MATH102 mid-term quiz 2
Friday 27th April, 2012

Name: ________________________________ ID: __________________

Circle your recitation section: 5 (8am) 6 (9am) 7 (11am) 8 (12am)

Question:	1	2	3	4	Bonus	Total
Points:	5	15	12	18	5	50
Score:						

Time: 50 minutes.
You must show your working for all questions.

1. (5 points) Evaluate the indefinite integral $\int \sec^4 x \, \mathrm{d}x$, giving your answer in terms of x.

Solution:

$$\int \sec^4 x \, \mathrm{d}x = \int (1 + \tan^2 x) \sec^2 x \, \mathrm{d}x$$

so we can substitute $u = \tan x$ and $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$ to get

$$\cdots = \int (1 + u^2) \, \mathrm{d}u = u + \frac{1}{3}u^3 + C = \tan x + \frac{1}{3}\tan^3 x + C.$$

2. (a) (8 points) Evaluate the definite integral $\displaystyle\int_0^2 \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x$.

Solution: Substitute $x = 3\sin\theta$, so $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$ and $\theta = \sin^{-1}(x/3)$. Remember to change the limits as well: $x = 0$ gives $\theta = 0$, and $x = 2$ gives $\theta = \sin^{-1}(2/3)$. We get

$$\begin{aligned}
&\int_0^{\sin^{-1}(2/3)} \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin\theta)^2}}(3\cos\theta)\,\mathrm{d}\theta \\
&\quad= \int_0^{\sin^{-1}(2/3)} \frac{(3\sin\theta)^2}{|3\cos\theta|}(3\cos\theta)\,\mathrm{d}\theta \\
&\quad= \int_0^{\sin^{-1}(2/3)} (3\sin\theta)^2\,\mathrm{d}\theta \qquad\qquad (\text{because } \cos\theta \geq 0) \\
&\quad= 9\int_0^{\sin^{-1}(2/3)} \frac{1-\cos(2\theta)}{2}\,\mathrm{d}\theta \\
&\quad= \frac{9}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\sin^{-1}(2/3)} \\
&\quad\approx 1.047 \quad .
\end{aligned}$$

Question 2 continues . . .

Question 2 (continued)

(b) (5 points) Use the trapezoid rule, with 10 intervals of width 0.2, to work out an approximation to the definite integral of part (a). The following table gives values of the function $f(x) = x^2/\sqrt{9-x^2}$ at the values you will need.

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0.00	0.01	0.05	0.12	0.22	0.35	0.52	0.74	1.01	1.35	1.79

Solution: Using the trapezoid rule:

$$\begin{aligned} A_T &= \frac{0.2}{2}(0.00 + 2\times 0.01 + 2\times 0.05 + 2\times 0.12 + 2\times 0.22 + 2\times 0.35 \\ &\qquad + 2\times 0.52 + 2\times 0.74 + 2\times 1.01 + 2\times 1.35 + 1.79) \\ &= 1.053 \quad . \end{aligned}$$

(c) (2 points) Give two ways in which you could modify your calculation in part (b) to give a better approximation to the actual value of the integral.

Solution: Either use more smaller intervals, or use an improved method such as Simpson's Rule.

3. (a) (7 points) Use the method of partial fractions to express the rational function $\dfrac{3x^2-6x+4}{(x^2+4)(x-3)}$ as a sum of simpler fractions.

Solution: Write

$$\frac{3x^2-6x+4}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3}$$

and solve for A, B and C:

$$\begin{aligned}\frac{3x^2-6x+4}{(x^2+4)(x-3)} &= \frac{(Ax+B)(x-3)+C(x^2+4)}{(x^2+4)(x+3)} \\ &= \frac{(A+C)x^2+(-3A+B)x+(-3B+4C)}{(x^2+4)(x-3)}\end{aligned}$$

and so $A+C=3$, $-3A+B=-6$ and $-3B+4C=4$. The first two give $C=3-A$ and $B=3A-6$, respectively; substituting these for B and C in the third equation gives

$$-3(3A-6)+4(3-A) = 30-13A = 4$$

and so $A=2$. Substituting back gives $B=0$ and $C=1$. So

$$\frac{3x^2-6x+4}{(x^2+4)(x-3)} = \frac{2x}{x^2+4} + \frac{1}{x-3}.$$

Question 3 continues . . .

Question 3 (continued)

(b) (5 points) Does the improper integral $\int_4^\infty \frac{3x^2 - 6x + 4}{(x^2+4)(x-3)}\,\mathrm{d}x$ converge? Justify your answer.

Solution: The integral diverges, and there are several ways to show it.
One way is to directly compute the limit:

$$\begin{aligned}
\int_4^\infty \frac{3x^2 - 6x + 4}{(x^2+4)(x-3)}\,\mathrm{d}x &= \lim_{a\to\infty} \int_4^a \frac{3x^2 - 6x + 4}{(x^2+4)(x-3)}\,\mathrm{d}x \\
&= \lim_{a\to\infty} \int_4^a \left(\frac{2x}{x^2+4} + \frac{1}{x-3} \right) \mathrm{d}x \\
&= \lim_{a\to\infty} \left[\ln(x^2+4) + \ln(x-3) \right]_4^a \\
&= \infty.
\end{aligned}$$

Another way is to use the direct comparison test: looking at the partial fraction expression, the integrand is definitely larger than $1/x$, because the first term is positive and the second term is greater than $1/x$. Since we know that $\int_4^\infty \mathrm{d}x/x$ diverges, so this integral diverges as well.

4. This question is about the parametric curve defined by

$$x = e^{-t}\cos t, \qquad y = e^{-t}\sin t, \qquad 0 \le t < \infty.$$

   (a) (4 points) At what points does the curve cut the x-axis? (Give the coordinates of each point, as well as the value of t).

   Solution: Setting $y = 0$ gives us $e^{-t}\sin t = 0$, and since e^{-t} cannot be zero, we deduce that $\sin t = 0$ and so $t = 0, \pi, 2\pi, 3\pi, \ldots$. So the points where the curve meets the x-axis are

$$(1, 0), (-e^{-\pi}, 0), (e^{-2\pi}, 0), (-e^{-3\pi}, 0), \ldots .$$

   (b) (4 points) Show that the slope of the curve at time t is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin t - \cos t}{\sin t + \cos t}.$$

   Solution: The derivatives of x and y are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -e^{-t}\cos t - e^{-t}\sin t = -e^{-t}(\sin t + \cos t)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t}\sin t + e^{-t}\cos t = -e^{-t}(\sin t - \cos t).$$

   So we compute

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-e^{-t}(\sin t - \cos t)}{-e^{-t}(\sin t + \cos t)} = \frac{\sin t + \cos t}{\sin t - \cos t}.$$

   (c) (2 points) Find the slope of the curve each time it cuts the x-axis. What is unusual about your answer?

   Solution: At the points of interest ($t = 0, \pi, 2\pi, 3\pi, \ldots$) we have $\sin t = 0$ and so $\mathrm{d}y/\mathrm{d}x = -1$. The unusual fact is that the slope is *the same* every time the curve cuts the x-axis.

Question 4 continues …

Question 4 (continued)

(d) (8 points) Calculate the length of the curve between $t = 0$ and $t = 2\pi$.

Solution: The length is given by

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{\big(-e^{-t}(\sin t + \cos t)\big)^2 + \big(-e^{-t}(\sin t - \cos t)\big)^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{e^{-2t}\big((\sin^2 t + 2\sin t\cos t + \cos^2 t) + (\sin^2 t - 2\sin t\cos t + \cos^2 t)\big)}\,\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{2e^{-2t}(\sin^2 t + \cos^2 t)}\,\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{2e^{-2t}}\,\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{2}e^{-t}\,\mathrm{d}t \\
&= \sqrt{2}\left[-e^{-t}\right]_0^{2\pi} \\
&= \sqrt{2}(1 - e^{-2\pi}) \approx 1.412 \quad .
\end{aligned}$$

Bonus question (5 points)

What can you say about the length of the *whole* curve in question 4?

Solution: By the same calculation as in question 4, we have

$$\int_0^{\infty} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \lim_{a\to\infty} \sqrt{2}(1 - e^{-a}) = \sqrt{2}$$

and so the length of the whole curve is finite, and equals $\sqrt{2}$.