Solution

American University of Beirut
Mathematics Department
Spring Semester 2011-2012
Math 101 Quiz I

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Version 1

Name : ______________________ **ID#:** ________________________

- **Circle your problem solving section:**

  Section 1 : M @8:00 **Section 2 : M @ 9:00** **Section 3 : M @ 11:00**

- **Answer table for Part I**

1	c		6	a
2	b		7	d
3	c		8	b
4	d		9	c
5	d		10	b

# of correct answers : ----------------------- **# of wrong answers :** ------------------------					**Grade of Part I** 30%
1.	**2.**	**3.**	**4.**	**5.**	**Grade of Part II** 70%

Final Grade

Part I (30%) 10 multiple choices with 3 points for each correct answer and -0.5 penalty on each wrong answer.

Circle the correct answer then, copy your answers as a, b, c or d on the table provided on page 1:

1. The graph of the function is the graph of $F(x)=\sqrt{x-1}+3$ is the graph of $f(x)=\sqrt{x}$ Shifted :

   a. 1 unit to the right and 3 units down b. 1 unit to the left and 3 units down

   c. 1 unit to the right and 3 units up d. 1 unit to the left and 3 units up

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2. Let $f(x)=3\sqrt{x}+1$ and $g(x)=\frac{x^2-1}{3}$ then domain of $\left(\frac{f}{g}\right)(x)=$

   a. $(1;+\infty)$ b. $[0;1)\cup(1;+\infty)$

   c. $[1;+\infty)$ d. $[0;1]\cup[1;+\infty)$

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3. The range of $f(x)=3+\frac{x^2}{x^2+5}$ is

   a. $(3,+\infty)$ b. $(-\infty,3)$ c. $[3,4)$ d. $(3,4)$

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4. $\lim\limits_{x\to 0}\frac{7x^3-2x^2+3x+5}{5x^3-6x-2}=$

   a. $\frac{7}{5}$ b. ∞ c. 0 d. $\frac{-5}{2}$

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5. The function $f(x)=x\cos x$ is :

   a. a polynomial b. an even function c. a cubic function d. an odd function

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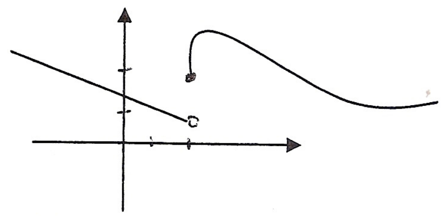
6. The function $f(x)$ is represented graphically as:



$f(x)$ is not continuous at $x=2$ because :

a. $\lim_{x\to 2} f(x)$ doesn't exist

b. $\lim_{x\to 2} f(x)=f(2)$

c. it is not defined at 2

d. $f(2)=1$

If $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{x-1}$ then: (Answer questions 7 and 8)

7. $f\circ g(10)=$

a. 3

b. 9

c. $\dfrac{1}{9}$

d. $\dfrac{1}{3}$

8. the domain of the function $g\circ f(x)$ is:

a. $(0,1]\cup[1;+\infty)$

b. $[0,1)\cup(1;+\infty)$

c. $[1;+\infty)$

d. $(1;+\infty)$

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9. If $\tan x=\dfrac{1}{2}$, $x\in\left[\pi,\dfrac{3\pi}{2}\right]$ then $\sec x=$

a. $\dfrac{2}{\sqrt{5}}$

b. $\dfrac{\sqrt{5}}{2}$

c. $-\dfrac{\sqrt{5}}{2}$

d. $-\dfrac{2}{\sqrt{5}}$

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10. The equation of the axis of symmetry of the parabola $y=x^2-6x-7$ is

a. $x=-3$

b. $x=3$

c. $x=7$

d. $x=-1$

Part II (70%) Answer each of the following questions. **Explain and show your work.**

1. a) Express the given in terms $\cos x$ and $\sin x$

(4 %) $\sin(\frac{3\pi}{2} - x) + \cos(x + \frac{\pi}{2}) =$

$\sin \frac{3\pi}{2} \cos x - \cos \frac{3\pi}{2} \sin x + \cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2}$

-1 2 0 0 2 1

$= -\cos x - \sin x$

b) Show that: $\frac{1-\cos x}{\sin x} - \frac{\sin x}{1+\cos x} = 0$

(4 %)

$\frac{1-\cos x}{\sin x} - \frac{\sin x}{1+\cos x} = \frac{(1-\cos x)(1+\cos x) - \sin^2 x}{\sin x(1+\cos x)}$ $1\frac{1}{2}$

$= \frac{(1-\cos^2 x) - \sin^2 x}{\sin x(1+\cos x)}$ 1

$= \frac{\sin^2 x - \sin^2 x}{\sin x(1+\cos x)} = 0$ 1 ½

c) Show that $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ and then find the value of $\cos^2 \frac{\pi}{8}$

(4 %) ½ +

$\cos^2 x + \sin^2 x = 1$ ½

$\cos^2 x - \sin^2 x = \cos 2x$ ½

$2\cos^2 x = 1 + \cos 2x$ ½

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$ ½

* $\cos^2 \frac{\pi}{8} = \frac{1}{2}(1 + \cos \frac{2\pi}{8})$

½

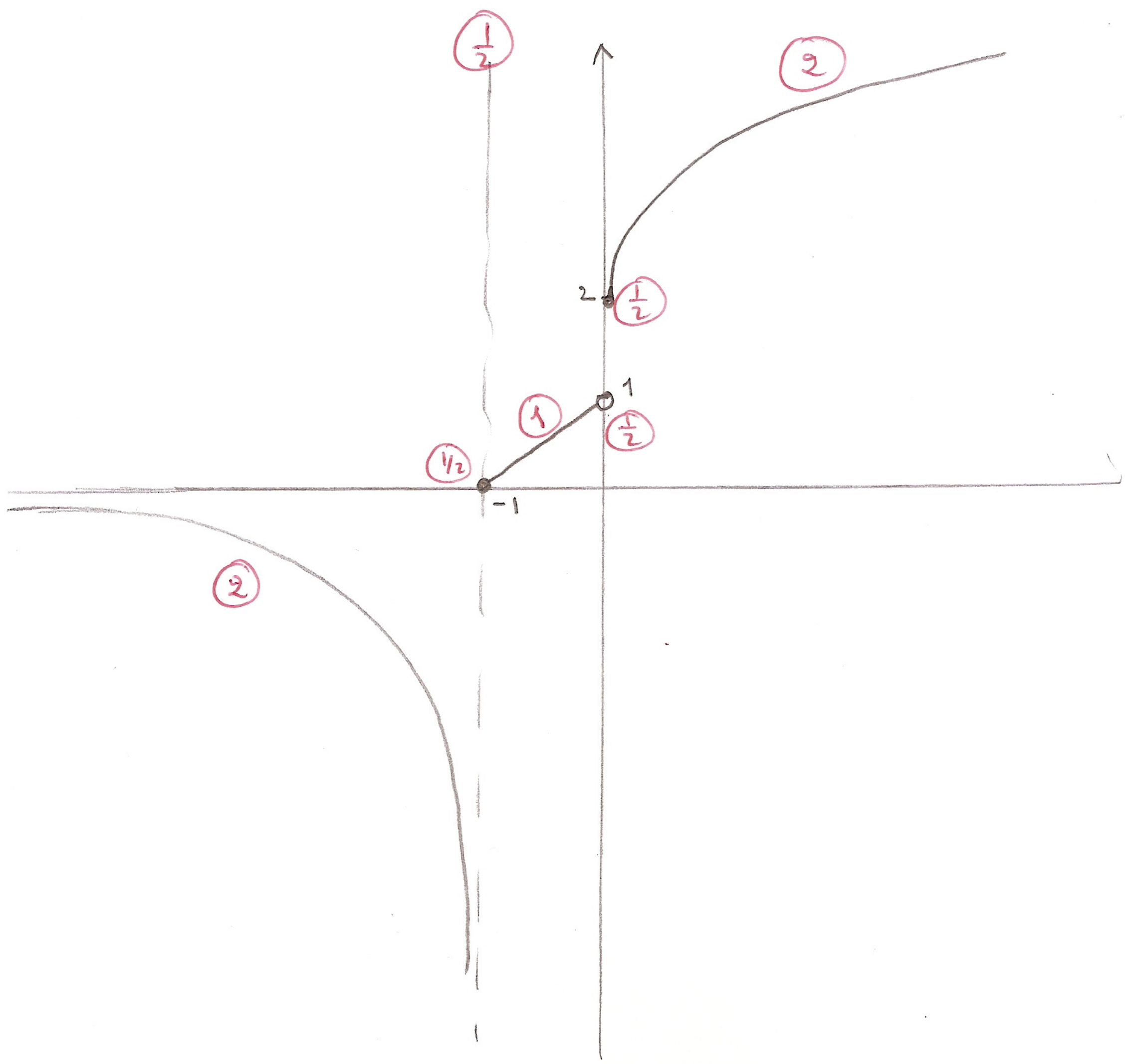
$= \frac{1}{2}(1 + \cos \frac{\pi}{4})$

1

$= \frac{1}{2}(1 + \frac{\sqrt{2}}{2}) = \frac{2+\sqrt{2}}{4}$

2. Sketch the graph of the functions $f(x)=\begin{cases}\dfrac{1}{x+1} & if\ x<-1\\ x+1 & if\ -1\le x<0\\ \sqrt{x}+2 & if\ x\ge 0\end{cases}$

(7%)



3. Assume that for all values of x near 0 the function $f(x)$ satisfies:

(8 %) $$\cot 4x \sin 2x < f(x) < \sin\left(\frac{\pi}{6}(\sin\left(x+\frac{\pi}{2}\right)\right)$$

Use the sandwich theorem to find $\lim_{x\to 0} f(x)$

$$* \quad \lim_{x\to 0} \cot 4x \sin 2x = \lim_{x\to 0} \frac{1}{\tan 4x} \cdot \sin 2x$$

$$= \lim_{x\to 0} \underbrace{\frac{4x}{4\tan 4x}}_{1} \cdot \overbrace{\frac{2\sin 2x}{2x}}^{1}$$

3½

$$= \frac{2}{4} = \boxed{\frac{1}{2}}$$

$$* \quad \lim_{x\to 0} \sin\left(\frac{\pi}{6} \sin\left(x+\frac{\pi}{2}\right)\right)$$

$$= \sin\left(\frac{\pi}{6} \overset{1}{\cancel{\sin\frac{\pi}{2}}}\right)$$

3½

$$= \sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$$

$$\lim_{x\to 0} \cot 4x \cdot \sin 2x = \lim \sin\left(\frac{\pi}{6}\sin\left(x+\frac{\pi}{2}\right)\right) = \frac{1}{2}$$

1

$$\Rightarrow \lim_{x\to 0} f(x) = \frac{1}{2}$$ by the sandwich thm.

4. For each of the following find the limit (if it exists)

a) $\lim\limits_{x\to 5}\dfrac{x^2-4x-5}{x^2-5x} = \dfrac{0}{0}$ (1½) factorize

(4 %)

$= \lim\limits_{x\to 5}\dfrac{(x+1)\cancel{(x-5)}}{x\cancel{(x-5)}}$ (1) cancel $= \lim\limits_{x\to 5}\dfrac{(x+1)}{x}$ (1) subst. $= \boxed{\dfrac{6}{5}}$ (1½)

b) $\lim\limits_{x\to 3}\dfrac{\sin(x-3)}{x^2-5x+6} = \lim\limits_{x\to 3}\dfrac{\sin(x-3)}{(x-3)(x-2)}$ (1½)

(5%)

Let $\theta = x-3$

as $x\to 3$

$\theta \to 0$

$= \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ (2) $\cdot \lim\limits_{x\to 3}\dfrac{1}{x-2}$

$= 1 \cdot \dfrac{1}{3-2}$ (1) $= \boxed{1}$ (½)

c) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 2x} =$

(5 %)

$= \lim\limits_{x\to 0} 3\underbrace{\dfrac{\sin 3x}{3x}}_{\to 1}$ (1) (1) $\cdot \dfrac{1}{2}\underbrace{\dfrac{2x}{\sin 2x}}_{\to 1}$ (1) (1)

$= 3 \cdot \dfrac{1}{2} = \boxed{\dfrac{3}{2}}$ (1)

d) $\lim\limits_{x\to 3^-}\dfrac{|x-3|(x+5)}{(x^2-9)\,x} = \lim\limits_{\substack{x\to 3^- \\ x<3}} \dfrac{-\cancel{(x-3)}(x+5)}{\cancel{(x-3)}(x+3)x}$ (2) (1) factorize.

(5 %)

$= \lim\limits_{x\to 3^-} -\dfrac{(x+5)}{(x+3)x}$ (1)

$= -\dfrac{8}{6\times 3}$ (1) $= \boxed{-\dfrac{4}{9}}$

e) $\lim_{x \to 0} \dfrac{\sqrt{3}-\sqrt{3}\cos x}{\sin^2 x}$ $= \lim_{x\to 0} \dfrac{(\sqrt{3}-\sqrt{3}\cos x)}{\sin^2 x} \dfrac{(\sqrt{3}+\sqrt{3}\cos x)}{(\sqrt{3}+\sqrt{3}\cos x)}$ 2

(5 %)

$= \lim_{x\to 0} \dfrac{3-3\cos^2 x}{\sin^2 x\,(\sqrt{3}+\sqrt{3}\cos x)}$ 1

$= \lim_{x\to 0} \dfrac{3(1-\cos^2 x)}{\sin^2 x\,(\sqrt{3}+\sqrt{3}\cos x)}$ 1

$= \lim_{x\to 0} \dfrac{3\cancel{\sin^2 x}}{\cancel{\sin^2 x}\,(\sqrt{3}+\sqrt{3}\cos x)} = \boxed{\dfrac{3}{2\sqrt{3}}}$ 1

OR

$\lim_{x\to 0} \dfrac{\sqrt{3}(1-\cos x)}{1-\cos^2 x} = \lim_{x\to 0} \dfrac{\sqrt{3}\,\cancel{(1-\cos x)}}{\cancel{(1-\cos x)}(1+\cos x)} = \boxed{\dfrac{\sqrt{3}}{2}}$

f) $\lim_{x \to 0} \dfrac{\sin\left(\frac{\pi}{2}+x\right)}{\left(\frac{\pi}{2}+x\right)}$

(4 %)

$= \lim_{x\to 0} \dfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}$ 2 $= \dfrac{1}{\frac{\pi}{2}}$ 2 $= \boxed{\dfrac{2}{\pi}}$

5. consider the function $f(x) = \begin{cases} 3-2x & \text{if } -1 \le x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ \dfrac{x^3-4x}{x^2-2x} & \text{if } 2 < x < 5 \\ 7 & \text{if } x = 2 \end{cases}$

(15 %)

a) For what values of x is $f(x)$ continuous ?(write the answer in interval form)

b) If $f(x)$ is not continuous at an interior point c determine whether discontinuity at c is removable, if not find the continuous extension of f at $x = c$ if possible? (Explain)

a

cont. at $x=-1$

- $f(-1) = 3-(-2) = 5$
- $\lim\limits_{x \to -1^+} (3-2x) = 5$
- $f(-1) = \lim\limits_{x \to -1^+} 3-2x$

$\therefore$ $f(x)$ is cont. at $x=-1$

2

at $x=0$

- $f(0) = 0$
- $\lim\limits_{x \to 0} f(x) = \begin{cases} \lim\limits_{x \to 0^-} 3-2x = 3 \\ \lim\limits_{x \to 0^+} x^2 = 0 \end{cases}$

L.H.L $\neq$ R.H.L

$\therefore \lim\limits_{x \to 0} f(x)$ doesn't exist

3

$\therefore$ $f(x)$ is not cont. at $x=0$

at $x = 2$

- $f(2) = 7$
- $\lim\limits_{x \to 2} f(x)$ $\begin{cases} \lim\limits_{x \to 2^-} x^2 = 4 \\ \lim\limits_{x \to 2^+} \dfrac{x^3 - 4x}{x^2 - 2x} \quad \dfrac{0}{0} \end{cases}$

$$= \lim_{x \to 2^+} \frac{\cancel{x}\cancel{(x-2)}(x+2)}{\cancel{x}\cancel{(x-2)}} = 2 + 2 = 4$$

L.H.L = R.H.L

$\therefore \lim\limits_{x \to 2} f(x) = 4$

4

- $f(2) \neq \lim\limits_{x \to 2} f(x)$

$\therefore$ $f(x)$ is not cont. at $x = 2$

at $x = 5$

- $f(5)$ is undefined 1½

$\therefore$ $f(x)$ is not cont. at $x = 5$

1

$f(x)$ is cont. $\forall\, x \in [-1, 0) \cup (0, 2) \cup (2, 5)$

(b) * discontinuity at $x = 0$ is not removable

1½ bec $\lim\limits_{x \to 0} f(x)$ doesn't exist

* discont. at $x = 2$ is removable bec. $\lim\limits_{x \to 2} f(x) = 4$

2 and the cont. extension $F(x) = \begin{cases} 3 - 2x & -1 \leq x < 0 \\ x^2 & 0 < x \leq 2 \\ \dfrac{x^3 - 4x}{x^2 - 2x} & 2 < x < 5 \end{cases}$