MATH102 mid-term quiz 1
Saturday 17 March 2012

Name: ______________________________ ID: ________________

Circle your recitation section: 5 (8am) 6 (9am) 7 (11am) 8 (12am)

Question:	1	2	3	4	5	6	7	Bonus	Total
Points:	18	12	12	12	19	12	15	5	100
Score:									

1. (18 points) Evaluate the following indefinite integrals. Show your working.

   (a) $\displaystyle\int \frac{x\,\mathrm{d}x}{x^2-1}$

   Solution: Substitute $u = x^2 - 1$, so that $\mathrm{d}u = 2x\,\mathrm{d}x$. Then

   $$\int \frac{x\,\mathrm{d}x}{x^2-1} = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

   (b) $\displaystyle\int \frac{\mathrm{d}x}{x^2-2x+2}$

   Solution: Firstly, complete the square: $\displaystyle\int \frac{\mathrm{d}x}{x^2-2x+2} = \int \frac{\mathrm{d}x}{(x-1)^2+1}$. Now substitute $u = x - 1$ to obtain

   $$\int \frac{\mathrm{d}x}{(x-1)^2+1} = \int \frac{\mathrm{d}u}{u^2+1} = \tan^{-1}(u) + C = \tan^{-1}(x-1) + C.$$

   (c) $\displaystyle\int 10^x\,\mathrm{d}x$

   Solution: Use the definition of 10^x as $e^{(\ln 10)x}$:

   $$\int 10^x\,\mathrm{d}x = \int e^{(\ln 10)x}\,\mathrm{d}x = \frac{1}{\ln 10}e^{(\ln 10)x} + C = \frac{1}{\ln 10}10^x + C.$$

   (You may like to know that $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}x) = \frac{1}{1+x^2}$.)

2. (12 points) Evaluate the following limits. Show your working.

   (a) $\lim_{x \to 0} \frac{e^x - 1}{x}$

   Solution:

$$\begin{aligned} \lim_{x \to 0} \frac{e^x - 1}{x} &= \lim_{x \to 0} \frac{e^x}{1} && \text{by l'Hôpital's rule} \\ &= 1. \end{aligned}$$

   (b) $\lim_{x \to \infty} \frac{(\ln x)^2}{x + 1}$

   Solution:

$$\begin{aligned} \lim_{x \to \infty} \frac{(\ln x)^2}{x+1} &= \lim_{x \to \infty} \frac{2 \ln x (1/x)}{1} = \lim_{x \to \infty} \frac{2 \ln x}{x} && \text{by l'Hôpital's rule} \\ &= \lim_{x \to \infty} \frac{2/x}{1} && \text{by l'Hôpital's rule} \\ &= 0. \end{aligned}$$

3. (a) (6 points) Indicate which of the following functions are one-to-one. For each function which is *not* one-to-one, find two different values of x for which the function gives the same result.

   i. $f(x) = (x - 1)^3$

   Solution: This is one-to-one.

   ii. $g(x) = (x - 1)^4$

   Solution: This is not one-to-one. For example, $g(0) = g(2) = 1$.

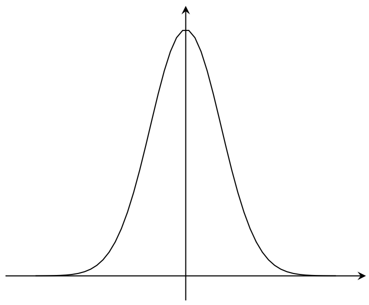
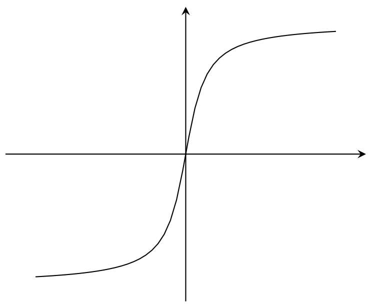
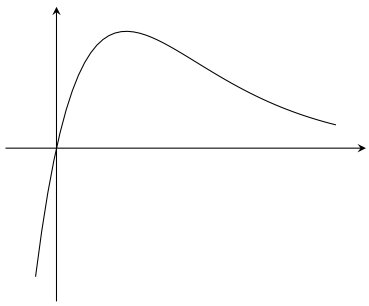
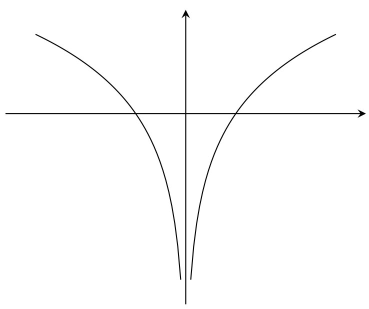
   iii. $h(x) = 1/(x - 1)$

   Solution: This is one-to-one.

   (b) (6 points) I am thinking of a function f, which satisfies $f(2) = 3$ and $f'(2) = -2$. If I tell you that f has an inverse f^{-1}, write numbers in the four blank spaces to show two facts you can deduce about f^{-1}:

$$f^{-1}(\ 3\ ) = \ 2\ ; \qquad (f^{-1})'(\ 3\ ) = \ -1/2\ .$$

4. (12 points) Write letters in the boxes to match the following functions to their graphs (which are not all to the same scale).

Functions		**Graphs**
A. $y = \ln \lvert x \rvert$	C	
B. $y = xe^{-x}$	D	
C. $y = e^{-x^2}$	B	
D. $y = \tan^{-1} x$	A	

5. (a) (9 points) Suppose a hot object is placed in a cool environment. If we let $H(t)$ denote the temperature of the object at time t, and H_S the constant temperature of the environment, then Newton's law of cooling states that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -k(H - H_S),$$

where k is some constant depending on the physical conditions. Show how to solve this differential equation to get the solution

$$H(t) = H_S + Ae^{-kt}$$

for some constant A.

Solution:

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= -k(H - H_S) \\
\frac{1}{H - H_S}\frac{\mathrm{d}H}{\mathrm{d}t} &= -k && \text{(rearranging)} \\
\int \frac{\mathrm{d}H}{H - H_S} &= \int (-k)\,\mathrm{d}t && \text{(integrating both sides with respect to } t\text{)} \\
\ln(H - H_S) &= -kt + C && \text{(no need for absolute value since } H - H_S > 0\text{)} \\
H - H_S &= e^{-kt+C} && \text{(exponentiating both sides)} \\
H - H_S &= Ae^{-kt} && \text{(letting } A = e^C\text{)} \\
H &= H_S + Ae^{-kt}.
\end{aligned}$$

Question 5 continues . . .

Question 5 (continued)

(b) (10 points) I like to drink my coffee at precisely 55 ℃. This morning I took my coffee at 95 ℃ and left it to cool in a room at 15 ℃. Unfortunately, I left it too long: after 6 minutes, the coffee had already cooled to 35 ℃. Assuming that my coffee obeys Newton's law of cooling, how long should I have left the coffee before drinking it?

Solution: From the question, $H_S = 15$. We have two further pieces of information: when $t = 0$, $H = 95$; and when $t = 6$, $H = 35$.

1. Substituting $t = 0$ and $H = 95$ into the equation from part (a) gives $95 = 15 + A$, so $A = 80$.
2. Substituting $t = 6$ and $H = 35$ gives $35 = 15 + 80e^{-6k}$, so $e^{-6k} = 20/80 = 1/4$ and $k = -\ln(1/4)/6$.
3. Now substitute $H = 55$ and solve for t: we get $55 = 15 + 80e^{-kt}$, so $e^{-kt} = 40/80 = 1/2$ and therefore

$$t = \frac{\ln(1/2)}{-k} = 6 \times \frac{\ln(1/2)}{\ln(1/4)} = 6 \times \frac{1}{2} = 3.$$

6. (12 points) Using integration by parts, evaluate the definite integral $\int_0^1 x^2 e^{-x}\,\mathrm{d}x$. Show your working.

Solution:

$$\begin{aligned}
\int_0^1 x^2 e^{-x}\,\mathrm{d}x &= \left[-x^2e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,\mathrm{d}x && \text{(integration by parts)}\\
&= \left[-x^2e^{-x}\right]_0^1 + \left[-2xe^{-x}\right]_0^1 + \int_0^1 2e^{-x}\,\mathrm{d}x && \text{(integration by parts again)}\\
&= \left[-x^2e^{-x}\right]_0^1 - \left[2xe^{-x}\right]_0^1 + \left[-2e^{-x}\right]_0^1\\
&= \left[e^{-x}(-x^2 - 2x - 2)\right]_0^1\\
&= e^{-1}(-1 - 2 - 2) - e^0(-2)\\
&= -5e^{-1} + 2 \approx 0.1606.
\end{aligned}$$

7. (15 points) Indicate whether each of the following statements is true or false. You do *not* need to justify your answers. **Each correct answer scores 3 points, but if you answer and get it wrong then you *lose* one point.**

   [T] **F** 3^x grows faster then 2^x as $x \to \infty$.

   T **[F]** $\log_2 x$ grows faster than $\log_3 x$ as $x \to \infty$.

   T **[F]** For any value of x, we have $\sin^{-1}(\sin x) = x$.

   T **F** For any value of x, we have $\sin(\sin^{-1} x) = x$. *It could be argued that this is only true for $-1 \leq x \leq 1$, since it doesn't even make sense for other values of x, so either answer to this question was counted as correct.*

   T **[F]** The function $y = \cos 2x$ satisfies the differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$.

Bonus question (5 points)

Find two functions f and g such that $f \circ g$ is one-to-one, but f is not one-to-one.

Solution: One possible answer is $g(x) = e^x$ and $f(x) = x^2$.