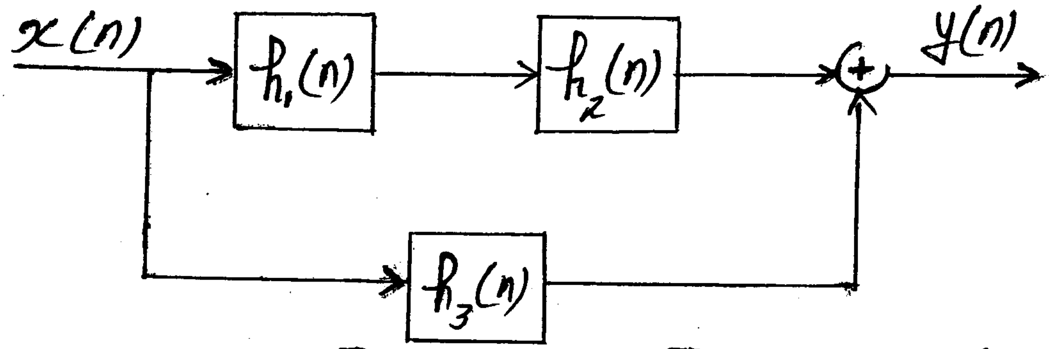
EECE 340 Final Exam Solution Spring 2009

Problem #1



$$y(n) = x(n) * [h_1(n) * h_2(n)] + x(n) * h_3(n)$$

$$= x(n) * [(h_1(n) * h_2(n)) + h_3(n)]$$

Since convolution is distributive with respect to addition.

$$h(n) = h_1(n) * h_2(n) + h_3(n)$$

$$= [-3\delta(n+1) + 2\delta(n-2)] * [2\delta(n+2) + \delta(n-1)] + h_3(n)$$

$$= -6\delta(n+3) - 3\delta(n) + 4\delta(n) + 2\delta(n-3)$$
$$+ 3\delta(n+1) - \delta(n) + 2\delta(n-1) + 7\delta(n-3)$$
$$+ 5\delta(n-5)$$

$$= -6\delta(n+3) + 3\delta(n+1) + 2\delta(n-1) + 9\delta(n-3)$$
$$+ 5\delta(n-5)$$

Problem #2

For a LTI discrete-time system, the input-output relationship is given by:

$$y(n) = 2^{-n} \sum_{k=-\infty}^{\infty} 2^{k} x(k)$$

$$= \sum_{k=-\infty}^{\infty} 2^{-(n-k)} x(k)$$

$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad \text{discrete convolution}$$

$\Rightarrow h(n-k) = 2^{-(n-k)}$ and $h(n) = 2^{-n}$ for all n.

$\Rightarrow$ the system is non-causal and unstable.

Since $h(n) \neq 0$ for $n<0$

and $\sum_{n=-\infty}^{\infty} |h(n)| \to \infty$.

Problem #3

$$G(z) = 2X(z) + 4Y(z)$$

$$= \frac{4z^2}{(z+1)^2} + \frac{12z^2}{(z+\ )^2(z-2)}$$

$$= \frac{4z^2(z-2) + 12z^2}{(z+1)^2(z-2)}$$

$$= \frac{4z^2(z+1)}{(z+1)^2(z-2)} = \frac{4z^2}{(z+1)(z-2)}; 1<|z|<2$$

Problem # 4

$$H(Z) = \frac{1-2Z^{-1}}{(Z-1)(Z+2)}, \quad |Z|>2$$

$$h_1(n) = \delta(n) - \delta(n-1)$$

$$H_1(Z) = 1 - Z^{-1} = \frac{Z-1}{Z}, \quad |Z|>0$$

$$H_{eq}(Z) = H(Z)H_1(Z) = \frac{1-2Z^{-1}}{Z(Z+2)}$$

$$= \frac{(Z-2)}{Z^2(Z+2)}, \quad |Z|>2$$

$$= \frac{A}{Z} + \frac{B}{Z^2} + \frac{C}{Z+2} = \frac{1}{Z} - \frac{1}{Z^2} - \frac{1}{Z+2}$$

$$= Z^{-1} - Z^{-2} - \frac{Z^{-1}}{1-(-2Z^{-1})}$$

$$\Rightarrow h_{eq}(n) = \delta(n-1) - \delta(n-2) - (-2)^{n-1}u(n-1)$$

$$= \delta(n-2) - (-2)^{n-1}u(n-3)$$

Problem # 4 (Alternative Solution)

$$H(Z) = \frac{1 - 2Z^{-1}}{(Z-1)(Z+2)}, \; |Z| > 2$$

$$h_1(n) = \delta(n) - \delta(n-1)$$

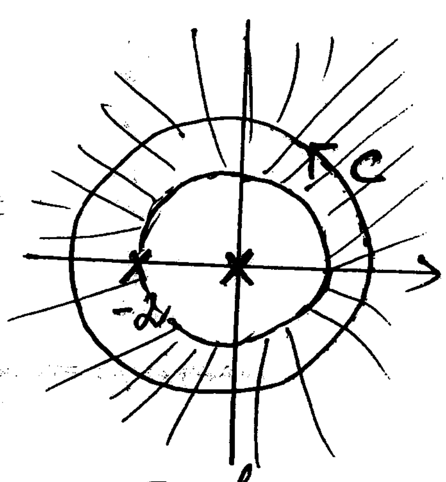
$$H_1(Z) = 1 - Z^{-1} = \frac{(Z-1)}{Z}, \; |Z| > 0$$

The transfer function of the system equivalent to the cascaded connection of $H(Z)$ and $H_1(Z)$ is:

$$H_{eq}(Z) = H(Z)\,H_1(Z) = \frac{1 - 2Z^{-1}}{Z(Z+2)}$$

$$= \frac{(Z-2)}{Z^2(Z+2)}, \; |Z| > 2.$$

$$h_{eq}(n) = \frac{1}{2\pi j} \oint_C \frac{(Z-2)Z^{n-1}}{Z^2(Z+2)}\,dZ$$



$$= \frac{1}{2\pi j} \oint_C \frac{(Z-2)Z^{n-3}}{(Z+2)}\,dZ$$

Z- plane

For $n \geq 3$, $h_{eq}(n) = (Z-2)Z^{n-3}\Big|_{Z=-2}$

$$= -4(-2)^{n-3}$$

$$= -(-2)^2(-2)^{n-3}$$

$$= -(-2)^{n-1}$$

For $\underline{n=2}$, $h_{eq}(n) = \frac{1}{2\pi j}\oint_c \frac{z-2}{z(z+2)}dz$

$$= \left.\frac{z-2}{z+2}\right|_{z=0} + \left.\frac{z-2}{z}\right|_{z=-2}$$

$$= -1 + 2 = 1$$

For $\underline{n=1}$, $h_{eq}(n) = \frac{1}{2\pi j}\oint_c \frac{z-2}{z^2(z+2)}\cancel{dz}$

$$= \left.\frac{z-2}{z^2}\right|_{z=-2} + \left[\frac{d}{dz}\left(\frac{z-2}{z+2}\right)\right]_{z=0}$$

$$= -1 + \left.\frac{(z+2)-(z-2)}{(z+2)^2}\right|_{z=0}$$

$$= -1 + \left.\frac{4}{(z+2)^2}\right|_{z=0} = 0$$

For $\underline{n=0}$, $h_{eq}(n) = \frac{1}{2\pi j}\oint_c \frac{z-2}{z^3(z+2)}dz$

$$= \left.\frac{z-2}{z^3}\right|_{z=-2} + \frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{z-2}{z+2}\right)\right]_{z=0}$$

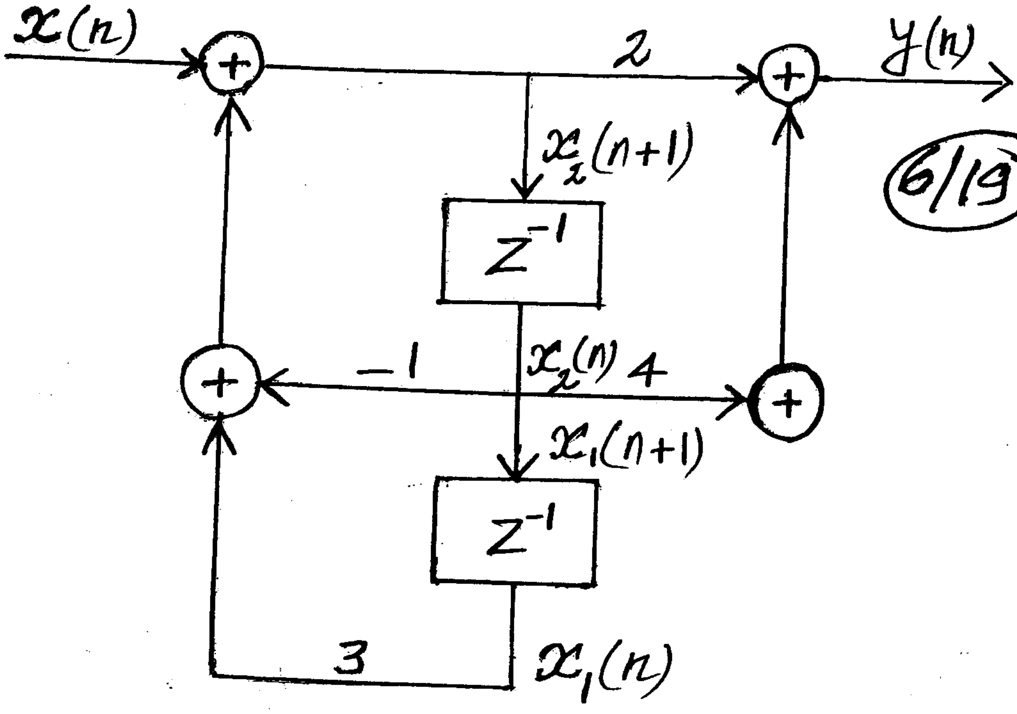
$$= \frac{4}{8} + \frac{1}{2}\left\{\frac{d}{dz}\left[\frac{4}{(z+2)^2}\right]\right\}_{z=0}$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{-2(z+2)\times 4}{(z+2)^4}\right)_{z=0}$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$\Rightarrow h_{eq}(n) = \delta(n-2) - (-2)^{n-1}u(n-3)$$

Problem #5



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$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = 3x_1(n) - x_2(n) + x(n)$$
$$y(n) = 2\left[3x_1(n) - x_2(n) + x(n)\right] + 4x_2(n)$$
$$= 6x_1(n) + 2x_2(n) + 2x(n)$$

In matrix form:

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + 2x(n)$$

Problem # 6

$$H(\omega) = \frac{1+e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

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$$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$$

$$\Rightarrow H(z) = \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{z+1}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\Rightarrow h(n) = \left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$

OR, using the method of residues, we write:

$$h(n) = \frac{1}{2\pi j}\oint_c \frac{z+1}{z-\frac{1}{2}} z^{n-1}\,dz$$

For $n \geq 1$, $h(n) = (z+1)z^{n-1}\Big]_{z=\frac{1}{2}}$

$$= \frac{3}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$= 3\left(\frac{1}{2}\right)^n$$

For $n=0$, $h(n) = \frac{z+1}{z}\Big|_{z=\frac{1}{2}} + \frac{z+1}{z-\frac{1}{2}}\Big|_{z=0}$

$$= 3 - 2 = 1.$$

$$\Rightarrow h(n) = \delta(n) + 3\left(\tfrac{1}{2}\right)^n u(n-1)$$

This $h(n)$ is the same as above which can be obtained as follows using the method of residues:

$$h(n) = \frac{1}{2\pi j}\oint_c \frac{z^n}{z-\frac{1}{2}}\,dz + \frac{1}{2\pi j}\oint_c \frac{z^{n-1}}{z-\frac{1}{2}}\,dz$$

$$= z^n\Big|_{z=\frac{1}{2}} + z^{n-1}\Big|_{z=\frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$= \delta(n) + 3\left(\frac{1}{2}\right)^n u(n-1)$$

Alternative but equal answers.

Problem #7

$$0.25y(n-2) + y(n-1) + y(n) = x(n-2)$$

Applying the Z-transform to both sides of the above difference equation we obtain:

$$Y(z)\left[0.25z^{-2} + z^{-1} + 1\right] = z^{-2}X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{0.25z^{-2} + z^{-1} + 1}$$

$$= \frac{1}{z^2 + z + \frac{1}{4}} = \frac{1}{\left(z+\frac{1}{2}\right)^2}$$

Since the system is causal and stable, then the ROC is: $|z| > \frac{1}{2}$.

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$$

$$= \frac{1}{\left(e^{j\omega}+\frac{1}{2}\right)^2} = \frac{1}{\left(\cos\omega + j\sin\omega + \frac{1}{2}\right)^2}$$

$$\Rightarrow |H(\omega)| = \frac{1}{\left(\cos\omega+\frac{1}{2}\right)^2 + \sin^2\omega}$$

$$= \frac{1}{\cos\omega + \frac{1}{4} + \cos^2\omega + \sin^2\omega}$$

$$= \frac{1}{\cos\omega + \frac{5}{4}}$$

Problem #8

$$x(n) = \begin{cases} \frac{1}{4}, & n = 0, 1, 2, 3 \\ 0, & \text{elsewhere} \end{cases}$$

The 4-point DFT of $x(n)$ is:

$$X(k) = \sum_{n=0}^{3} \frac{1}{4} W_4^{kn}, \quad k = 0, 1, 2, 3.$$

$$= \frac{1}{4}\left[\frac{1 - W_4^{4k}}{1 - W_4^{k}}\right]$$

with $W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$

$$W_4^{4k} = e^{-j\frac{\pi}{2}4k} = e^{-j2\pi k} = 1.$$

$$\Rightarrow X(k) = \frac{1}{4}\left[\frac{1-e^{-j2\pi k}}{1-e^{-j\frac{\pi}{2}k}}\right]$$

$$X(0) = \frac{1}{4}\left[\frac{1-1}{1-1}\right] = \frac{0}{0}$$

Applying L'Hopital's rule:

$$X(0) = \frac{1}{4}\left[\frac{j2\pi\, e^{-j2\pi k}}{j\frac{\pi}{2}\, e^{-j\frac{\pi}{2}k}}\right]_{k=0} = 1 = X(\omega)\Big|_{\omega=0}$$

$$X(1) = \frac{1}{4}\left(\frac{1-1}{1-(-j)}\right) = 0 = X(\omega)\Big|_{\omega=\frac{\pi}{2}}$$

$$X(2) = \frac{1}{4}\left(\frac{1-1}{1-(-j)^2}\right) = 0 = X(\omega)\Big|_{\omega=\pi}$$

$$X(3) = \frac{1}{4}\left(\frac{1-1}{1-(-j)^3}\right) = 0 = X(\omega)\Big|_{\omega=\frac{3\pi}{2}}.$$

Problem #9

$$f(n) = 3\delta(n) + \delta(n-1) + 2\delta(n-2) - \delta(n-3)$$

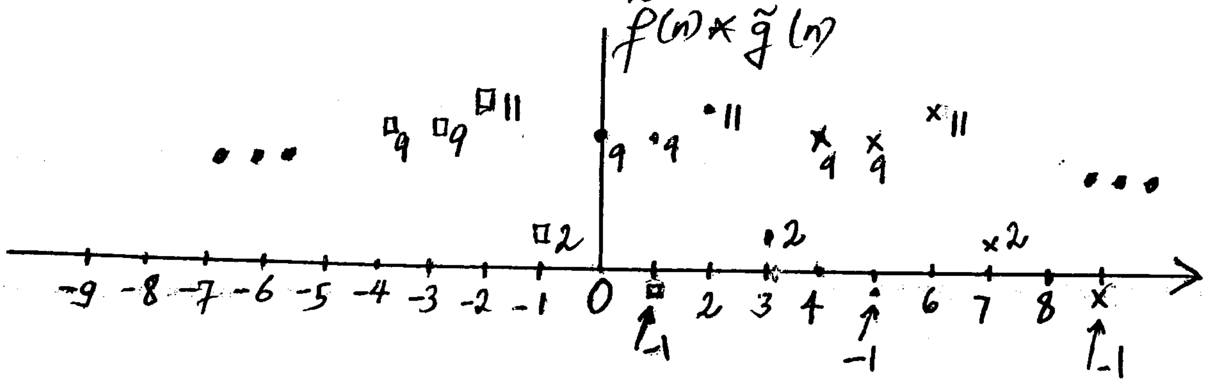
$$g(n) = 3\delta(n) + 2\delta(n-1) + \delta(n-2)$$

$$y(n) = \left[\tilde{f}(n) * \tilde{g}(n)\right] R_4(n)$$

Where $\tilde{f}(n)$ and $\tilde{g}(n)$ are the periodic repetitions of $f(n)$ and $g(n)$ respectively with period 4.

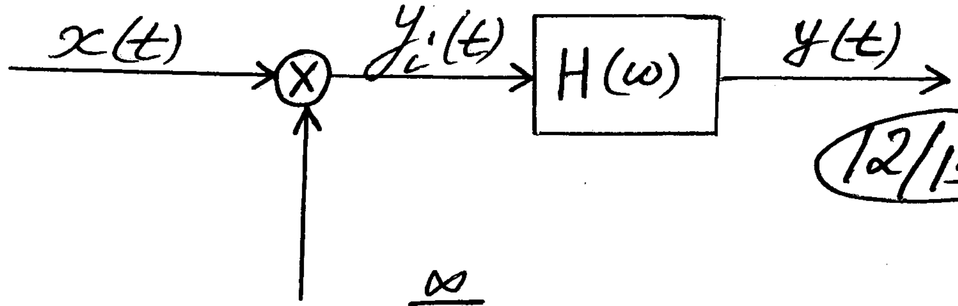
To obtain $\tilde{f}(n) * \tilde{g}(n)$, we do the linear convolution of $f(n)$ and $g(n)$ and then repeat the result periodically with period equal to 4.

$$
\begin{aligned}
f(n) * g(n) &= 9\delta(n) + 6\delta(n-1) + 3\delta(n-2) \\
&\quad + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3) \\
&\quad + 6\delta(n-2) + 4\delta(n-3) + 2\delta(n-4) \\
&\quad - 3\delta(n-3) - 2\delta(n-4) - \delta(n-5) \\
&= 9\delta(n) + 9\delta(n-1) + 11\delta(n-2) \\
&\quad + 2\delta(n-3) - \delta(n-5)
\end{aligned}
$$



$$
\Rightarrow y(n) = 9\delta(n) + 8\delta(n-1) + 11\delta(n-2) + 2\delta(n-3)
$$

Problem # 10



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x(t) = \cos\left(\frac{3\pi t}{2}\right)$ and $H(\omega)$ is an ideal LPF with bandwidth equal to π rad/sec. $T = 1$.

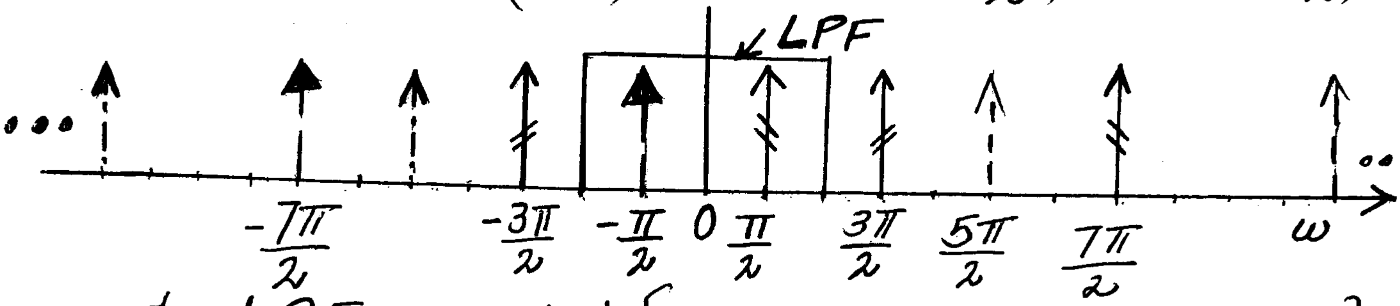
$x(t) \times p(t)$ represents the instantaneous sampling of $x(t)$. Hence, the input, $y_i(t)$, of the LPF has the following Fourier spectrum:

$$Y_i(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$= \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0),$$

where $\omega_0 = \frac{2\pi}{T} = 2\pi$ since $T = 1$.

$$X(\omega) = \mathcal{F}(x(t)) = \pi\,\delta\left(\omega - \frac{3\pi}{2}\right) + \pi\,\delta\left(\omega + \frac{3\pi}{2}\right)$$



The LPF output is $\left\{\pi\,\delta\left(\omega - \frac{\pi}{2}\right) + \pi\,\delta\left(\omega + \frac{\pi}{2}\right)\right\}$

$\Rightarrow y(t) = \cos\left(\frac{\pi}{2} t\right).$

Problem # 11

$h_1(n) = \frac{4}{7} 2^{n-1} u(-n)$ and $h_2(n) = \delta(n) - 3^{n-1} u(-n)$

$h(n) = h_1(n) * h_2(n)$

$\Rightarrow H(z) = H_1(z) \cdot H_2(z)$

$$H_1(z) = \frac{4}{7} \sum_{n=-\infty}^{0} 2^{n-1} z^{-n} = \frac{2}{7} \sum_{n=-\infty}^{0} \left(\frac{z}{2}\right)^{-n}$$

$$= \frac{2}{7} \sum_{n=0}^{\infty} \left(\frac{1}{2} z\right)^{n} = \frac{2}{7} \frac{1}{1 - \frac{1}{2} z}$$

$$= -\frac{4}{7(z-2)}, \quad |z| < 2$$

$$H_2(z) = 1 - \frac{1}{3} \sum_{n=-\infty}^{0} \left(\frac{z}{3}\right)^{-n} = 1 - \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^{n}$$

$$= 1 - \frac{1}{3} \times \frac{1}{1 - \frac{1}{3} z} = 1 + \frac{1}{z-3}$$

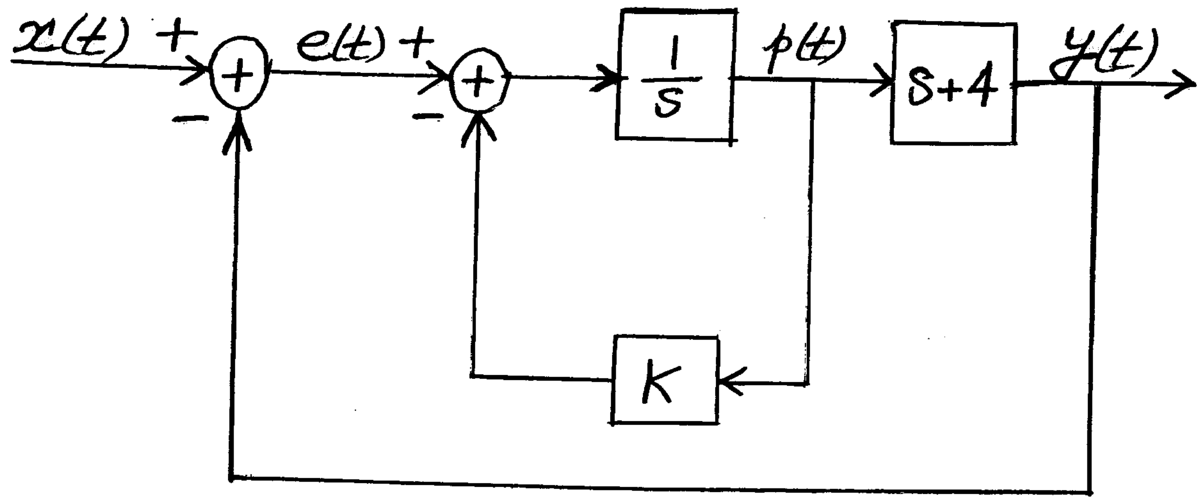
$$= \frac{z-2}{z-3}, \quad |z| < 3$$

Hence, $$H(z) = -\frac{4}{7(z-2)} \times \frac{z-2}{z-3}$$

$$= -\frac{4}{7(z-3)}, \quad |z| < 3$$

Since the ROC of $H(z)$ is the inside of a circle in the z-plane, then $h(n)$ is a left-sided sequence and the system is non-causal. The system is stable since the ROC contains the unit circle in the z-plane.

Problem # 12



For the inner loop, the following can be written:

$$P(s) = \frac{1}{s}\left[E(s) - K\,P(s)\right]$$

$$\Rightarrow P(s)\left[1 + K/s\right] = \frac{E(s)}{s}$$

$$\Rightarrow \frac{P(s)}{E(s)} = \frac{1/s}{\frac{s+K}{s}} = \frac{1}{s+K}$$

Now, for the outer loop we can write:

$$Y(s) = \left[X(s) - Y(s)\right]\left[\frac{s+4}{s+K}\right]$$

$$\Rightarrow Y(s)\left[1 + \frac{s+4}{s+K}\right] = X(s)\left[\frac{s+4}{s+K}\right]$$

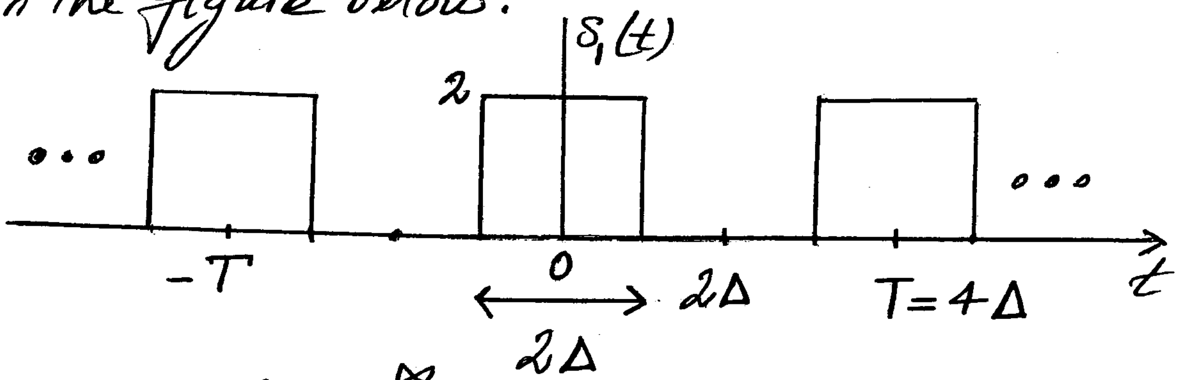
$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{(s+4)/(s+K)}{(2s+4+K)/(s+K)}$$

$$= \frac{s+4}{2s+4+K}$$

The system is causal and has a pole at $s = -\left(\frac{K+4}{2}\right)$. For stability, the pole must be in the LHS p. Hence, $K+4>0$ or $K>-4$.

Problem #13

With $s(t)$ as given the problem figure, then with $s(t) = s_1(t) - 1$, $s_1(t)$ can be drawn as in the figure below.



$$S_1(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} P(n\omega_0)\, \delta(\omega - n\omega_0)$$

with $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4\Delta} = \frac{\pi}{2\Delta}$

$$P(\omega) = 2\int_{-\Delta}^{\Delta} e^{-j\omega t}\,dt = \frac{-2}{j\omega}\left(e^{-j\omega t}\Big|_{-\Delta}^{\Delta}\right)$$

$$= \frac{2}{j\omega}\left[e^{j\omega\Delta} - e^{-j\omega\Delta}\right] = 4\Delta\,\frac{\sin(\omega\Delta)}{(\omega\Delta)}$$

$$P(n\omega_0) = P\left(\frac{n\pi}{2\Delta}\right) = 4\Delta\,\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}$$

$$S_1(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}\, \delta(\omega - n\omega_0)$$

$$S(\omega) = S_1(\omega) - 2\pi\delta(\omega)$$

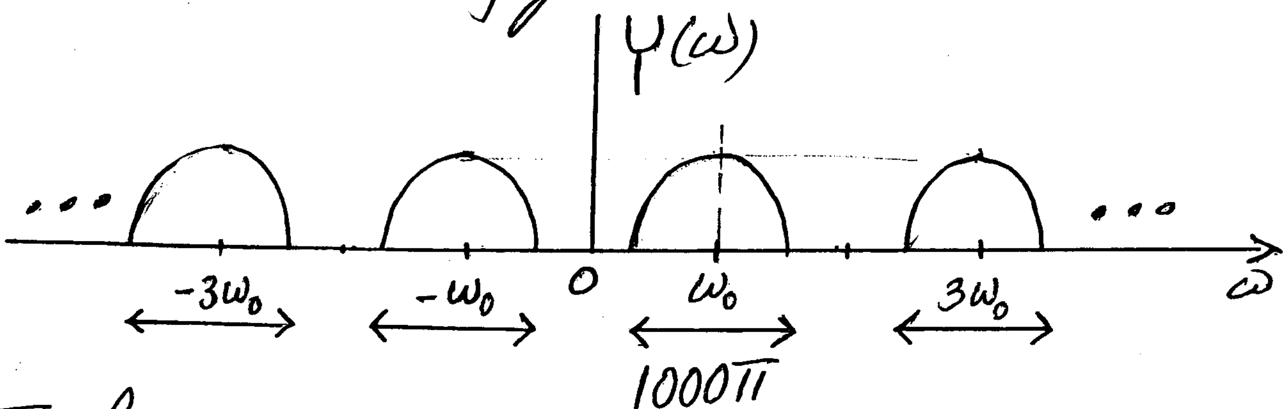
$$= 2\pi \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}\, \delta(\omega - n\omega_0)$$

$y(t) = x(t) \cdot s(t)$

$$\Rightarrow Y(\omega) = \frac{1}{2\pi} X(\omega) * S(\omega)$$

$$= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(n\pi/2)}{\left(\frac{n\pi}{2}\right)} X(\omega - n\omega_0)$$

Withe $X(\omega)$ as shown in the problem figure and $\sin\left(\frac{n\pi}{2}\right) = 0$ for n even, then $Y(\omega)$ becomes as in the figure below



To have no aliasing in the spectrum of $Y(\omega)$, the following condition needs to be satisfied:

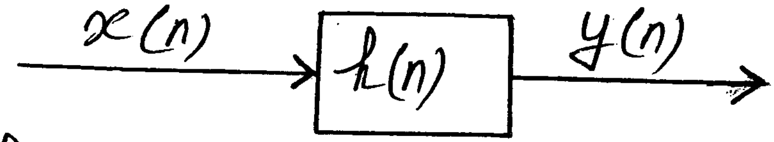
$$3\omega_0 - 5000\pi \geq \omega_0 + 5000\pi$$

$$\Rightarrow 2\omega_0 \geq 1000\pi \Rightarrow \omega_0 = \frac{2\pi}{T} \geq 5000\pi$$

$$\Rightarrow \frac{1}{T} \geq 2500 \text{ samples/sec.}$$

The smallest sampling rate $\left(\frac{1}{T}\right)$ is 2500 samples/sec which is half the Nyquist rate, which is equal to 5000 samples/sec.

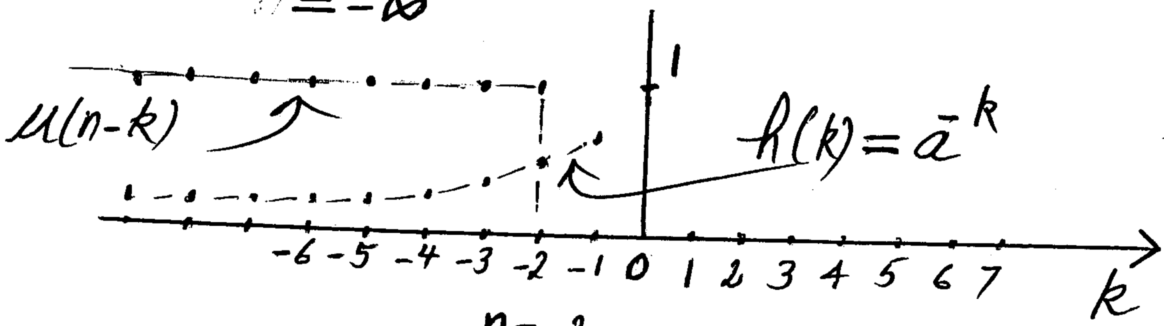
Problem # 14



$$h(n) = a^{-n} u(-n-1) \text{ with } |a|<1$$

$$x(n) = u(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$



For $n \leq -1$, $$y(n) = \sum_{k=-\infty}^{n} a^{-k} = \sum_{k=-n}^{\infty} a^{k}$$

$$\Rightarrow y(n) = \sum_{k=0}^{\infty} a^{k} - \sum_{k=0}^{-n-1} a^{k}$$

$$= \frac{1}{1-a} - \frac{1-a^{-n}}{1-a} = \frac{a^{-n}}{1-a}$$

For $n \geq 0$, $$y(n) = \sum_{k=-\infty}^{-1} a^{-k} = \sum_{k=1}^{\infty} a^{k}$$

$$= \sum_{k=0}^{\infty} a^{k} - 1 = \frac{1}{1-a} - 1$$

$$= \frac{a}{1-a}$$

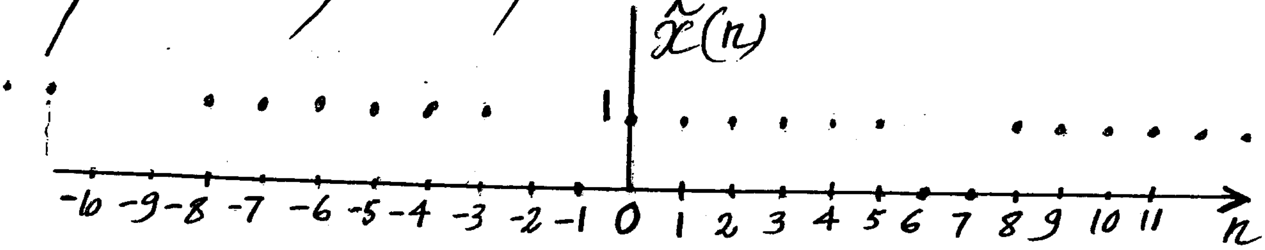
$$\Rightarrow y(n) = \frac{1}{1-a}\left[a\, u(n) + a^{-n} u(-n-1)\right]$$

Problem #15

$$x(n) = u(n) - u(n-6)$$

$$\Rightarrow x(n) = \begin{cases} 1, & n = 0, 1, 2, 3, 4, 5 \\ 0, & \text{elsewhere.} \end{cases}$$

If $X(z)$ is sampled at $z = e^{j\frac{2\pi}{8}k}$, $k = 0, 1, \ldots 7$, i.e., 8 samples are taken from $X(z)$ at equally spaced angles, then we obtain the coefficients of the Fourier series representation of the periodic signal $\tilde{x}(n)$ obtained from $x(n)$ by repeating it periodically with period 8.



Hence, $\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n+8r)$ is as shown above.

$$\Rightarrow \tilde{x}(n) R_8(n) = \begin{cases} 1, & n = 0, 1, 2, 3, 4, 5 \\ 0, & \text{elsewhere.} \end{cases}$$

$$= x(n).$$

$$= u(n) - u(n-6)$$

Problem #16

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$$H(z) = \frac{z}{(z-1)(z-2)}, \quad |z|>2$$

The system is causal and unstable.

The transfer function of the system whose impulse response is $h_1(n) = a^n h(n)$ is:

$$H_1(z) = H(z/a)$$

$$= \frac{z/a}{\left(\frac{z}{a}-1\right)\left(\frac{z}{a}-2\right)}, \quad \left|\frac{z}{a}\right| > 2$$

$$= \frac{az}{(z-a)(z-2a)}, \quad |z| > 2|a|$$

For this system to be stable, the region of convergence must contain the unit circle or equivalently the poles of $H_1(z)$ must be inside the unit circle. Hence, we need to have $2|a| < 1$ or $|a| < 1/2$.