MATH 102: Calculus and Analytic Geometry II

Spring 2017-2018, Quiz 1, Duration: 60 min.

Exercise 0. (2 points)
Write your name and section and circle the name of your instructor.

Name: ____________________________

Section: ____________________________

Circle the name of your instructor:

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Exercise	Points	Scores
0	2	
1	20	
2	12	
3	14	
4	12	
5	40	
Total	100	

INSTRUCTIONS:

(a) Explain your answers in detail and clearly to ensure full credit.

(b) No book. No notes. No calculator.

(c) Reasonable answer attempts will be taken into account and may result in partial credit, even if they fail to lead to the solution.

(d) The back of the pages are meant for rough work and will not be corrected unless you clealy indicate otherwise.

Exercise 1. (20 points) Let $f(x) = e^x + x$.

(a) **(5 points)** Show that f is 1-to-1.

We have $f'(x) = e^x + 1$ which is > 1 for all x, so f is increasing, so it is one-to-one.

(b) **(5 points)** What are the domain and range of f?

e^x is defined for all x, so the domain of f is $\mathbb{R}$.
Since f is increasing, we can figure out its range by checking its limits:
We have $\lim_{x\to+\infty} e^x = +\infty$ so $\lim_{x\to+\infty} f(x) = +\infty$, and $\lim_{x\to-\infty} e^x = 0$ so $\lim_{x\to-\infty} f(x) = -\infty$.
So the range of f is also $\mathbb{R}$.

(c) **(5 points)** What are the domain and range of f^{-1}?

The domain of f^{-1} is the range of f, which is $\mathbb{R}$ by the previous question.
The range of f^{-1} is the domain of f, which is $\mathbb{R}$ by the previous question.

(d) **(5 points)** Find $\dfrac{df^{-1}}{dx}$ at the point $x = f(\ln 2)$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

so

$$(f^{-1})'(f(\ln 2)) = \frac{1}{f'(f^{-1}(f(\ln 2)))} = \frac{1}{f'(\ln 2)} = \frac{1}{e^{\ln 2} + 1} = \frac{1}{2+1} = \frac{1}{3}.$$

Exercise 2. (12 points) Let $y = \dfrac{2x(2^x)}{\sqrt{1+x^2}}$. Find the value of $\dfrac{dy}{dx}$ at $x = \sqrt{3}$.

Hint: Logarithmic differentiation.

We have

$$\ln y = \ln 2 + \ln x + \ln(2^x) - \ln(\sqrt{x^2+1}) = \ln 2 + \ln x + x\ln 2 - \frac{1}{2}\ln(x^2+1)$$

so

$$\frac{y'}{y} = (\ln y)' = 0 + \frac{1}{x} + \ln 2 - \frac{1}{2}\frac{(x^2+1)'}{x^2+1} = \frac{1}{x} + \ln 2 - \frac{x}{x^2+1}$$

so

$$y' = y\frac{y'}{y} = \left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)\frac{2x(2^x)}{\sqrt{1+x^2}}.$$

At $x = \sqrt{3}$, we get

$$y'(\sqrt{3}) = \left(\frac{1}{\sqrt{3}} + \ln 2 - \frac{\sqrt{3}}{4}\right)\frac{2\sqrt{3}2^{\sqrt{3}}}{2} = \left(\sqrt{3}\ln 2 + \frac{1}{4}\right)2^{\sqrt{3}}.$$

Exercise 3. (14 points)

Solve the initial value problem (a.k.a. Cauchy problem) $\begin{cases} \dfrac{dy}{dx} = e^{-x-y-2}, \\ \\ y(0) = -2. \end{cases}$

$\frac{dy}{dx} = e^{-x-y-2} = e^{-x}e^{-y-2}$ so $\frac{dy}{e^{-y-2}} = e^{-x}dx$ so $e^{y+2}dy = e^{-x}dx$ so

$$\int e^{y+2}dy = \int e^{-x}dx$$

so $e^{y+2} = -e^{-x} + C$, C a constant.

In order to have $y(0) = -2$, we must have $e^{-2+2} = -e^0 + C$, that is $1 = -1 + C$, whence $C = 2$. Thus $e^{y+2} = 2 - e^{-x}$. Taking $\ln$, we get $y + 2 = \ln(2 - e^{-x})$, so finally

$$y = \ln(2 - e^{-x}) - 2.$$

Exercise 4. (12 points)

Determine whether $\lim_{x\to 4} \dfrac{\sin^2(\pi x)}{e^{x-4}+3-x}$ exists, and compute its value if it does.

$\frac{\sin^2(\pi x)}{e^{x-4}+3-x}$ assumes the indeterminate form $\frac{0}{0}$ when $x \to 4$. Let try to apply l'Hospital:
After differentiating the numerator and the denominator, we get

$$\frac{2\pi\cos(\pi x)\sin(\pi x)}{e^{x-4}-1} = \frac{\pi\sin(2\pi x)}{e^{x-4}-1}$$

which still assumes the indeterminate form $\frac{0}{0}$ when $x \to 4$. Differentiating once more, we get

$$\frac{2\pi^2\cos(2\pi x)}{e^{x-4}}$$

which this time clearly tends to $\frac{2\pi^2}{1}$. Thus, by a double application of l'Hosital's rule, we deduce that $\lim_{x\to 4} \dfrac{\sin^2(\pi x)}{e^{x-4}+3-x}$ exists and is $2\pi^2$.

Exercise 5. (40 points)
Compute the following integrals:

(a) **(10 points)** $\displaystyle\int_0^{\pi/4} \sin^2(2\theta)\cos^3(2\theta)d\theta$,

Since 3 is odd, we rewrite the integral as

$$\int_0^{\pi/4} \sin^2(2\theta)\cos^2(2\theta)\cos(2\theta)d\theta$$

and we let $s = \sin(2\theta)$, whence $ds = 2\cos(2\theta)d\theta$ so that $\cos(2\theta)d\theta = \frac{1}{2}ds$. Since furthermore $\cos^(2\theta) = 1 - \sin^2(2\theta) = 1 - s^2$, we get

$$\int_{\sin 0}^{\sin\frac{\pi}{2}} s^2(1-s^2)\frac{1}{2}ds = \frac{1}{2}\int_0^1 (s^2 - s^4)ds$$

which is

$$\frac{1}{2}\left[\frac{s^3}{3} - \frac{s^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}.$$

(b) **(10 points)** $\int_0^{\pi/9} \sqrt{1+\cos(3x)}dx$,

We have $1+\cos(3x) = 2\cos^2\left(\frac{3}{2}x\right)$ so this integral is

$$\int_0^{\pi/9} \sqrt{2\cos^2\left(\frac{3}{2}x\right)}dx = \sqrt{2}\int_0^{\pi/9} \left|\cos\left(\frac{3}{2}x\right)\right| dx.$$

For $0 \leq x \leq \pi/9$ we have $0 \leq \frac{3}{2}x \leq \pi/6$ whence, by looking at the graph of $\cos$, $1 \geq \cos\frac{3}{2}x \geq \frac{\sqrt{3}}{2}$ (it is **NOT** enough to just check the values of cos at the endpoints). So the cos in the integral is always positive, so we can drop the absolute value:

$$\sqrt{2}\int_0^{\pi/9} \left|\cos\left(\frac{3}{2}x\right)\right| dx = \sqrt{2}\int_0^{\pi/9} \cos\left(\frac{3}{2}x\right) dx = \sqrt{2}\left[\frac{1}{3/2}\sin\left(\frac{3}{2}x\right)\right]_0^{\pi/9} = \frac{2\sqrt{2}}{3}\sin(\pi/6)$$

$$= \frac{\sqrt{2}}{3}.$$

(c) **(10 points)** $\displaystyle\int_3^4 \frac{dy}{\sqrt{-y^2+6y-5}}$,

Complete the square: $-y^2+6y-5 = -(y-3)^2+4$ so

$$\int_3^4 \frac{dy}{\sqrt{-y^2+6y-5}} = \int_3^4 \frac{dy}{\sqrt{4-(y-3)^2}} = \int_3^4 \frac{\frac{1}{2}dy}{\sqrt{1-\left(\frac{1}{2}y-\frac{3}{2}\right)^2}}$$

Letting $z = \frac{1}{2}y - \frac{3}{2}$, we have $dz = \frac{1}{2}dy$, so

$$= \int_0^{1/2} \frac{dz}{\sqrt{1-z^2}} = [\sin^{-1} z]_0^{1/2} = \sin^{-1}\frac{1}{2}$$

$$= \frac{\pi}{6}.$$

(d) **(10 points)** $\int (t^2 - 3t + 10)\cos^2(t)dt$.

We have $\cos^2(t) = \frac{1+\cos(2t)}{2}$ so

$$\int (t^2 - 3t + 10)\cos^2(t)dt = \int (t^2 - 3t + 10)\frac{1+\cos(2t)}{2}dt = \frac{1}{2}\int (t^2 - 3t + 10)dt + \frac{1}{2}\int (t^2 - 3t + 10)\cos(2t)dt.$$

The first integral is easy; for the second one, we integrate by parts repeatedly by differentiating the polynomial until is becomes 0 (and hence integrating the cos):

$'$		$\int$
$t^2 - 3t + 10$		$\cos(2t)$
	$+$	
$2t - 3$		$\frac{1}{2}\sin(2t)$
	$-$	
2		$-\frac{1}{4}\cos(2t)$
	$+$	
0		$-\frac{1}{8}\sin(2t)$

Finally we get

$$\int (t^2 - 3t + 10)\cos^2(t)dt$$

$$= \frac{1}{2}\left(\frac{t^3}{3} - \frac{3}{2}t^2 + 10t\right) + \frac{1}{2}\left(\frac{t^2 - 3t + 10}{2}\sin(2t) + \frac{2t-3}{4}\cos(2t) - \frac{1}{4}\sin(2t)\right) + cst.$$

$$= \frac{t^3}{6} - \frac{3}{4}t^2 + 5t + \frac{2t^2 - 6t + 19}{8}\sin(2t) + \frac{2t-3}{8}\cos(2t) + cst.$$

END