AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

Math 101 - Quiz II

Fall 2005-2006

Name: Key ID:

Lecture: Ms. Marwa El Houri
Section 1: T 08:00 - 09:00
Section 2: T 11:00 - 12:00

Lecure: Ms. Diana Audi
Section 3: Th 08:00 - 09:00
Section 4: Th 02:00 - 03:00
Section 5: Th 11:00 - 12:00
Section 6: Th 09:30 - 10:30

Time: 60 min

Directions: Write your name and ID number and circle your section number. Answer the questions in the allocated spaces, if more space is needed continue on the back. NO CALCULATORS ARE ALLOWED!!!

I- **(20 points)** Find the derivative $\frac{dy}{dx}$ of the followings:

a) $y = \sin(\frac{1}{2+x})$

$$\frac{dy}{dx} = \cos\left(\frac{1}{2+x}\right) \cdot \left(-\frac{1}{2+x}\right)$$

$$= -\frac{\cos\left(\frac{1}{2+x}\right)}{2+x}$$

b) $y = -x^2 \cot^7(\frac{x}{3})$

$$\frac{dy}{dx} = -2x\cot^7\left(\frac{x}{3}\right) + \left(-x^2 \cdot \left(7\cot^6\left(\frac{x}{3}\right) \cdot \left(-\csc^2\frac{x}{3}\right) \cdot \frac{1}{3}\right)\right)$$

$$= -2x\cot^7\left(\frac{x}{3}\right) + \frac{7}{3}x^2\cot^6\left(\frac{x}{3}\right)\csc^2\left(\frac{x}{3}\right)$$

c) $\sin(xy^2) + 2y^3 = 3x + 2$

Implicit differentiation

$\cos(xy^2)\cdot(y^2 + 2xyy') + 6y^2y' = 3$

$\Rightarrow y^2\cos(xy^2) + 2xyy'\cos(xy^2) + 6y^2y' = 3$

$\Rightarrow y'[2xy\cos(xy^2) + 6y^2] = 3 - y^2\cos(xy^2)$

$\Rightarrow y' = \dfrac{3 - y^2\cos(xy^2)}{2xy\cos(xy^2) + 6y^2}$

II- **(15 points)** Find the equation of the tangent line to the parametric curve.

$$x = \sqrt[3]{t^4} + t \quad = t^{4/3} + t$$
$$y = \sin(3t + \pi) + 2$$

at $t = 0$

$\dfrac{dx}{dt} = \dfrac{4}{3}t^{1/3} + 1$

$\dfrac{dy}{dt} = 3\cos(3t + \pi)$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3\cos(3t + \pi)}{\frac{4}{3}\sqrt[3]{t} + 1}$

at $t = 0$

$x = 0$

$y = \sin(0 + \pi) + 2 = 2.$

$\dfrac{dy}{dt} = \dfrac{3\cos\pi}{1} = -3$

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Equation of tg:

$y - 2 = -3(x - 0)$

$\boxed{y = -3x + 2}$

III- (20 points) Let $f(x) = x^3 - 3x^2 + 2$

1. Find f'.
2. Find critical values of f.
3. Find intervals where the graph of f is increasing or decreasing.
4. Find local extremes.
5. Find intervals where the graph of f is concave up or concave down.
6. Does f have an inflection point? If yes where?
7. Sketch the graph of f.

Domain $(-\infty, \infty)$

1) $f'(x) = 3x^2 - 6x$

2) $3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

Critical pts: $\boxed{x = 0}$ or $\boxed{x = 2.}$

3)

	$-\infty$	0	2	$+\infty$
$3x$	$-$	$+$	$+$	
$x-2$	$-$	$-$	$+$	
$f'(x)$	$+$	$-$	$+$	
f	$\nearrow$	$\searrow$	$\nearrow$	

4) f has a local max at $x = 0$

$f(0) = 2.$

f has a local min at $x = 2.$

$f(2) = 8 - 12 + 2 = -2.$

3

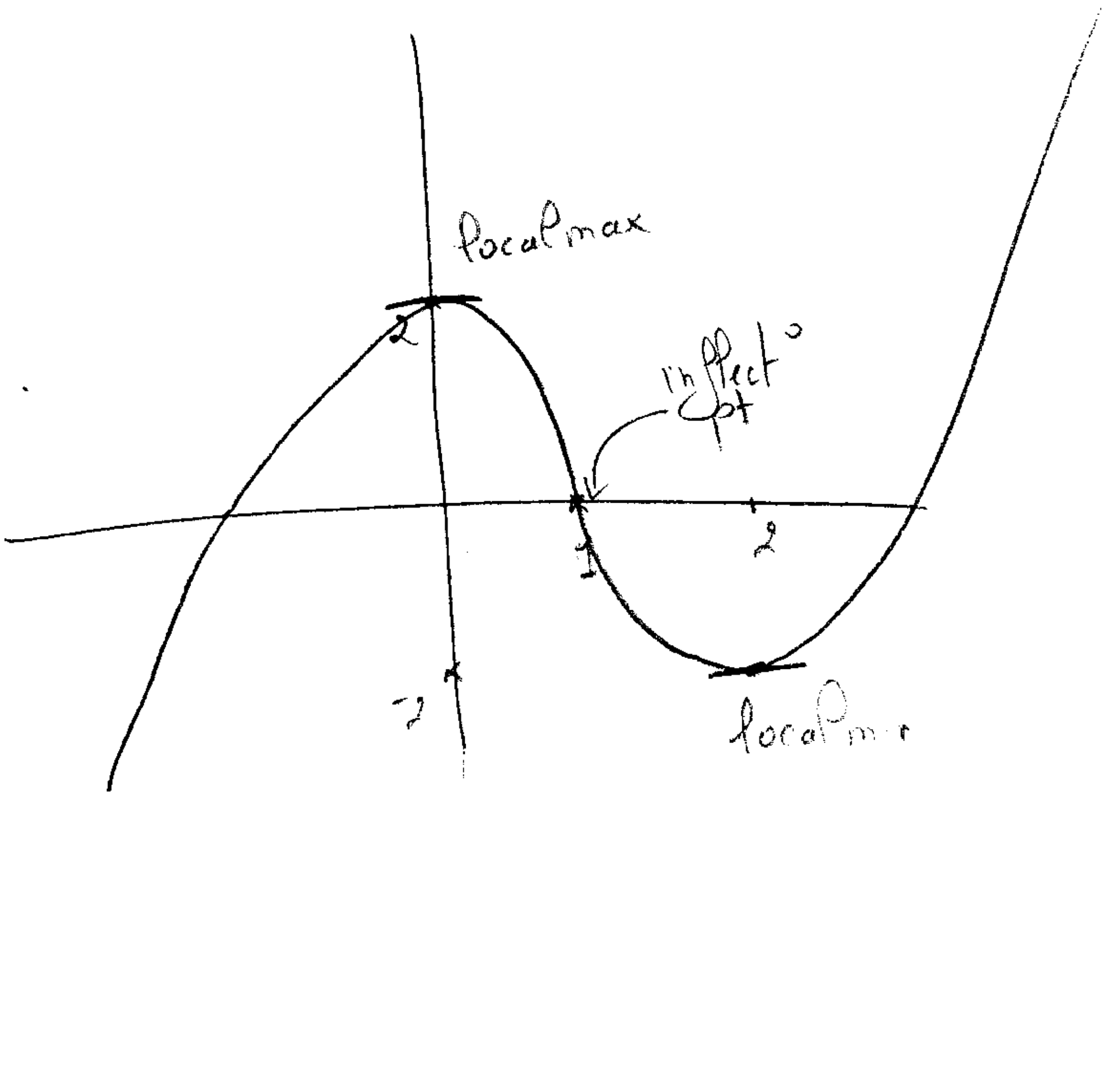
5) $f''(x) = 6x - 6$

$6(x-1) = 0 \Rightarrow x = 1$

		1	
$f''(x)$	$-$		$+$
$f(x)$	concave down		concave up

6] f has an inflection point at $x = 1$
since $f''(x)$ changes sign at $x = 1$.

x	y
0	2
1	0
2	-2



IV- a) **(6 points)** Show that $\sqrt[5]{1-5x} \approx 1 - x$ at $x = 0$

$$L(x) = f(0) + f'(0)(x-0)$$

$$f'(x) = -(1-5x)^{-4/5}$$

$$f'(0) = -1.$$

$$f(0) = 1$$

$$L(x) = 1 - x$$

b) **(4 points)** Use part a) to find

$$\lim_{x \to 0} \frac{\sqrt[5]{1-5x} - 1}{x}$$

$$\lim_{x \to 0} \frac{\sqrt[5]{1-5x} - 1}{x} = \lim_{x \to 0} \frac{1 - x - 1}{x} = 1.$$

V- **(15 points)** Let $f(x) = \begin{cases} 2x & x \leq 0 \\ x^2 + 2x & 0 < x \leq 1 \\ 3x & x > 1 \end{cases}$

where is f differentiable? Explain.

$$f'(x) = \begin{cases} 2 & x < 0 \\ 2x + 2 & 0 < x < 1 \\ 3 & x > 1. \end{cases}$$

at $x = 0$

$$\lim_{x \to 0^-} f'(x) = 2.$$

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} 2x + 2 = 2.$$

~~then~~ Go to definition.

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 + 2h - 0}{h} = \lim_{h \to 0^+} h + 2 = 2.$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{2h - 0}{h} = 2.$$

$\Rightarrow f'(0) = 2$ and f is diff at $x = 0$

at $x = 1$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 2x + 2 = 4$$

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 3 = 3$$

$\lim_{x \to 1^-} f'(x) \neq \lim_{x \to 1^+} f'(x) \Rightarrow$ f is not diff at $x = 1$ by Intermediate Value thm for derivatives.

* f is diff on $(-\infty, 1) \cup (1, +\infty)$

VI- a) **(6 points)** Let $g(x) = \dfrac{1}{1+x^2}$.

Show that g has an absolute maximum and an absolute minimum on [0, 1] and find them.

g is cont on a closed interval then g has abs min and max.

$$g'(x) = \frac{-2x}{(1+x^2)^2} \leq 0 \quad \text{on } [0,1].$$

critical point $x = 0$

bd points $x = 0$ and $x = 1$

$g(0) = 1$ ~~local max~~ absolute max

$g(1) = \frac{1}{2}$ absolute min.

b) **(4 points)** Let f be a function such that $f(0) = 0$ and its derivative $f'(x) = \dfrac{1}{1+x^2}$. Use the Mean Value Theorem to show that

$$\frac{1}{2} \leq f(1) \leq 1$$

f is cont on [0,1] diff on (0,1) then by MVT there is a point c in (0,1) such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = f(1)$$

~~but~~ $f'(x) = \dfrac{1}{1+x^2} \Rightarrow f(1) = \dfrac{1}{1+c^2}$

by part a)

$$\frac{1}{2} \leq \frac{1}{1+c^2} \leq 1 \Rightarrow \frac{1}{2} \leq f(1) \leq 1.$$

VII- a) **(5 points)** Let f and g be two continuous functions on $[a, b]$ and differentiable on (a, b) such that $f(a) = g(a)$ and $f(b) = g(b)$ for some real numbers a and b. Show that there exists a number c in (a, b) such that

$$f'(c) = g'(c)$$

(need to show that $f'(c) - g'(c) = 0$ or $h'(c) = 0$ where $h'(c) = f'(c) - g'(c)$)

Let $h = f - g$

h is cont on $[a, b]$

diff on (a, b)

$h(a) = f(a) - g(a) = 0$

$h(b) = f(b) - g(b) = 0$

by Rolle's theorem there is a point c in (a, b) such that $h'(c) = 0$

$\Rightarrow f'(c) - g'(c) = 0$

$\Rightarrow f'(c) = g'(c)$.

b) **(5 points)** Let f and g be two continuous functions on $[0, 2]$ and differentiable on $(0, 2)$ such that $f(0) = 2$, $g(0) = 3$, $f(2) = 1$, and $g(2) = \frac{3}{2}$. Show that there is a point c in $(0, 2)$ such that

$$3f'(c) = 2g'(c).$$

(need to show that $3f'(c) - 2g'(c) = 0$ or $h'(c) = 0$ where $h'(c) = 3f'(c) - 2g'(c)$)

Let $h = 3f - 2g$.

h is cont on $[0, 2]$ and ~~cont~~ diff on $(0, 2)$

$h(0) = 3f(0) - 2g(0) = 6 - 6 = 0$

$h(2) = 3f(2) - 2g(2) = 3 - 3 = 0$

by Rolle's thm. there is a point c in $(0, 2)$ such that $h'(c) = 0$

$\Rightarrow 3f'(c) - 2g'(c) = 0$

$\Rightarrow 3f'(c) = 2g'(c)$.

GOOD LUCK!!