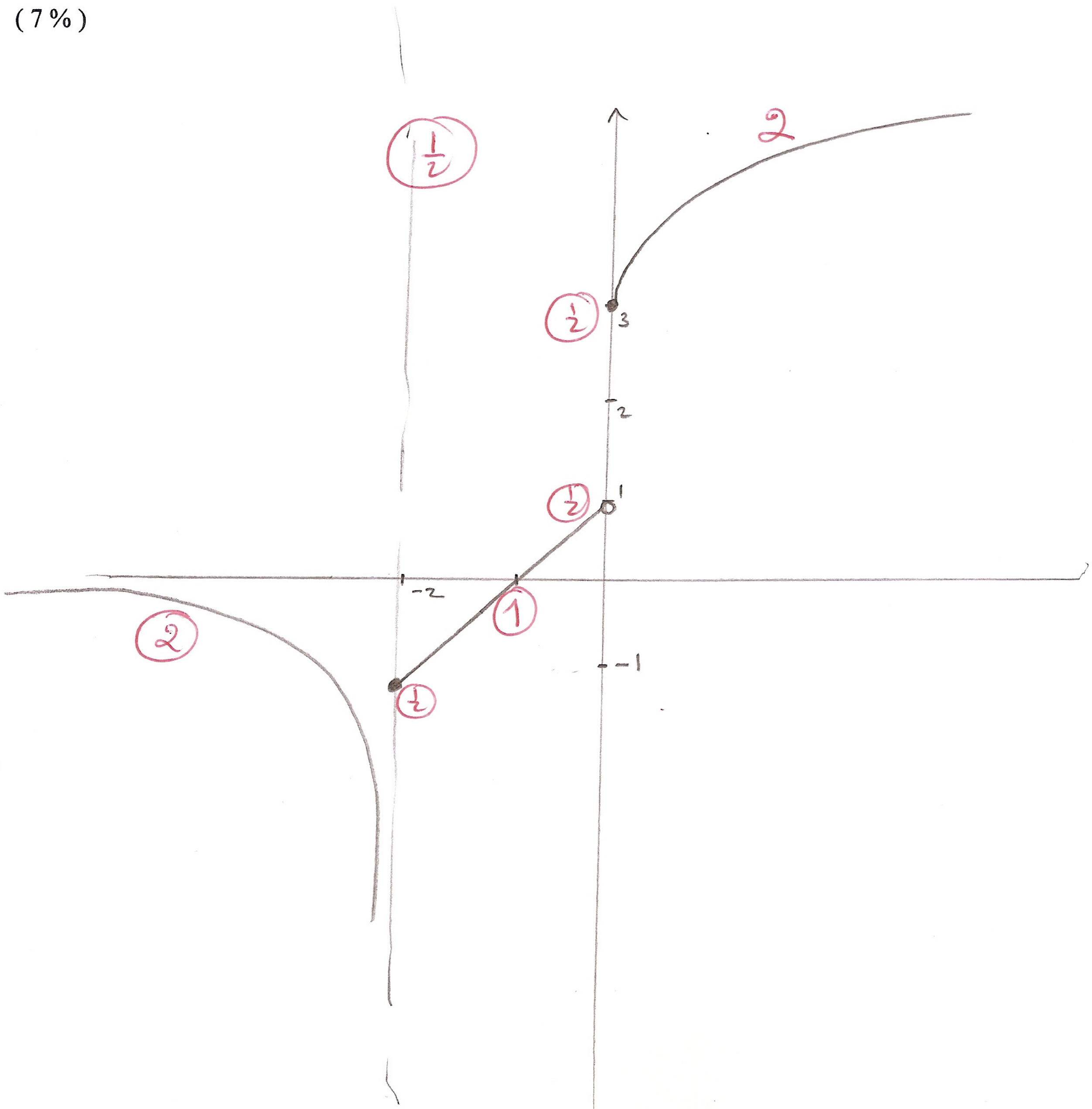
$* \quad \cos^2\frac{\pi}{12} = \frac{1}{2}\left(1 + \cos\frac{2\pi}{12}\right)$

$= \frac{1}{2}\left(1 + \cos\frac{\pi}{6}\right)$

$= \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}+2}{4}$

2. Sketch the graph of the functions $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x < -2 \\ x+1 & \text{if } -2 \le x < 0 \\ \sqrt{x}+3 & \text{if } x \ge 0 \end{cases}$

(7 %)



3. Assume that for all values of x near 0 the function $f(x)$ satisfies:

(8 %) $$\cot 4x \sin 2x < f(x) < \sin\left(\frac{\pi}{6}(\sin\left(x+\frac{\pi}{2}\right)\right)$$

Use the sandwich theorem to find $\lim_{x\to 0} f(x)$

$$* \lim_{x\to 0} \cot 4x \sin 2x = \lim_{x\to 0} \frac{1}{\tan 4x} \cdot \sin 2x$$

$$= \lim_{x\to 0} \underbrace{\frac{4x}{4\tan 4x}}_{1} \cdot \underbrace{\frac{2\sin 2x}{2x}}_{\to 1}$$

$3\frac{1}{2}$

$$= \frac{2}{4} = \boxed{\frac{1}{2}}$$

$$* \lim_{x\to 0} \sin\left(\frac{\pi}{6} \sin\left(x+\frac{\pi}{2}\right)\right)$$

$$= \sin\left(\frac{\pi}{6} \underbrace{\sin\frac{\pi}{2}}_{1}\right)$$

$3\frac{1}{2}$

$$= \sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$$

$$\lim_{x\to 0} \cot 4x \cdot \sin 2x = \lim \sin\left(\frac{\pi}{6}\sin\left(x+\frac{\pi}{2}\right)\right) = \frac{1}{2}$$

(1)

$\Rightarrow \lim_{x\to 0} f(x) = \frac{1}{2}$ by the sandwich thm.

4. For each of the following find the limit (if it exists)

a) $\lim_{x\to -5} \frac{x^2+4x-5}{x^2+5x} = \frac{0}{0}$ 1½ factorize

(4%) $\lim_{x\to -5} \frac{\cancel{(x+5)}(x-1)}{x\cancel{(x+5)}}$ 1 cancel $= \lim_{x\to -5} \frac{x-1}{x}$ 1 subst. $= \frac{-5-1}{-5}$ ½ $= \boxed{\frac{6}{5}}$

b) $\lim_{x\to 2} \frac{\sin(x-2)}{x^2-5x+6} = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)(x-3)}$ 1½

(5%)

Let $\theta = x-2$

as $x \to 2$

$\theta \to 0$

$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$ 2 $\cdot \lim_{x\to 2} \frac{1}{x-3}$

$= 1 \cdot \frac{1}{2-3}$ 1 $= \boxed{-1}$ ½

c) $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} =$

$= \lim_{x\to 0} 2\frac{\sin 2x}{2x} \cdot \frac{3x}{3\sin 3x}$ 1 1 1 1 $\frac{\sin 2x}{2x}\to 1$, $\frac{3x}{\sin 3x}\to 1$

(5%)

$= 2 \cdot \frac{1}{3} = \boxed{\frac{2}{3}}$ 1

d) $\lim_{x\to 6^-} \frac{|x-6|(x+5)}{(x^2-36)} = \lim_{x\to 6^-} \frac{-\cancel{(x-6)}(x+5)}{\cancel{(x-6)}(x+6)}$ 2 1 factorize.

(5%)

$= \lim_{x\to 6^-} -\frac{(x+5)}{(x+6)}$ 1

$= -\frac{(6+5)}{6+6}$ 1 $= \boxed{\frac{-11}{12}}$

e) $\lim_{x \longrightarrow 0} \dfrac{\sqrt{5}-\sqrt{5}\cos x}{\sin^2 x}$ $= \lim_{x \to 0} \dfrac{(\sqrt{5}-\sqrt{5}\cos x)}{\sin^2 x} \dfrac{(\sqrt{5}+\sqrt{5}\cos x)}{(\sqrt{5}+\sqrt{5}\cos x)}$ 2

(5 %)

$$= \lim_{x \to 0} \frac{5-5\cos^2 x}{\sin^2 x(\sqrt{5}+\sqrt{5}\cos x)}$$ 1

$$= \lim_{x \to 0} \frac{5.(1-\cos^2 x)}{\sin^2 x(\sqrt{5}+\sqrt{5}\cos x)}$$ 1

$$= \lim \frac{5\cancel{\sin^2 x}}{\cancel{\sin^2 x}(\sqrt{5}+\sqrt{5}\cos x)} = \frac{5}{2\sqrt{5}} = \boxed{\frac{\sqrt{5}}{2}}$$ 1

OR $\lim_{x \to 0} \dfrac{\sqrt{5}(1-\cos x)}{1-\cos^2 x} = \lim_{x \to 0} \dfrac{\sqrt{5}\,\cancel{(1-\cos x)}}{(1+\cos x)\cancel{(1-\cos x)}} = \boxed{\dfrac{\sqrt{5}}{2}}$

f) $\lim_{x \longrightarrow 0} \dfrac{\sin\left(\dfrac{\pi}{2}-x\right)}{\left(\dfrac{\pi}{2}-x\right)}$

(4 %)

$$= \lim_{x \to 0} \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

2 2

5. consider the function $f(x) = \begin{cases} 3-5x & \text{if } -1 \le x < 0 \\ x^3 & \text{if } 0 \le x < 2 \\ \dfrac{x^3-4x}{x-2} & \text{if } 2 < x < 5 \\ 5 & \text{if } x = 2 \end{cases}$

(15 %)

a) For what values of x is $f(x)$ continuous ?(write the answer in interval form)

b) If $f(x)$ is not continuous at an interior point c determine whether discontinuity at c is removable, if not find the continuous extension of f at $x = c$ if possible? (Explain)

(a) Cont. at $x=-1$

- $f(-1) = 3+5 = 8$
- $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 3-5x = 8$
- $f(-1) = \lim_{x \to -1^+} f(x)$

$\therefore f(x)$ is cont. at $x=-1$

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at $x=0$

- $f(0) = 0$
- $\lim_{x \to 0} f(x)$ $\begin{cases} \lim_{x \to 0^-} 3-5x = 3 \\ \lim_{x \to 0^+} x^3 = 0 \end{cases}$ L.H.L $\neq$ R.H.L $\therefore \lim_{x \to 0} f(x)$ doesn't exist

3

$\therefore f(x)$ is not Cont. at $x=0$

at $x=5$

- $f(5)$ is undefined

$1\frac{1}{2}$

$\therefore f(x)$ is not cont. at $x=5$

at $x=2$

- $f(2)=5$
- $\lim_{x\to 2} f(x)$

$$\lim_{x\to 2^-} x^3 = 8$$

4

$$\lim_{x\to 2^+} \frac{x^3-4x}{x-2} = \lim_{x\to 2^+} \frac{x\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$= \lim_{x\to 2^+} x(x+2)$$

$$= 8$$

L.H.L = R.H.L

$\therefore \lim_{x\to 2} f(x) = 8$

- $f(2) \neq \lim_{x\to 2} f(x)$

$\therefore$ $f(x)$ is <u>not</u> cont. at $x=2$

$f(x)$ is cont. $\forall\, x \in [-1,0)\cup(0,2)\cup(2,5)$

(b) * Discontinuity at $x=0$ is not removable

bec $\lim_{x\to 0} f(x)$ doesn't exist.

$1\frac{1}{2}$

* Discont. at $x=2$ is removable bec

$\lim_{x\to 2} f(x) = 8$ exists

2

The cont. extension $F(x) = \begin{cases} 3-5x & -1\le x<0 \\ x^3 & 0<x\le 2 \\ \frac{x^3-4x}{x-2} & 2<x<5 \end{cases}$