MATH 102: Calculus and Analytic Geometry II

Spring 2017-2018, Quiz 2, Duration: 60 min.

Write your name and section and circle the name of your instructor.

Name: ____________________________

Section: ____________________________

Circle the name of your instructor:

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Exercise	Points	Scores
1	20	
2	25	
3	20	
4	20	
5	15	
Total	100	

INSTRUCTIONS:

(a) Explain your answers in detail and clearly to ensure full credit.

(b) Some exercises include multiple choice questions. For these questions only, you do **not** have to justify anything, there will be no partial credit, and no penalty for giving a wrong answer.

(c) No book. No notes.

(d) Reasonable answer attempts will be taken into account and may result in partial credit, even if they fail to lead to the solution.

(e) The back of the pages are meant for rough work and will **not be corrected** unless you clearly indicate otherwise.

Exercise 1. (20 points) Let C be the curve of cartesian equation $9x^2+25y^2+54x-100y=44$. We factor, then complete the squares:

$$9(x^2+6x)+25(y^2-4y)=44$$
$$9\big((x+3)^2-3^2\big)+25\big((y-2)^2-2^2\big)=44$$

Then we move the constants to the right, and divide so the constant becomes 1:

$$9(x+3)^2-81+25(y-2)^2-100=44$$
$$9(x+3)^2+25(y-2)^2=44+81+100=225$$
$$\frac{9}{225}(x+3)^2+\frac{25}{225}(y-2)^2=1$$
$$\frac{(x+3)^2}{25}+\frac{(y-2)^2}{9}=1$$
$$\frac{(x+3)^2}{5^2}+\frac{(y-2)^2}{3^2}=1$$

This is the question of an ellipse centered at $x=-3$, $y=2$, with horizontal semi-major axis $a=5$, and vertical semi-minor axis $b=3$. So the vertices are at $(-3\pm 5,2)$, that is to say $(-8,2)$ and $(2,2)$, and the foci are at $-3\pm 4,2)$, that is to say $(-7,2)$ and $(1,2)$.

Exercise 2. (25 points) Consider the parametric curve defined by $x=t^3-3t$, $y=3t^2$, and let P be the point corresponding to $t=0$.

(a) **(5 points)** Compute x^2, and deduce a cartesian equation for this curve.

$$x^2=(t^3-3t)^2=(t(t^2-3))^2=t^2(t^2-3)^2$$

Besides, $t^2=y/3$, so a cartesian equation for this curve is

$$x^2=\frac{y}{3}\left(\frac{y}{3}-3\right)^2.$$

(b) **(3+2=5 points)** Find the value of dy/dx **and** d^2y/dx^2 at P.

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{6t}{3t^2-3} = \frac{2t}{t^2-1},\text{ which is 0 at } t=0.$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{y'}{x'}\right)'}{x'} = \frac{\frac{2(t^2-1)-2t2t}{(t^2-1)^2}}{t^2-1},\text{ which is } 2/3 \text{ at } t=0.$$

(c) **(5 points)** Give the equation of the tangent at P.

By the previous question, the tangent has slope 0, so has an equation of the form $y =$constant (in other words, the tangent is a horizontal line). Besides it goes through P, which has $y = 0$, so this equation is actually $y = 0$ (in other words, the tangent is the x axis).

(d) **(10 points)** Compute the length of the portion of the curve corresponding to $-1 \leqslant t \leqslant 1$.

The length is

$$\begin{aligned}
&\int_{-1}^{1} \sqrt{x'^2 + y'^2} dt \\
=&\int_{-1}^{1} \sqrt{(3t^2-3)^2 + (6t)^2} dt \\
=&3\int_{-1}^{1} \sqrt{(t^2-1)^2 + (2t)^2} dt \\
=&3\int_{-1}^{1} \sqrt{t^4 - 2t^2 + 1 + 4t^2} dt \\
=&3\int_{-1}^{1} \sqrt{t^4 + 2t^2 + 1} dt \\
=&3\int_{-1}^{1} \sqrt{(t^2+1)} dt \\
=&3\int_{-1}^{1} |t^2+1| dt \\
=&3\int_{-1}^{1} (t^2+1) dt \qquad \text{as } t^2+1 > 0 \\
=&3\left[\frac{t^3}{3} + t\right]_{-1}^{1} \\
=&\left[t^3 + 3t\right]_{-1}^{1} \\
=&8.
\end{aligned}$$

Exercise 3. (20 points)

(a) **(10 points)** Is the integral $\int_0^{+\infty} e^{-t}\cos^2(5t)dt$ convergent or divergent? (**You must justify your answer in order to receive credit**)

We have $-1 \leq \cos(5t) \leq 1$ so $0 \leq \cos(5t) \leq 1$ so $0 \leq e^{-t}\cos^2(5t) \leq e^{-t}$. Besides, we know that $\int_0^{+\infty} e^{-t}dt$ converges. So Since $e^{-t}\cos^2(5t)$ is less than something whose integral converges, its integral converges.

(b) **(5 points)** Circle the correct answer: The integral $\int_1^{+\infty} \frac{x^2+\sqrt{x}}{x^a}dx$ converges if and only if:

$a > 1$ $a \geqslant 1$ $a > 3$ $a \geqslant 3$

The only danger is at $+\infty$. For $x \to +\infty$, we expect x^2 to be dominant compared to $\sqrt{x}$, so that the integrand would behave like x^2/x^a. Let's make this rigorous by using the LCT:

$$\frac{(x^2+\sqrt{x})/x^a}{x^2/x^a} = \frac{x^2+\sqrt{x}}{x^2} = 1 + x^{-3/2}$$

which tends to 1 when $x \to \infty$, confirming that the integrand behaves like x^2/x^a at $+\infty$.

Thus the integral in question has the same nature as $\int_1^{+\infty} \frac{x^2}{x^a}dx = \int_1^{+\infty} \frac{1}{x^{a-2}}dx$, and the latter converges if and only if $a - 2 > 1$, that is to say $a > 3$.

(c) **(5 points)** Circle the correct answer: the integral $\displaystyle\int_1^9 \frac{dx}{(x-1)^{2/3}}$

- converges, and its value is 6
- converges, and its value is $6\sqrt[3]{2}$
- converges, and its value is $3(1+\sqrt[3]{2})$
- diverges

This time the danger is at $x = 1$ (division by 0, causing a vertical asymptote). However

$$\int \frac{dx}{(x-1)^{2/3}} = \int (x-1)^{-2/3} dx = \frac{1}{-\frac{2}{3}+1}(x-1)^{-\frac{2}{3}+1} + C = 3\sqrt[3]{x-1}$$

where C is a constant, so

$$\int_1^9 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \int_a^9 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} [3\sqrt[3]{x-1}]_a^9 = 3\sqrt[3]{8} - 3\sqrt[3]{0} = 6,$$

so the integral converges and its value is 6.

Remark: if we perform the substitution $t = x - 1$ so as to bring the problem at $x = 1$ to $t = 0$, we get

$$\int_1^9 \frac{dx}{(x-1)^{2/3}} = \int_0^8 \frac{dt}{t^{2/3}}$$

and we immediately see that this converges since $2/3 < 1$. However this does not give us the value of the integral, that still needs to be computed.

Exercise 4. (20 points)
Compute the following integrals:

(a) **(10 points)** $\int e^t\sqrt{100-e^{2t}}dt$,

Let us start by the substitution $x = e^t$, which gives $dx = e^t dt$ and turns the integral into $\int \sqrt{100-x^2}dx$.

This hints at the right-angled triangle with sides x, $\sqrt{100-x^2}$ and hypotenuse 10. Calling θ one of the angles, we get $\sqrt{100-x^2} = 10\cos\theta$ and $x = 10\sin\theta$, whence $dx = 10\cos\theta d\theta$, so that the integral becomes

$$\begin{aligned}
&\int 10\cos\theta 10\cos\theta d\theta \\
=&100\int \cos^2\theta d\theta \\
=&100\int \frac{1+\cos 2\theta}{2}d\theta \\
=&50\int (1+\cos 2\theta)d\theta \\
=&50(\theta+\frac{1}{2}\sin 2\theta)+C \\
=&50(\sin^{-1}\frac{x}{10}+\frac{1}{2}2\sin\theta\cos\theta)+C \\
=&50\left(\sin^{-1}\frac{x}{10}+\frac{x}{10}\frac{\sqrt{100-x^2}}{10}\right)+C \\
=&50\sin^{-1}\frac{x}{10}+\frac{1}{2}x\sqrt{100-x^2}+C \\
=&50\sin^{-1}\frac{e^t}{10}+\frac{1}{2}e^t\sqrt{100-e^{2t}}+C,
\end{aligned}$$

C a constant.

(b) **(10 points)** $\int \frac{x^3-2}{x^2+x} dx.$

The degree of the numerator is not less than that of the denominator, so we must first perform a long division. We find

$$x^3 - 2 = (x-1)(x^2+x) + (x-2),$$

so

$$\int \frac{x^3-2}{x^2+x} dx = \int \left(x - 1 + \frac{x-2}{x^2+1} \right) dx = \frac{x^2}{2} - x + \int \frac{x-2}{x^2+x} dx.$$

Now $x^2 + x = x(x+1)$, so by partial fractions

$$\frac{x-2}{x^2+1} = \frac{A}{x} + \frac{B}{x+1}$$

for some constants A and B. By covering, we find $A = -2$ and $B = 3$, so

$$\int \frac{x-2}{x^2+x} dx = \int \left(\frac{-2}{x} + \frac{3}{x+1} \right) dx = -2 \ln |x| + 3 \ln |x+1| + C,$$

C a constant. Therefore

$$\int \frac{x^3-2}{x^2+x} dx = \frac{x^2}{2} - x - 2 \ln |x| + 3 \ln |x+1| + C,$$

C a constant.

Exercise 5. (15 points)

Let $R(x) = \dfrac{3x-7}{(x^2+4)(x+1)}$.

(a) **(10 points)** Compute $\displaystyle\int R(x)dx$.

This time the degree of the numerator his less than that of the denominator, so no long division is needed. As x^2+4 is irreducible (it has $\Delta = -16 < 0$), the partial fraction decomposition is of the form

$$\frac{3x-7}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$

with A, B and C constants. Reducing to the same denominator yields

$$\frac{Ax+B}{x^2+4} + \frac{C}{x+1} = \frac{(A+C)x^2+(A+B)x+(B+4C)}{(x^2+4)(x+1)},$$

whence $A+C=0$, $A+B=3$, and $B+4C=-7$ by identifying the coefficients. Solving this system results in $A=2$, $B=1$, and $C=-2$, so that

$$\begin{aligned}
&\int \frac{3x-7}{(x^2+4)(x+1)}dx \\
&= \int \left(\frac{2x+1}{x^2+4} - \frac{2}{x+1}\right) dx \\
&= \int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx - 2\int \frac{dx}{x+1} \\
&= \ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} - 2\ln|x+1| + D,
\end{aligned}$$

D a constant.

Remark: One could also get the value of A, B and C by doing a bit of covering, as

$$(Ax+B)(x+1) + C(x^2+4) = 3x-7$$

yields $5C=-10$ at $x=-1$ whence $C=-2$, and $B+4C=-7$ at $x=0$ whence $B=1$, and finally $2(A+B)+5C=-4$ at $x=1$ whence $A=-2$.

(b) **(5 points)** Circle the correct answer: the integral $\int_0^{+\infty} R(x)dx$

- converges, and its value is $\frac{\pi}{2} - 2\ln 2$
- converges, and its value is $\frac{\pi}{4} - \ln 2$
- converges, and its value is $\frac{\pi}{4} - 2\ln 2$
- diverges

The denominator of $R(x)$ does not vanish on $[0, +\infty)$ so the only danger is at $+\infty$. Checking the degrees, we guess that $R(x)$ behaves like $\frac{3x}{x^3} = \frac{3}{x^2}$ when $x \to \infty$ (we need an LCT to justify this properly), so the integral converges since $2 > 1$. However this does not give us its value.

So we compute

$$\begin{aligned}
&\int_0^{+\infty} \frac{3x-7}{(x^2+4)(x+1)}dx \\
=&\lim_{b\to+\infty} \int_0^{b} \frac{3x-7}{(x^2+4)(x+1)}dx \\
=&\lim_{b\to+\infty} \left[\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} - 2\ln|x+1|\right]_0^b \\
=&\lim_{b\to+\infty} \ln(b^2+4) + \frac{1}{2}\tan^{-1}\frac{b}{2} - 2\ln|b+1| - \ln 4 - \frac{1}{2}\tan^{-1}0 + 2\ln 1 \\
=&\lim_{b\to+\infty} \ln(b^2+4) - \ln(b+1)^2 + \frac{1}{2}\tan^{-1}\frac{b}{2} - \ln 4 \\
=&\lim_{b\to+\infty} \ln\frac{b^2+4}{(b+1)^2} + \frac{1}{2}\tan^{-1}\frac{b}{2} - \ln 4 \\
=&\lim_{b\to+\infty} \ln\frac{1+\frac{4}{b^2}}{\left(1+\frac{1}{b}\right)^2} + \frac{1}{2}\tan^{-1}\frac{b}{2} - \ln 4 \\
=&\ln 1 + \frac{1}{2}\frac{\pi}{2} - \ln 4 \\
=&\frac{\pi}{4} - 2\ln 2.
\end{aligned}$$

END