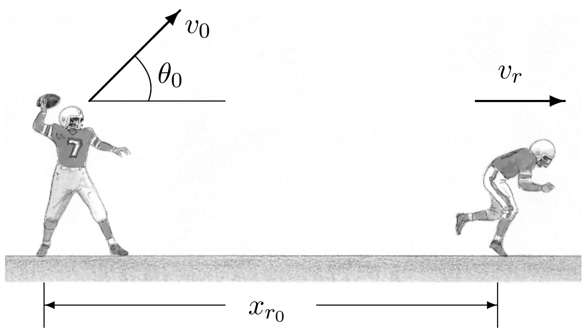
EM 311M - Dynamics
Exam 1 - **Solutions**
February 15, 2012

Write your solutions only on the **FRONT** side of the sheets provided. Show all your work.

1. In practice, a quarterback throws the football with a velocity v_0 at a given angle θ_0 above the horizontal. At the same instant, a receiver standing at a given distance x_{r_0} in front of him starts running straight down field with a given constant velocity v_r and catches the ball. Assume that the ball is thrown and caught at the same height above the ground.

i. Find the expression for the initial velocity v_0 as a function of the given data and g? (30 points)
Note: It is OK to express the final result as two possible solutions.

ii. Explain why do you get two possible solutions and which one represent the physical solution of the problem. (5 points)



Solution:
i)

(5 points) Motion of ball:

$$x_b = v_0 \cos(\theta_0) t$$
$$y_b = v_0 \sin(\theta_0) t - \frac{g}{2} t^2$$

(5 points) Motion of receiver:

$$x_r = x_{r_0} + v_r t$$
$$y_r = 0$$

(10 points) At some time t, the ball and the receiver must meet each other:

$$(1)\ \ x_b = x_r \Rightarrow v_0 \cos(\theta_0) t = x_{r_0} + v_r t$$
$$(2)\ \ y_b = y_r \Rightarrow v_0 \sin(\theta_0) t - \frac{g}{2} t^2 = 0$$

From (2), we obtain the time when the ball is caught:

$$v_0 \sin(\theta_0) t - \frac{g}{2} t^2 = 0 \Rightarrow t\left(v_0 \sin(\theta_0) - \frac{g}{2} t\right) = 0.$$

The values of t here represent the times when the ball is at the same height as when it was thrown. The value $t = 0$, represents the initial time when the ball is thrown. The second value

$$t = \frac{2}{g} v_0 \sin(\theta_0)$$

is the time when the ball is caught.

(10 points) We substitute the expression of the time in (1) to get

$$v_0 \cos(\theta_0) \left(\frac{2}{g} v_0 \sin(\theta_0) \right) = x_{r_0} + v_r \left(\frac{2}{g} v_0 \sin(\theta_0) \right).$$

Reordering the terms we get

$$2 \sin(\theta_0) \cos(\theta_0)\, v_0^2 - 2 v_r \sin(\theta_0) v_0 - g\, x_{r_0} = 0.$$

This is a quadratic equation for v_0 of the form

$$a\, v_0^2 + b v_0 + c = 0$$

with coefficients: $a = 2\sin(\theta_0)\cos(\theta_0)$, $b = -2v_r \sin(\theta_0)$ and $c = -g\, x_{r_0}$. Using the formula for the solution of the quadratic equation:

$$v_0 = \frac{-b \pm \sqrt{b^2 - 4\, a\, c}}{2a},$$

we obtain

$$v_0 = \frac{2v_r \sin(\theta_0) \pm \sqrt{(2v_r \sin(\theta_0))^2 + 4 \cdot 2 \sin(\theta_0) \cos(\theta_0)\, g\, x_{r_0}}}{2 \cdot 2 \sin(\theta_0) \cos(\theta_0)},$$

and by reordering the expression we get the result

$$\boxed{v_0 = \frac{v_r \sin(\theta_0) \pm \sqrt{v_r^2 \sin^2(\theta_0) + g\, x_{r_0} \sin(2\theta_0)}}{\sin(2\theta_0)}},$$

where we used the identity $\sin(2\theta_0) = 2\sin(\theta_0)\cos(\theta_0)$.

ii)

(2 points) Mathematically, we obtain two possible solution: one using the "+" sign in the result and the other using the "−" sign. Since for $0 \leq \theta \leq 90^o$, we get $\sin(2\theta) \geq 0$, we have that

$$\sqrt{v_r^2 \sin^2(\theta_0) + g\, x_{r_0} \sin(2\theta_0)} > v_r \sin(\theta_0)$$

assuming $g > 0$ and $x_{r_0} > 0$. Then, we will get one positive solution for v_0 and one negative solution for v_0.

The question is what is the meaning of those two solutions. In order to understand what the two mathematical solutions may mean, we can think of two different scenarios as follows:

- Scenario I: At $t = 0$ the football is at $x = 0$ with a velocity $v_0 > 0$ and angle θ_0 above the horizontal and, at the same instant, a receiver at a distance x_{r_0} has a constant velocity v_r. The ball meets the receiver at some time $t > 0$. (see Figure 1)
- Scenario II: A virtual meeting between the ball and the receiver happened at some past time $t < 0$. The ball has a path in which later, once the receiver is at the position x_{r_0} at $t = 0$, it reaches the quarterback with a velocity having $v_0 < 0$. (see Figure 1)

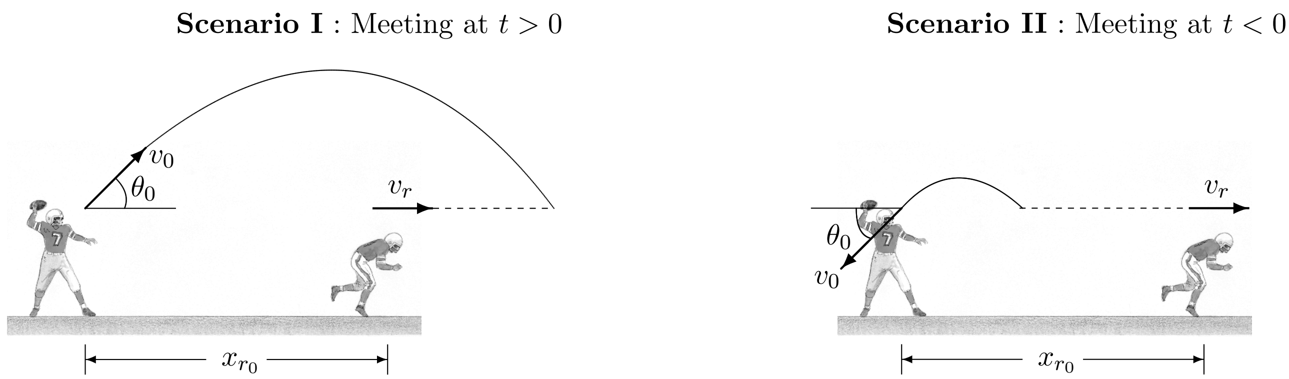


Figure 1: Two different scenarios for which at $t = 0$ the football is at $x = 0$ with a velocity v_0 with angle θ_0 with respect to the horizontal and at the same time a receiver at a distance x_{r_0} has a constant velocity v_r.

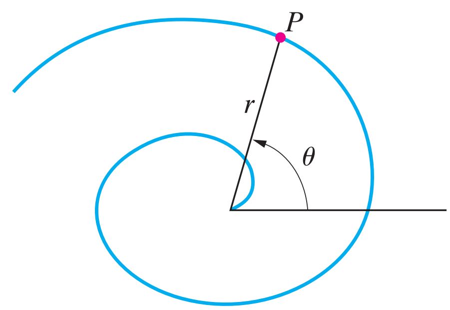
(3 points) The physical solution of the problem is given by the scenario I, for which $v_0 > 0$, since the problem indicates that the receiver catches the ball *after* it is thrown by the quarterback, at some time $t > 0$.

2. i. Derive the formulas for the acceleration vector components a_r and a_θ in the polar coordinate system. Express the results using the angular velocity ω and the angular acceleration α. (15 points)

ii. An object P of mass m moves along the spiral path $r = 2\theta$ ft, where θ is in radians. Its angular position is given as a function of time by $\theta = \omega_0 t$ rad, with ω_0 the object's constant angular velocity.

a) Determine the expression of the polar components of the total force acting on the object, and express the results *only* as a function of m, ω_0, and t, with the appropriate units. (15 points)

b) Find the expression of the ratio of the magnitude of the total force to its transverse component *only* as a function of ω_0, for $t = 2$ s ? (5 points)



Solution:

i)

(3 points) In polar coordinates

$$\vec{r} = r\hat{e}_r.$$

(5 points) The velocity vector is calculated as

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} = \frac{dr}{dt}\hat{e}_r + r\omega\hat{e}_\theta,$$

where we use the product rule for derivatives. Furthermore, the unit vectors in polar coordinates, $\hat{e}_r$ and $\hat{e}_\theta$, satisfy

$$\frac{d\hat{e}_r}{dt} = \omega\hat{e}_\theta \quad ; \quad \frac{d\hat{e}_\theta}{dt} = -\omega\hat{e}_r.$$

(5 points) Similarly, we compute the acceleration as

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}\omega\hat{e}_\theta + r\frac{d\omega}{dt}\hat{e}_\theta + r\omega\frac{d\hat{e}_\theta}{dt}\\ &= \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\omega\hat{e}_\theta + \frac{dr}{dt}\omega\hat{e}_\theta + r\frac{d\omega}{dt}\hat{e}_\theta + r\omega(-\omega\hat{e}_r)\\ &= \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{e}_r + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{e}_\theta,\end{aligned}$$

where $\omega = \frac{d\theta}{dt}$ and $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$.

(2 points) The polar components of the acceleration vector are:

$$\boxed{a_r = \frac{d^2r}{dt^2} - r\omega^2} \quad ; \quad \boxed{a_\theta = r\alpha + 2\frac{dr}{dt}\omega}$$

ii a)

(5 points) Given $r = 2\theta$ ft and $\theta = \omega_0 t$ rad with ω_0 constant, we calculate:

$$\omega = \frac{d\theta}{dt} = \omega_0 \ \text{rad/s} \qquad ; \qquad \frac{dr}{dt} = 2\frac{d\theta}{dt} = 2\omega_0 \ \text{ft/s},$$
$$\alpha = \frac{d\omega}{dt} = 0 \ \text{rad/s}^2 \qquad ; \qquad \frac{d^2r}{dt^2} = 0 \ \text{ft/s}^2.$$

(5 points) Using the expressions for a_r and a_θ above, we get

$$a_r = \cancel{\frac{d^2r}{dt^2}} - r\omega^2 = -r\omega_0^2 = -(2\theta)\omega_0^2 = -(2\omega_0 t)\omega_0^2 = -2\omega_0^3 t \ \text{ft/s}^2,$$
$$a_\theta = r\cancel{\alpha} + 2\frac{dr}{dt}\omega = 2(2\omega_0)\omega_0 = 4\omega_0^2 \ \text{ft/s}^2.$$

(5 points) By Newton's 2nd law

$$\vec{F} = m\vec{a}.$$

In the polar coordinate system:

$$\vec{F} = F_r\hat{e}_r + F_\theta\hat{e}_\theta = m(a_r\hat{e}_r + a_\theta\hat{e}_\theta).$$

The polar components of the total force are:

$$F_r = ma_r = -2m\omega_0^3 t \ \text{slug ft/s}^2$$
$$F_\theta = ma_\theta = 4m\omega_0^2 \ \text{slug ft/s}^2$$

$$\boxed{F_r = -2m\omega_0^3 t \ \text{lb}} \quad ; \quad \boxed{F_\theta = 4m\omega_0^2 \ \text{lb}}$$

ii b)

(2 points) The ratio of the magnitude of the total force to its transverse component is

$$\frac{||\vec{F}||}{F_\theta} = \frac{\sqrt{F_r^2 + F_\theta^2}}{F_\theta} = \sqrt{\left(\frac{F_r}{F_\theta}\right)^2 + 1}.$$

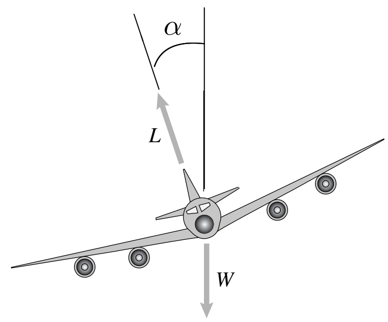
(2 points)Using the expressions from above

$$\frac{||\vec{F}||}{F_\theta} = \sqrt{\left(\frac{-2m\omega_0^3 t}{4m\omega_0^2}\right)^2 + 1} = \sqrt{\left(\frac{\omega_0 t}{2}\right)^2 + 1}.$$

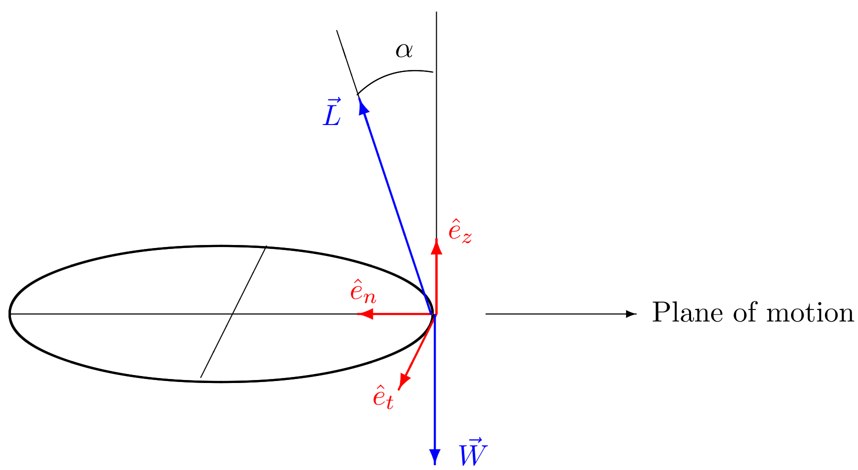
(1 points) For $t = 2$ s, we obtain

$$\boxed{\frac{||\vec{F}||}{F_\theta} = \sqrt{\omega_0^2 + 1}}.$$

3. An airplane of weight W makes a turn at constant altitude and at constant velocity v. The bank angle is α. Find the expression for the radius of curvature ρ of the plane's path as a function *only* of v, g, and α. (30 points)



Solution:
In order to analyze this problem we need to realize that, since the airplane travels at a constant altitude, its path occurs on a horizontal plane. We then choose standard normal and tangential unit vectors: $\hat{e}_n$ and $\hat{e}_t$, on the plane, and since the lift and weight forces has a component perpendicular to the plane, we use an additional unit vector, $\hat{e}_z$, in the vertical direction; see the illustration below:



Note: the forces acting on the airplane are

$$\vec{L} = L\sin(\alpha)\hat{e}_n + L\cos(\alpha)\hat{e}_z$$
$$\vec{W} = -mg\hat{e}_z$$

There is no force acting on the tangential direction.

(10 points) By Newton's 2^{nd} law in the z-direction

$$\Sigma F_z = L\cos(\alpha) - W = ma_z = 0.$$

Then, we obtain

$$(1) \quad L = \frac{W}{\cos(\alpha)}.$$

(5 points) We know that the velocity $\vec{v} = v\hat{e}_t$, then the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} = \frac{dv}{dt}\hat{e}_t + v\omega\,\hat{e}_n,$$

where $\frac{d\hat{e}_t}{dt} = \omega\hat{e}_n$. Furthermore, $v = \rho\omega$ with ρ the radius of curvature, and since v is constant $\frac{dv}{dt} = 0$. Then,

$$\vec{a} = \frac{v^2}{\rho}\,\hat{e}_n.$$

(7 points) By Newton's 2^{nd} law in the normal direction

$$(2) \quad \Sigma F_n = L\sin(\alpha) = ma_n = m\frac{v^2}{\rho}.$$

(8 points) Using equation (1) into (2), we obtain

$$L\sin(\alpha) = m\frac{v^2}{\rho} \Rightarrow \frac{W}{\cos(\alpha)}\sin(\alpha) = m\frac{v^2}{\rho}$$

and by replacing W by mg, we get

$$\frac{\cancel{m}g}{\cos(\alpha)}\sin(\alpha) = \cancel{m}\frac{v^2}{\rho}$$

giving the result

$$\boxed{\rho = \frac{v^2}{g\tan\alpha}}.$$