Systems: abstractions of phenomena where the focus is on I/O relationships.

Signals: "objects" by which Systems "communicate".

⇒ Discrete-Time

eg// 1) "Clock", "Bell ring"
2) more & more important.
3) "easier" than CT (continuous-time)

representation(s) of a system:

1- "mathematical relation" b/w output & input. Difference Equation
2- Block Diagrams
3-
4- } later.
5-
⋮

* Eg ① "Modern Burgers"

- On Dec. 31, Invest Money: open 1st branch. year: xxx0.
- On Jan 1st, xxxn ← "Time".
- * let $y[n]$: Stores owned on Jan 1st, xxxn.
  Every store takes 1 year to break even.
  After another year, I have enough money & invest in a new store which I do.

→ $y[0] = 1$
→ $y[1] = 1$
→ $y[2] = 2$
→ $y[3] = 3$
⋮
→ $y[n] = ??$

& use auxiliary signals to know for $y[n]$.

ii) Taylor w/ $X(z) = \ln(1 - az^{-1})$ ROC $= \{ z \in \mathbb{C} \;\; |z| > |a| \}$

By Taylor exp. $X(z) = \sum_{-\infty}^{+\infty} \boxed{x[n]}\, z^{-n}$.

$$\ln(1-az^{-1}) = -\sum_{1}^{\infty} \frac{(az^{-1})^n}{n} = -\sum_{1}^{\infty} \frac{a^n}{n} z^{-n} = \sum_{-\infty}^{+\infty} -\frac{a^n}{n} z^{-n} u[n-1]$$

$\Rightarrow x[n] = -\frac{a^n}{n} u[n-1]$

using properties

$$\frac{dX(z)}{dz} = \frac{-a(-z^{-2})}{1-az^{-1}} = \frac{az^{-2}}{1-az^{-1}}.$$

$n\,x[n] \longleftrightarrow -z \frac{dX(z)}{dz}$

$$-z\frac{dX(z)}{dz} = \frac{-az^{-1}}{1-az^{-1}}$$

- time shift by 1.
- multiply by a.
- $a^n u[n]$.

$\rightarrow n\,x[n] = -a^n u[n-1]$.

$\Rightarrow x[n] = -\frac{a^n}{n} u[n-1]$.

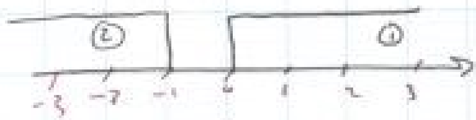
iii) Plug & arrange $x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$.

⋮

(complex analysis).

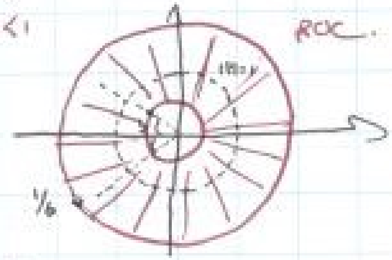
eg// $x[n] = b^{|n|}$

$$= b^n u[n] + b^{-n} u[-n-1]$$
$\quad\quad$ ① $\quad\quad\quad\quad$ ②



$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}$$

$|z| > |b| \quad\quad |z| < 1/b$



$\Rightarrow$ if $|b| < 1 \to$ ROC: $z \in \mathbb{C}$ s.t. $|b| < |z| < \frac{1}{|b|}$

- if $|b| > 1 \to$ ROC: $\emptyset$ no Z-transform $\quad\quad$ $|b|>1$ $\Rightarrow$ no intersection

Computing Z-transform:

1-) use basic pairs + properties.
2-) Taylor Series.
3-) Plug & arrange (mathematical model).

Recall $\quad x[n] \leftrightarrow X(z)$ + ROC $\quad$ "one-to-one"

Going Backward to $x[n]$:

i) use of pairs + properties.
ii) Taylor
iii) plug & arrange.

i) pairs + properties:

eg// $X(z) = \frac{1}{(1-az^{-1})^2}$ $\quad$ ROC: $\{z \in \mathbb{C}$ s.t. $|z| > |a|\}$

$$= \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az^{-1}}$$

$$= Y(z) \cdot Y(z).$$

$$\Rightarrow x[n] = y[n] * y[n] = (a^n u[n]) * (a^n u[n])$$

eg// $X(z) = \frac{az^{-1}}{(1-az^{-1})^2} = \underset{\text{multiply by } a}{a} \; \underbrace{\frac{1}{(1-az^{-1})^2}}_{(n+1)a^n u[n]} \; z^{-1}$ $\quad$ time shift of 1.

$$\Rightarrow x[n] = (n+1-1)\, a^{n+1-1} u[n-1] = n a^n u[n-1].$$

Z-transforms:

	$f[n]$	$F(z)$	ROC
1-	$\delta[n]$	1	$\overline{\mathbb{C}}$
2-	$\delta[n-N]$	z^{-N}	$\overline{\mathbb{C}} - \{0\}$
3-	$\delta[n+N]$	z^{+N}	$\mathbb{C}$
4-	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$z \in \overline{\mathbb{C}}$ s.t. $\lvert z\rvert > \lvert a\rvert$.
5-	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$z \in \mathbb{C}$ s.t. $\lvert z\rvert < \lvert a\rvert$.
6-	$b^{\lvert n\rvert}$	$\frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}$	$\lvert b\rvert < 1 \rightarrow z \in \mathbb{C}$ s.t. $\lvert b\rvert < \lvert z\rvert < \frac{1}{\lvert b\rvert}$ $\lvert b\rvert > 1 \rightarrow$ No z-transform

4- Conjugation

$$\mathcal{Z}\{x^*[n]\} = X^*(z^*)$$

NB// $ROC' = ROC_x$.

pf// $\sum y[n] z^{-n} = \sum x^*[n] z^{-n}$

$$= \left(\sum x[n] (z^*)^{-n}\right)^*$$

5- Multiplication by 'n'

$$\mathcal{Z}\{n x[n]\} = -z \frac{dX}{dz}(z)$$

NB// $ROC' = ROC_x$

pf// $\sum n x[n] z^{-n}$

$$\frac{dX(z)}{d(z)} = \sum x[n](-n) z^{-n-1}$$

$$= -\sum n x[n] z^{-(n+1)}$$

$$= -z^{-1} \sum n x[n] z^{-n}$$

$$-z\frac{dX(z)}{d(z)} = \sum n x[n] z^{-n} = \mathcal{Z}[n x[n]].$$

6- Multiplication by exponential

$$\mathcal{Z}[x[n] a^n] = X(a^{-1}z)$$

NB// $ROC' = |a| \cdot ROC_x$.

$$z \in ROC_x \rightarrow \frac{z}{|a|} \in ROC'$$

eg// $x[n] = u[n] \rightarrow X(z) = \frac{1}{1-z^{-1}}$ $\quad |z| > 1$.

$x[n] = a^n u[n] \rightarrow X(z) = \frac{1}{1-az^{-1}}$ $\quad |z| > |a|$. $\left(\Leftrightarrow \frac{|z|}{|a|} > 1.\right)$

7- Convolution

$$\mathcal{Z}\{x * y[n]\} = X(z) \cdot Y(z)$$

$$(ROC_x \cap ROC_Y) \subset ROC'$$

pf// $\sum_n \left(\sum_k x[k] y[n-k]\right) z^{-n}$

$$= \sum_n \left(\sum_k x[k] y[n-k]\right) z^{-(n-k)} z^{-k}$$

$$= \sum_k x[n] z^{-k} \cdot \sum_n y[n-k] z^{-(n-k)}$$

$$= X(z) \cdot Y(z)$$

eg// ③ $x[n] = -3\left(\frac{1}{7}\right)^n u[-n-1] + 2\left(\frac{1}{8}\right)^n u[n]$.

$$X(z) = 3 \cdot \underbrace{\frac{1}{1-\left(\frac{1}{7}\right)z^{-1}}}_{|z| < 1/7} + 2 \cdot \underbrace{\frac{1}{1-\frac{1}{8}z^{-1}}}_{|z| > 1/8}$$

$ROC = \{ \frac{1}{8} < |z| < 1/7 \}$

eg// ④ $x[n] = 0 \leftrightarrow X(z) = 0 \quad ROC = \mathbb{C}$

but $x[n] = 2^n u[n] - 2^n u[n] \quad (=0)$.

$$= \underbrace{\frac{1}{1-2z^{-1}}}_{|z|>2} - \underbrace{\frac{1}{1-2z^{-1}}}_{|z|>2}$$

$ROC = \mathbb{C}$ not $\{|z| > 2\}$.

2 - Time Delays $\mathcal{Z}\{x[n-N]\} = X(z)\, z^{-N}$

NB// $ROC' = ROC_x$ ▷ check "0" & "∞"

pf//
$$\begin{aligned}\sum y[n] z^{-n} &= \sum x[n-N] z^{-n} \\ &= z^{-N} \sum x[n-N] z^{-(n-N)} \\ &= z^{-N} \sum x[m] z^{-m} \\ &= z^{-N} X(z).\end{aligned}$$

3 - Time Reverse $\mathcal{Z}\{x[-n]\} = X(z^{-1})$

NB// $ROC' = \frac{1}{ROC_x}$: $z \in ROC_x \leftrightarrow z^{-1} \in ROC'$

pf//
$$\begin{aligned}&\sum x[-n] z^{-n} \\ &= \sum x[m] z^{m} \\ &= \sum x[m] (z^{-1})^{-m} \\ &= X(z^{-1}).\end{aligned}$$

where : * $x[n] = \alpha^n u[n]$.

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad ROC = \{ |z| > |\alpha| \} \cup \{``\infty"\}$$

* $x[n] = -\alpha^n u[-n-1]$.

$$X(z) = -\sum_{-\infty}^{+\infty} \alpha^n u[-n-1] z^{-n}$$

$$= -\sum_{-\infty}^{-1} \alpha^n z^{-n} \qquad n = -m.$$

$$= -\sum_{m=1}^{\infty} \alpha^{-m} z^{m} = -a^{-1} z \frac{1}{1-a^{-1} z}$$

$$= -\frac{1}{a z^{-1} - 1} = \frac{1}{1 - a z^{-1}} \qquad |z| < |\alpha|.$$

$ROC = \{ |z| < |\alpha| \} \cup \{0\}$ (anti-causal).

Properties of Z-transform:

1 - Linearity $\mathcal{Z}\{a_1 f_1[n] + a_2 f_2[n]\} = a_1 F_1(z) + a_2 F_2(z)$

NB// $(ROC_x \cap ROC_y) \subset ROC$.

eg// ① $X(z) = 3 \times \underbrace{\frac{1}{1-\frac{1}{7}z^{-1}}}_{ROC:\ |z| > 1/7} + 2 \times \underbrace{\frac{1}{1-\frac{1}{8}z^{-1}}}_{ROC:\ |z| > 1/8}$ // $x[n] = 3\left(\frac{1}{7}\right)^n u[n] + 2\left(\frac{1}{8}\right)^n u[n]$

$\rightarrow$ ROC : $\{|z| > 1/7\} \cup \{``\infty"\}$.

② $x[n] = 3\left(\frac{1}{7}\right)^n u[n] - 2\left(\frac{1}{8}\right)^n u[-n-1]$.

$$X(z) = 3 \cdot \underbrace{\frac{1}{1-\frac{1}{7}z^{-1}}}_{|z| > 1/7} + 2 \cdot \underbrace{\frac{1}{1-\frac{1}{8}z^{-1}}}_{|z| < 1/8}$$

$ROC = \phi$ (No intersection) $\Rightarrow$ has no Z-transform.

9) If $X(z)$ is rational, its ROC is determined by poles.

i.e. If $X(z)$ has ℓ distinct poles (mag. distinct)

$\longrightarrow$ $\ell+1$ possible ROC's.

10) If $X(z)$ rational & $x[n]$ right-sided

ROC outermost pole extending outward. with/out $\{0\}$ o/& $\{"\infty"\}$

if causal $\longrightarrow$ $\{"\infty"\} \in$ ROC

11) If $X(z)$ rational & $x[n]$ left-sided.

ROC innermost pole extending inward with/out $\{0\}$ o/& $\{"\infty"\}$

if anti-causal $\rightarrow$ $\{0\} \in$ ROC

Proof

of ⑥: $x[n]$ right-sided $\rightarrow$ ROC extends outward.

i.e. if $z_0 \in$ ROC $\longrightarrow$ $\forall z$ s.t. $|z| > |z_0|$ $z \in$ ROC.

$\Rightarrow$ $\sum_{-\infty}^{+\infty} |x[n]||z_0|^{-n} < \infty$.

now $|z| \geq |z_0|$.

if $n > 0$ $\rightarrow |z|^n \geq |z_0|^n$ $\forall n \geq 0$.

if $n < 0$ $\rightarrow |z|^n \leq |z_0|^n$ $\forall n \leq 0$.

$$\Rightarrow \sum_{-\infty}^{+\infty} |x[n]||z|^{-n} = \underbrace{|x[n_0]||z|^{-n_0} + |x[n_0+1]||z|^{-(n_0+1)} + \cdots}_{\text{finite}} + \underbrace{|x[0]| + |x[-1]||z| + |x[-2]||z|^2}_{n \leq 0}$$

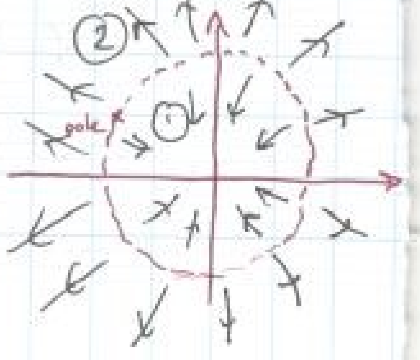
$\leq |x[0]| + |x[-1]||z| + |x[-2]||z|^2 + \ldots$

converges (finite)

eg on ⑦: ①: $x[n] = -a^n u[-n-1]$.

②: $x[n] = a^n u[n]$.

$$X(z) = \frac{1}{1 - a z^{-1}}$$



More on ROC: ⟨ROC⟩ - - -

1) $z_0 \in ROC \rightarrow \forall z$ s.t. $|z|=|z_0|$, $z \in ROC$.

ie. ROC is a collection of circles centered at zero.

N.B. Finding ROC: - Look at '0' & '∞'

- Look at the rest ∴ "Limits are defined" for summation.

$$X(z) = \underbrace{\cdots + x(-2)z^2 + x(-1)z}_{\text{effect of "}\infty\text{"}} + x[0] + \underbrace{x[1]\frac{1}{z} + x[2]\frac{1}{z^2} + \cdots}_{\text{effect of '0'}}$$

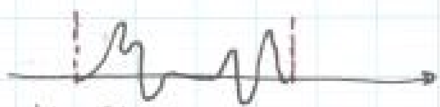
2) for zero: if $\exists n>0$ s.t. $x[n] \neq 0$.

$\rightarrow$ '0' $\notin$ ROC.

3) for ∞: if $\exists n<0$ s.t. $x[n] \neq 0$.

$\rightarrow$ "∞" $\notin$ ROC.

4) if $x[n]$ is finite duration.



$\rightarrow$ ROC = $\mathbb{C}$ except possibly $\{0\}$ and/or $\{\text{"}\infty\text{"}\}$

5) if $X(z)$ is rational, with some ROC.

No poles are elements of its ROC.

6) If $x[n]$ is rightsided, ROC extends outward

If it is causal, it also includes "∞".

7) If $x[n]$ is left-sided, ROC extends inward

If it is anti-causal, it also includes '0'.

8) If $x[n]$ is two-sided, ROC is a ring.

Z-transform			ROC.	
$\delta[n] \longrightarrow 1$			$\overline{\mathbb{C}}$	
$\alpha^n u[n] \longrightarrow \dfrac{1}{1-\alpha z^{-1}}$			$z \in \overline{\mathbb{C}}$ s.t. $\lvert z\rvert > \lvert\alpha\rvert$	
$\delta[n-N] \longrightarrow z^{-N}$		$N>0$	$\overline{\mathbb{C}} - \{0\}$	
$\delta[n+N] \longrightarrow z^{N}$		$N>0$	$\mathbb{C}$	("$\overline{\mathbb{C}} - \{$"$\infty$"$\}$")

Summary

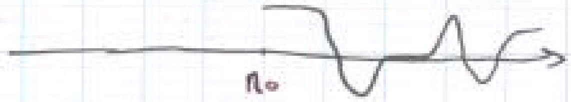
Def $\quad X(z) = \displaystyle\sum_{-\infty}^{+\infty} x[n]\, z^{-n}$

ROC $\quad \left\{ z \in \overline{\mathbb{C}} \text{ s.t. } \displaystyle\sum_{-\infty}^{+\infty} \lvert x[n]\rvert \lvert z\rvert^{-n} < \infty \right\}$

Notes:
- fn $\overline{\mathbb{C}} \longrightarrow \mathbb{C}$
- one-to-one fn.
- $x[n] \longleftrightarrow X(z)$ (uniqueness).

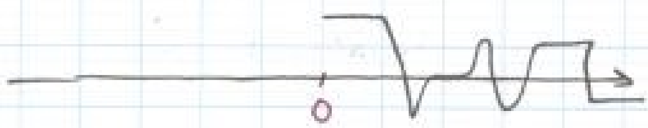
Terminology

i) Right-Sided: $\exists n_0$ s.t. $\forall n < n_0 \quad x[n] = 0$.

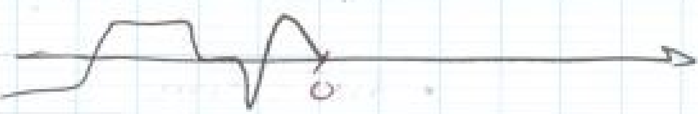


ii) Left-Sided: $\exists n_0$ s.t. $\forall n > n_0 \quad x[n] = 0$.

iii) Causal: iff $\forall n < 0 \quad x[n] = 0$.



iv) Anti-causal: iff $\forall n > 0 \quad x[n] = 0$



N.B// causal $\Rightarrow$ right sided.

anti-causal $\Rightarrow$ left sided.

Def// ROC: region of convergence : set of 'z' $\in \mathbb{C}$.

$\hookrightarrow$ $\text{ROC} = \left\{ z \in \overline{\mathbb{C}} \quad \text{s.t.} \quad \sum_{-\infty}^{+\infty} |x[n]|\,|z|^{-n} \text{ is finite} \right\}$

N.B// $\overline{\mathbb{C}} = \mathbb{C} \cup \{\text{"}\infty\text{"}\}$.

eg// $x[n] = \kappa^n u[n]$.

$$X(z) = \sum_{-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{-\infty}^{+\infty} \kappa^n u[n] z^{-n} = \sum_{n=0}^{\infty} \kappa^n z^{-n} = \sum_{n=0}^{\infty} (\kappa z^{-1})^n.$$

converges to $\dfrac{1}{1-\kappa z^{-1}}$

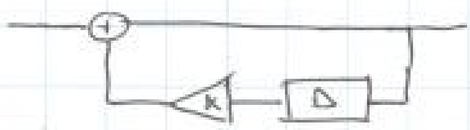
ROC: $|\kappa z^{-1}| < 1$

$|z| > |\kappa|$

$\hookrightarrow$ set of $z \in \overline{\mathbb{C}}$ s.t. $|z| > |\kappa|$.

Recall

sys. fn. $\dfrac{Y}{X} = \dfrac{1}{1-\kappa D}$.



IR $= \kappa^n u[n]$

But "$z^{-1} \Leftrightarrow D$."

eg// • $x[n] = \delta[n]$

$$X(z) = \sum_{-\infty}^{+\infty} x[n] z^{-n} = \sum_{-\infty}^{+\infty} \delta[n] z^{-n} = z^{-0} = 1.$$

$\hookrightarrow$ ROC : $\overline{\mathbb{C}}$

• $x[n] = \delta[n-1]$

$$X(z) = \sum_{-\infty}^{+\infty} \delta[n-1] z^{-n} = z^{-1}$$

$\hookrightarrow$ ROC: $z \in \overline{\mathbb{C}} - \{0\}$

• $x[n] = \delta[n+1]$

$$X(z) = \sum_{-\infty}^{+\infty} \delta[n+1] z^{-n} = z$$

$\hookrightarrow$ ROC: $z \in \mathbb{C}$ $\quad (\overline{\mathbb{C}} - \{\text{"}\infty\text{"}\})$.

Observation: if $x[n] \neq 0$ only on $N_1 \leq n \leq N_2$

$y[n] \neq 0$ only on $N_3 \leq n \leq N_4$

then $(x*y)[n] \neq 0$ only on $N_1+N_3 \leq n \leq N_2+N_4$.

eg// * $x[n] = \alpha^n u[n] \neq 0 \quad n \geq 0.$

$\rightarrow (x*x)[n] = (\alpha+1)\alpha^n u[n] \neq 0 \quad n \geq 0.$

* $x[n] \neq 0$ over $-1 \leq n \leq 1$

$y[n] = \alpha^n u[n] \neq 0$ over $n \geq 0$

$\rightarrow [x*y][n] \neq 0$ over $n \geq -1.$

* $x[n] = \delta[n+1] = y[n]. \quad \neq 0 \quad -1 \leq n \leq -1$

$\rightarrow (x*y)[n] = y[n+1] = \delta[n+2]. \quad \neq 0 \quad -2 \leq n \leq -2.$

Representations of LTI systems continued

Z-transform motivation: 1- "similar" to Laplace

$\rightarrow$ transforms convolution to products.

2- "Similar" to 'D'-operator

Def Let $x[n]$ (complex-value) be a mapping;

$x[n]$	its Z-transform: $X(z)$
$\mathbb{Z} \longrightarrow \mathbb{C}$	$\mathbb{C} \longrightarrow \mathbb{C}$
$n \longrightarrow x[n]$	$z \longrightarrow X(z)$

$$\Rightarrow X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n}$$

"Laplace: $\int x(t) e^{-st} dt$"

** whenever the limits at infinity are defined.

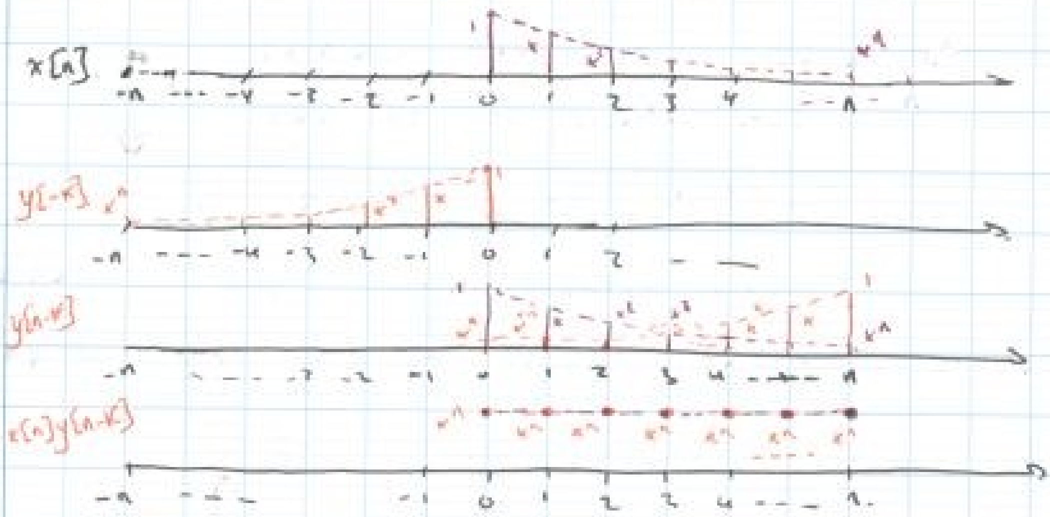
"$\lim_{\substack{N_1 \to \infty \\ N_2 \to \infty}} \sum_{n=-N_1}^{N_2} x[n] z^{-n}$"

Computing Convolution:

=> method - 1) Plug & arrange. "use mathematical expressions"

=> method - 2) Flip & slide
(+) multiply & add

eg// $x[n] = \alpha^n u[n]$ & $y[n] = \alpha^n u[n]$.



Flip: $y[-k]$

(by n) slide: $y[n-k]$

Multiply: $x[k]y[n-k]$

Add $\sum x[k]y[n-k]$ $\rightarrow \alpha^n (n+1) u[n]$.

=> method - 3) Think of a signal as a combination of "δ" impulses.

i) $x[n] = \delta[n]$. $(x*y)[n] = \sum x[k] y[n-k]$ $\forall y[\cdot]$

$(y*\delta)[n] = \sum_{k\in\mathbb{Z}} \delta[k] y[n-k] = y[n]$.

ii) $x[n] = \delta[n-N]$

$(y*x)[n] = y[n-N]$ s.t. $x[n] = \delta[n-N]$.

N.B// This is appropriate whenever a signal, $x[n]$, is "finite in duration".

eg// $x[n] = 2\delta[n+1] - \delta[n-1]$ & given $y[n]$.

=> $(x*y)[n] = 2y[n+1] - y[n-1]$

Properties

i- commutative: $(x * y)[n] = (y * x)[n]$

pf// RHS: $\sum_{k \in \mathbb{R}} y[k]\, x[n-k]$ let $k' = n-k$.

$= \sum_{k' \in \mathbb{R}} y[n-k']\, x[k'] = (x * y)[n]$

ii- distributative: $(x * (y+z))[n] = (x * y)[n] + (x * z)[n]$

pf// multiplication is distributative with addition.

iii- associative: $(x * (y * z))[n] = ((x * y) * z)[n]$

pf// LHS: $\sum_{k \in \mathbb{R}} x[k]\, (y * z)[n-k]$

$= \sum_{k \in \mathbb{R}} x[k] \left(\sum_{l \in \mathbb{R}} y[l]\, z[n-k-l] \right)$

let $l' = k+l$ $\quad l = l'-k$.

$= \sum_{k \in \mathbb{R}} x[k] \left(\sum_{l' \in \mathbb{R}} y[l'-k]\, z(n-l') \right)$

$= \sum_{l \in \mathbb{R}} \sum_{k \in \mathbb{R}} x[k]\, y[l'-k]\, z[n-l']$

$= \sum_{l' \in \mathbb{Z}} z[n-l'] \sum_{k \in \mathbb{R}} \underbrace{x[k]\, y[l'-k]}_{(x * y)[l']}$

$= \sum_{l' \in \mathbb{R}} (x * y)[l']\, z[n-l']$

$= ((x * y) * z)[n]$.

Convolution: let $x[n]$ & $y[n]$ be any 2 DT signals.

$\rightarrow$ their convolution is a DT signal.

$$(x * y)[n] = \sum_{x \in \mathbb{R}} x[k]\, y[x-k].$$

eg// $x[n] = \alpha^n u[n]$. & $y[n] = \alpha^n u[n]$.

$$Z[n] = (x * y)[n] = \sum_{k \in \mathbb{R}} x[k]\, y[n-k]$$

$$= \sum_{k \in \mathbb{R}} \alpha^k u[k]\, \alpha^{n-k} u[n-k] \qquad // \; u[k] = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0. \end{cases}$$

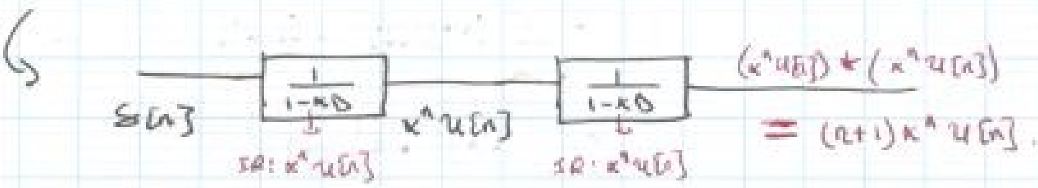
$$= \sum_{k=0}^{\infty} \alpha^k \alpha^{n-k} u[n-k] \qquad // \; u[n-k] = \begin{cases} 0 & k > n \\ 1 & k \leq n. \end{cases}$$

$$= \sum_{k=0}^{n} \alpha^k \alpha^{n-k}$$

$$= \begin{cases} \sum_{n=0}^{n} \alpha^n & n \geq 0 \\ 0 & \text{elsewhere}. \end{cases}$$

$$\Rightarrow (x * y)[n] = \begin{cases} 0 & n < 0 \\ (n+1)\alpha^n & n \geq 0. \end{cases}$$

Recall $\quad \dfrac{Y}{X} = \dfrac{1}{(1 - \alpha D)^2} \qquad$ IR: $(n+1)\, \alpha^n u[n]$.



So $\dfrac{Y}{X} = \dfrac{1}{(1 - \alpha D)^2}$

can be tackled by: – Taylor series expansion –

– Convolution of 1st order systems.

(III) Operator 'D'

$$\frac{Y}{X} = \frac{\sum_{l=0}^{M} b_l D^l}{\sum_{k=0}^{N} a_k D^k}$$

"resembled in DIRECT FORM (1)"
→ direct form (2) is same as (1)
but inverted order of cascade
& 2 chains of delay grouped in 1 chain.

(IV) Impulse Responses

Def

$\delta[n]$ — $\boxed{sys}$ — $h[n]$.

$h[n]$: is the output of the sys. for impulse input $\delta[n]$.

Lemma Any DT system $\underbrace{x[n]}_{signal} = \sum_{k \in \mathbb{R}} \underbrace{x[k]}_{scalar} \underbrace{\delta[n-k]}_{signal}$

⇒ any signal is a linear combination of shifted impulses.

* $\delta[n-k]$ — $\boxed{sys}$ — $h[n-k]$ (LTI sys.)

* $x[n]$ — $\boxed{sys}$ — ??

$\sum_{k \in \mathbb{R}} x[k]\, \delta[n-k]$ — $\boxed{sys}$ — $\sum_{k \in \mathbb{R}} x[k]\, h[n-k]$

Theorem

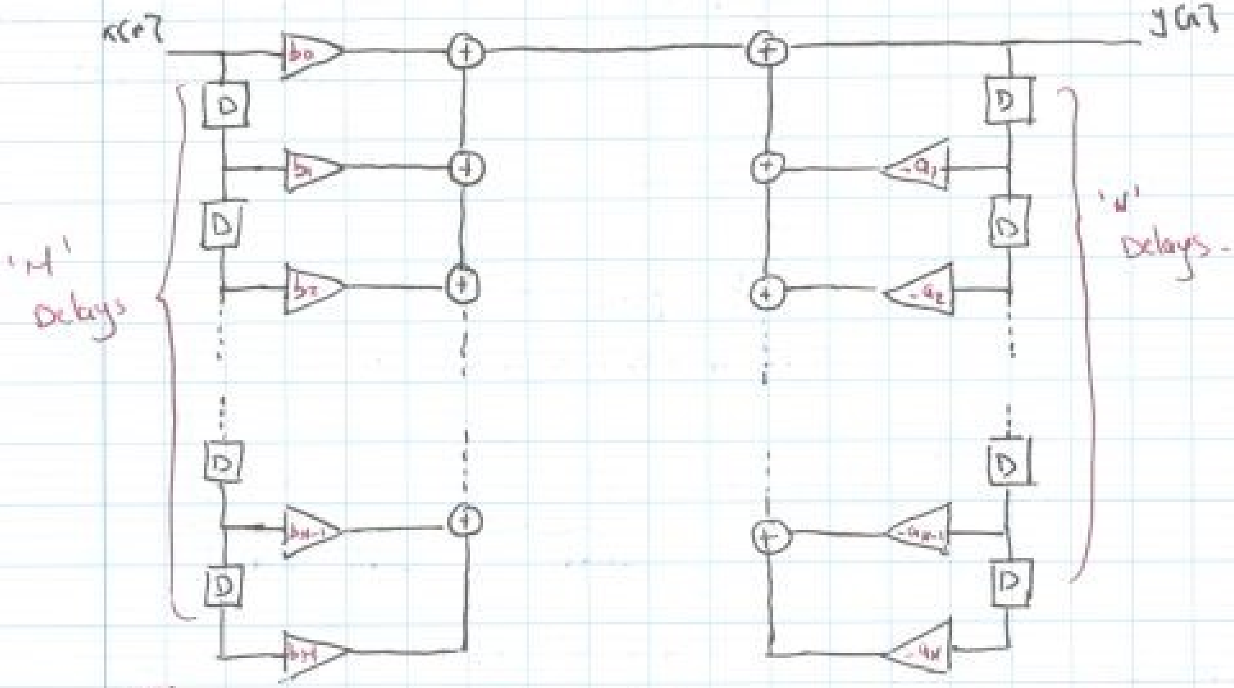
$\delta[n]$ — $\boxed{sys}$ — $h[n]$ (IR)

$x[n]$ — $\boxed{sys}$ — $y[n] = \sum_{k \in \mathbb{R}} x[k]\, h[n-k]$

"Similar in a sense to $y(n) = \int x(\tau)\, h(t-\tau)\, d\tau$"

II Block Diagrams: adders - delays - multipliers.

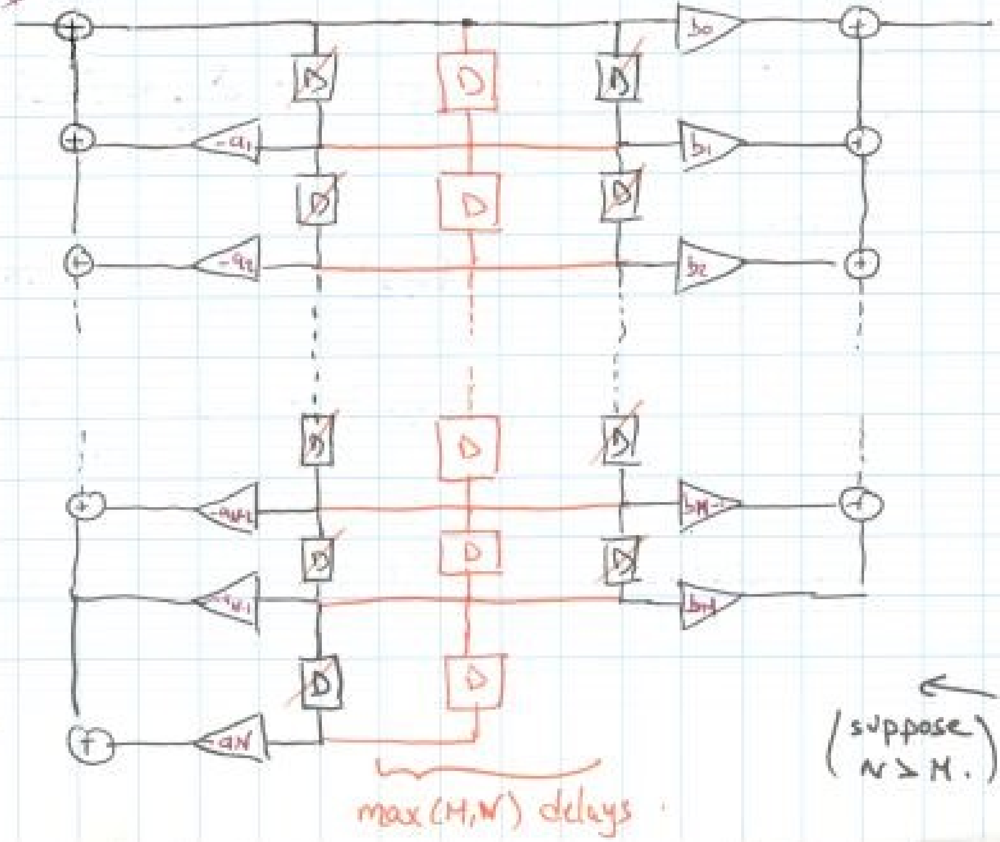
$$y[n] = \sum_{l=0}^{M} b_l \, x[n-l] - \sum_{k=1}^{N} a_k \, y[n-k].$$



⇒ DIRECT FORM ①

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{l=0}^{M} b_l \, x[n-l].$$

⇒ DIRECT FORM ②



(suppose N > M.)

Formal Treatment:

Def linear: a DT system is called linear if:

$$x_1[n],\ x_2[n] \longrightarrow \boxed{\text{sys}} \longrightarrow y_1[n],\ y_2[n]$$

then. $$a\,x_1[n] + b\,x_2[n] \longrightarrow \boxed{\text{sys}} \longrightarrow a\,y_1[n] + b\,y_2[n] \quad \forall a,b \in \mathbb{C}^2$$

Def Time-invariant: a DT system is time invariant if it is time-shift invariant. i.e:

$$x[n] \longrightarrow \boxed{\text{sys}} \longrightarrow y[n]$$

$$x[n-N] \longrightarrow \boxed{\text{sys}} \longrightarrow y[n-N] \quad \forall N \in \mathbb{Z}$$

$\Rightarrow$ We'll study (DT & CT) systems that are Linear & Time invari(ant) (LTI systems)

Representations of LTI sys:

(I.) Difference Equation — cts-coeff. diff. eqn. (CCDE)

Generally: $$\underbrace{\sum_{k=0}^{N} a_k\, y[n-k]}_{\text{much critical}} = \sum_{k=0}^{M} b_l\, x[n-l]$$

$\hookrightarrow$ $a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + \cdots + b_M [n-M]$.

N.B. ① $\sum a_l\, x[n-l] + \underset{\text{cts}}{\delta}$ is not linear.

② when $a_0 = 0$ or $k-1$ $\Big\}\rightarrow$ change indices $k' = k-1$ $(a_1 \neq 0)$

$\Rightarrow$ the 1st term in y's must be $y[n]$ (by change of indices) & its coeff. must be '1' (by dividing by a_0).

$x[n] = \delta[n]$.

Sensible

$\frac{Y}{X} = \alpha_0 + \alpha_1 D + \alpha_2 D^2 + \cdots$

$y[n] = \alpha_0 x[n] + \alpha_1 x[n-1] + \alpha_2 x[n-2] + \cdots$

$y[0] = \alpha_0$.

$y[1] = \alpha_1$

$\vdots$

$\Rightarrow \frac{1}{(1-pD)^2} = (1-pD)^{-2} = 1 + (-2)(-pD) + \frac{(-2)(-3)}{2!}(-pD)^2 + \frac{(-2)(-3)(-4)}{3!}(-pD)^3 \cdots$

$= 1 + 2pD + 3p^2D^2 + 4p^3D^3 + \cdots + (l+1)p^l D^l$.

$\Rightarrow$ **IR**

$$\left.\begin{array}{l} x[n] = \delta[n]. \\ \frac{Y}{X} = \frac{1}{(1-pD)^2} \end{array}\right\} \Rightarrow y[n] = (n+1)p^n \quad n \geq 0.$$

* **Claim** $\frac{1}{(1-pD)^k}$

$\hookrightarrow$ IR: $\text{poly}[n] \cdot p^n \quad n \geq 0$.

s.t. degree of $\text{poly}[n]$ is $k-1$

* $|p| > 1 \rightarrow$ still divergent.

* $|p| < 1 \rightarrow$ still converges to "0".

$\underline{\underline{IR}}$ $\frac{1}{(1-pD)^2}$

ii) Approximation.

$$\frac{1}{(1-pD)^2} \approx \frac{1}{1-(p+\varepsilon)D} \cdot \frac{1}{1-pD}$$ $\varepsilon \ll$ limit. (small).

→ get IR.

→ take ε to zero.

iii) Proved later:

Recall * $\frac{Y}{X} = 1 + pD$. $IR = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ p & n=1 \\ 0 & \text{elsewhere} \end{cases}$

* $\frac{Y}{X} = \frac{1}{1-pD} = (1-pD)^{-1} = \underbrace{1 + pD + p^2D^2 + p^3D^3 + ---}_{\text{Taylor series expansion}}$

$IR = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ p & n=1 \\ p^2 & n=2 \\ \vdots & \vdots \\ p^L & n=L \end{cases}$

Taylor Series Expansion:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + ---$$

By Taylor series expansion:

$$\frac{Y}{X} = \kappa_0 + \kappa_1 D + \kappa_2 D^2 + \kappa_3 D^3 + ---$$

$IR: \begin{cases} 0 & n<0 \\ \kappa_0 & n=0 \\ \kappa_1 & n=1 \\ \kappa_2 & n=2 \\ \vdots & \vdots \end{cases}$ $\boxed{IR[n] = \kappa_n}$

$IR[n] =$ coeff. D^n.

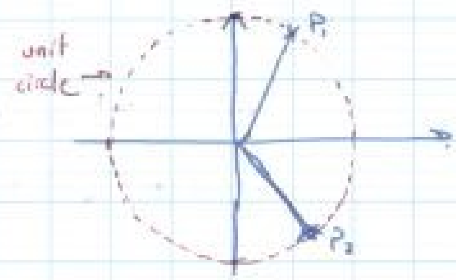
Papeteries

** N.B. $\frac{Y}{X}$ rational.

IR: combination of $(p_i)^n$.

↳ if roots are complex, oscillations appear in the output.

* often, we represent poles (roots of denominator of sys. fn) on complex plane.

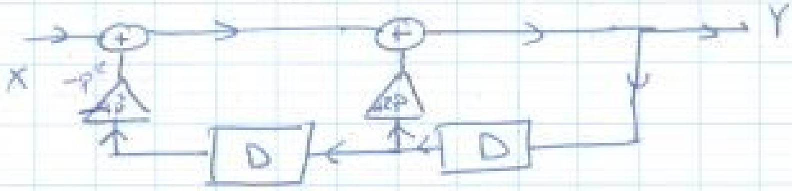


$\boxed{\text{Sol. for repeated Roots}}$

$$\frac{Y}{X} = \frac{1}{(1-pD)^2} \text{ ??}$$

* Diff eqn. $y[n] = x[n] + 2p\, y[n-1] - p^2 y[n-2]$.

* Block Diagram



* Sys. fn $\frac{Y}{X} = \frac{1}{(1-pD)^2} = \frac{1}{1-2pD+p^2D^2}$.

⇒ I.R $x[n] = \delta[n]$.

$y[n] = ??$

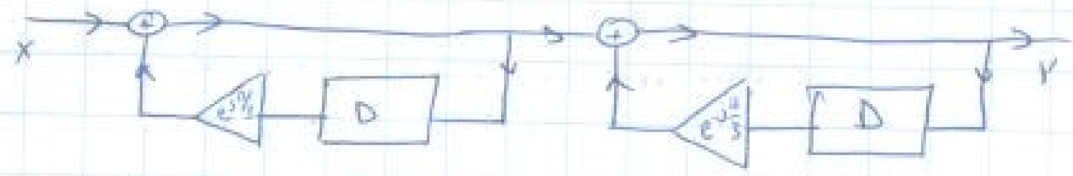
i) use recurrence: $y[0] = x[0] = 1$.

$y[1] = 2p\, y[0] = 2p$.

$y[2] = 2p\, y[1] - p^2 y[0] = 3p^2$.

- compute few terms.
- identify pattern / form an expression.
- prove it by induction.

Series $\ast$ $\frac{Y}{X} = \frac{1}{1 - D + D^2} = \frac{1}{(1 - P_1 D)} \cdot \frac{1}{(1 - P_2 D)} = \frac{1}{1 - e^{j\pi/3} D} \cdot \frac{1}{1 + e^{-j\pi/3} D}$



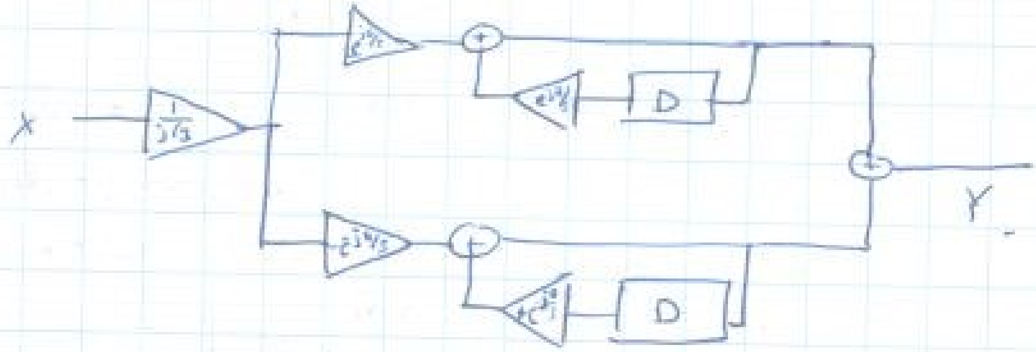
N.B// if $x \in \mathbb{R} \rightarrow y \in \mathbb{R}$. (above).

Parallel $\ast$ $\frac{Y}{X} = \frac{1}{1 - e^{j\pi/3} D} \cdot \frac{1}{1 + e^{-j\pi/3} D} = \frac{K}{1 - e^{j\pi/3} D} + \frac{\beta}{1 + e^{-j\pi/3} D}$.

$$K = \frac{1}{1 - e^{-j\pi/3} e^{-j\pi/3}} = \frac{e^{j\pi/3}}{e^{j\pi/3} - e^{-j\pi/3}} = \frac{e^{j\pi/3}}{e^{j\pi/3} - e^{-j\pi/3}} = \frac{e^{j\pi/3}}{j\sqrt{3}}.$$

$$\beta = \frac{1}{1 - e^{j\pi/3} e^{j\pi/3}} = --- = -\frac{e^{-j\pi/3}}{j\sqrt{3}}.$$

$$\Rightarrow \frac{Y}{X} = \frac{1}{j\sqrt{3}} \left[\frac{e^{j\pi/3}}{1 - e^{j\pi/3} D} - \frac{e^{-j\pi/3}}{1 - e^{-j\pi/3} D} \right].$$



I.R $x[n] = \delta[n] \rightarrow$.

$$y[n] = \frac{1}{j\sqrt{3}} \left[e^{j\pi/3} \cdot (e^{j\pi/3})^n - e^{-j\pi/3} (e^{-j\pi/3})^n \right] u[n]$$

$$= \frac{1}{j\sqrt{3}} \left[e^{j\pi/3 (n+1)} - e^{-j\pi/3 (n+1)} \right] u[n].$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\pi}{3} (n+1) \qquad n \geq 0. \quad \text{(Real)}$$

N.B// $\boxed{\text{poles:}}$ i.e. "roots" of the denominator of Y/X. are critical

$\Rightarrow$ obtain an output which is a combination of $(p_i)^n u[n]$.

II- Parallel

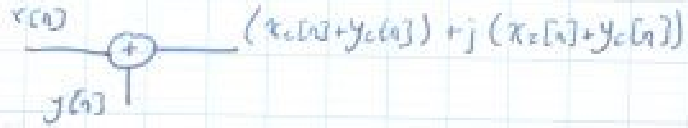
* potential problems: * complex roots.
 * repeated roots eg// $\frac{1}{(1-pD)^2}$.

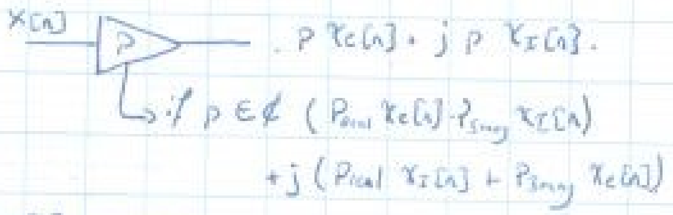
Sol for complex roots

i-) complex signals are 2 real signals.

$$x[n] \in \mathbb{C} \rightarrow x_R[n],\ x_I[n] \Rightarrow x[n] = x_R[n] + j\, x_I[n].$$

ii) complex operators:





see P.S. 2

Eg// $\frac{Y}{X} = \frac{1}{1-D+D^2}$

$1 - z^{-1} + z^{-2} \rightarrow z^{-2}(z^2 - z + 1)$

$p_1 = \frac{1+j\sqrt{3}}{2}$ $\quad p_2 = \frac{1-j\sqrt{3}}{2}$.

$p_1 = e^{j\pi/3}$ $\quad p_2 = e^{-j\pi/3}$.

N.B// poly. with real coeff. can still have complex roots
$\rightarrow$ would appear as conjugates.

I- Serial (if sys. fn is proper).

$$\frac{Y}{X} = \frac{P(D)}{Q(D)} = \sum \frac{k_i}{1 - p_i D} \quad \text{(Partial Fraction Expansion)}$$

NB// If not proper $\quad \frac{P(D)}{Q(D)} = \underbrace{\Gamma(D)}_{\text{later.}} + \sum \frac{k_i}{1 - p_i D}$

* potential problems: complex roots

Eg// $\frac{Y}{X} = \frac{1}{1 - D - D^2} = \frac{1}{(1 - p_1 D)(1 - p_2 D)}$

$$= \frac{k_1}{(1 - p_1 D)} + \frac{k_2}{(1 - p_2 D)}$$

$k_1 = \left.\frac{1}{1 - p_2 D}\right|_{D \to 1/p_1} \quad k_1 = \frac{1}{1 - p_2/p_1}$

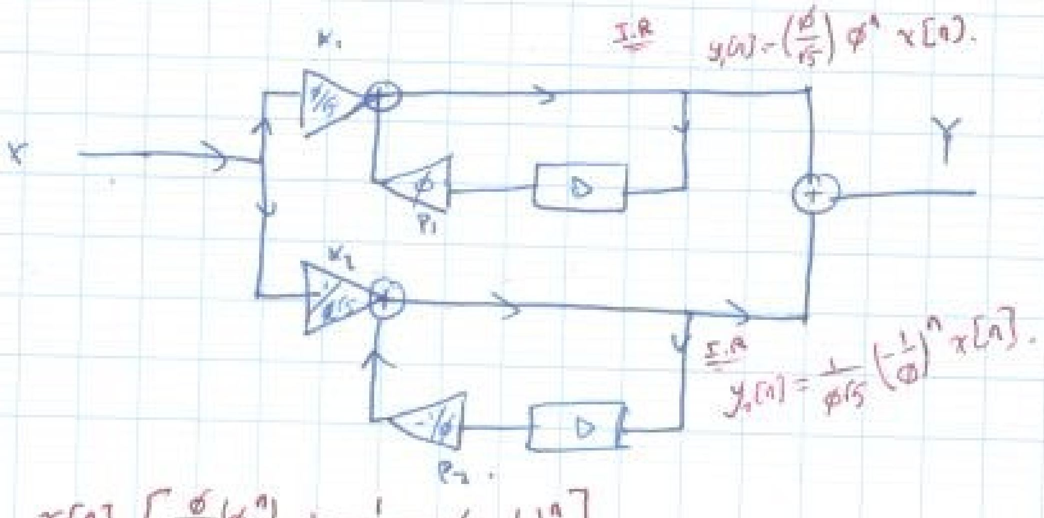
$k_2 = \left.\frac{1}{1 - p_1 D}\right|_{D \to 1/p_2} \quad k_2 = \frac{1}{1 - p_1/p_2}$.

$$\frac{Y}{X} = \frac{\phi/\sqrt{5}}{1 - \phi D} + \frac{1/\phi\sqrt{5}}{1 + \frac{1}{\phi} D}$$

$p_1 = \frac{1 + \sqrt{5}}{2} = \phi$, golden # (≈1.6).

$p_2 = \frac{1 - \sqrt{5}}{2} = -1/\phi$.

$\frac{Y}{X}$: "poles": roots of denominator are critical



$\Rightarrow$ I.R $\quad x[n] \left[\frac{\phi}{\sqrt{5}}(\phi^n) + \frac{1}{\phi\sqrt{5}}\left(-\frac{1}{\phi}\right)^n\right]$

$\Rightarrow$ Cascade

$$\frac{Y}{x} = \prod_{i=0}^{n} (\text{sys } P_n\, i).$$

✦ potential problem: complex roots

If $\quad \frac{Y}{x} = \frac{P(x)}{Q(x)}.$

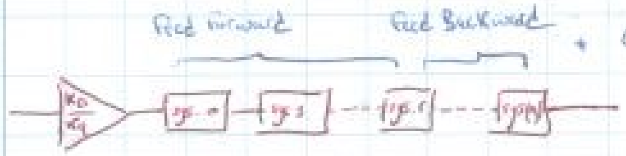
$$\boxed{\frac{Y}{X} = \frac{K_p}{K_q} \prod (1-r_i)D \cdot \prod \left(\frac{1}{1-p_i D}\right)}$$

+ $P(D)$: poly. factorization.

$D \leftrightarrow z^{-1}$

$z^{-b}(\text{Poly}(z))$

$\hookrightarrow P(D) = K_p (1-r_1 D)(1-r_2 D) --- (1-r_n D)$



+ $Q(D)$: poly fact.

$D \leftrightarrow z^{-1}$

$z^{-b}(\text{poly}(z))$

$\hookrightarrow Q(D) = K_q (1-p_1 D)(1-p_2 D) --- (1-p_n D).$

Ex// (the one before)

$$Y/x = \frac{1}{1-D-D^2} = \frac{P(D)}{Q(D)} = \frac{1}{(1-p_1 D)} \cdot \frac{1}{(1-p_2 D)}.$$

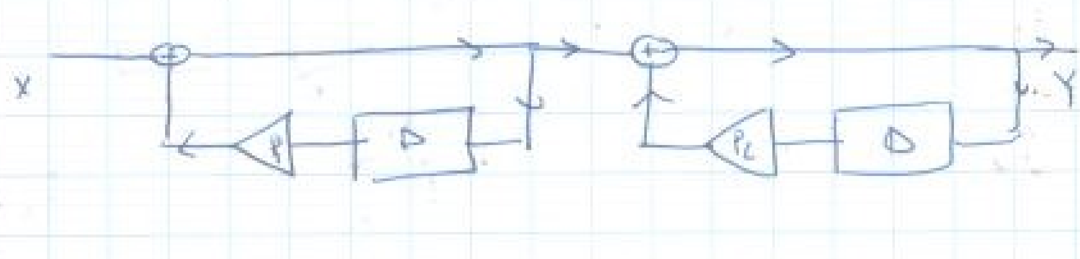
$\hookrightarrow Q(D): \quad 1 - D - D^2$

$1 - z^{-1} - z^{-2} \quad \rightarrow z^{-2}(z^2 - z - 1)$

$\hookrightarrow$ roots: $p_1 = \frac{-1-\sqrt{5}}{2}$ & $p_2 = \frac{-1+\sqrt{5}}{2}$.

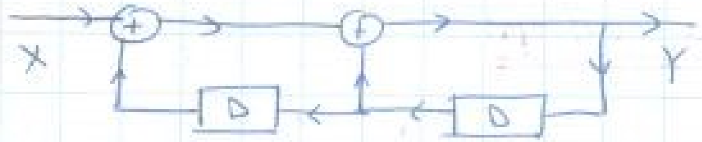
$\hookrightarrow Q(D) = (1-p_1 D)(1-p_2 D).$

$\hookrightarrow$ **Block Diagram**



Property Any DT system can be decomposed into the 2 previous fundamental Blocks. (Feed forward & Feed Backward.)

Eg//



$$\frac{Y}{X} = \frac{1}{1 - D - D^2}$$

$\Rightarrow$ write as a product of two components.

$$\left(\frac{1}{1 - pD}\right)\left(\frac{1}{1 - qD}\right)$$

$\Leftrightarrow$ cascade of 2 systems.

Procedure related to polynomial factorization.

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \prod (x - x_i)$$

↳ where x_i's are the roots of the eqn.

Steps:

① - Start with $P(D)$ (denominator or numerator). & replace D by z^{-1}

② - Make "P" a polynomial in z. (multiply by the most -ve power)

③ - Find the roots of $P(z) \longrightarrow p \,\&\, q$.

$\Rightarrow$ eg// $1 - D - D^2 \Leftrightarrow (1 - pD)(1 - qD)$

① $1 - z^{-1} - z^{-2}$.

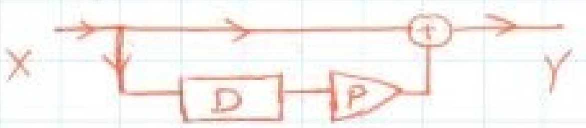
② $z^{-2}(z^2 - z - 1)$.

③ Roots of $(z^2 - z - 1)$ are $p \,\&\, q$.

$$p = \frac{-1 + \sqrt{5}}{2} \quad \& \quad q = \frac{-1 + \sqrt{5}}{2}.$$

Fundamental Systems:

(I) ⟹ (I) Feed Forward.



* "0th order" Diff eqn:

* $Y[n] = x[n] + p\, x[n-1]$.
* Block diagram given above.

* sys. fn $\boxed{\frac{Y}{X} = (1 + pD)}$

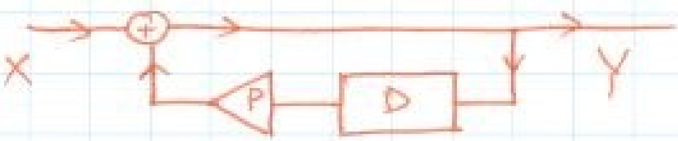
* IR : impulse response.

$$y[n] = \begin{cases} 0 & n < 0 \\ 1 & x = 0 \\ p & x = 1 \\ 0 & \text{elsewhere} \end{cases}$$

"FIR"

* "finite impulse response"

(II) ⟹ (II) Feed Back.

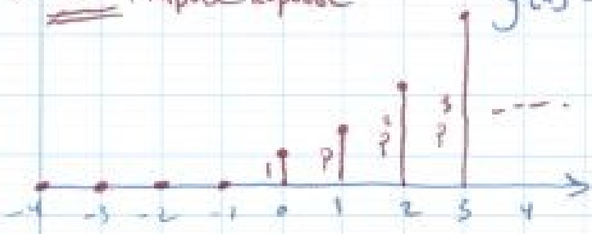


* 1st Order Diff eqn:

* $y[n] = x[n] + p \cdot y[n-1]$.
* Block diagram given above.

* sys. fn: $\boxed{\frac{Y}{X} = \frac{1}{1 - pD}}$

* IR : impulse response



$$y[n] = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ p & x = 1 \\ \vdots & \\ p^n & x = n \end{cases}$$

"IIR"

* "infinite impulse response"

S.R $\boxed{\Rightarrow y[n] = p^n\, y[n]}$

"p" is the "mode" or "pole"

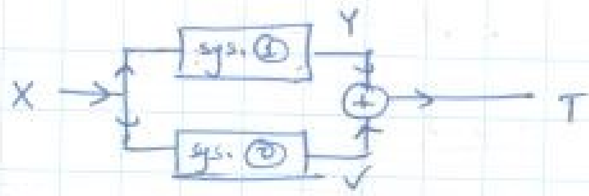
$|p| > 1$ ⟹ IIR is divergent.

$|p| < 1$ ⟹ IIR converges to zero; "0".

$p = 1$ ⟹ step. fn.

$p = -1$ ⟹ alternates +ve & -ve. . .

Parallel Systems



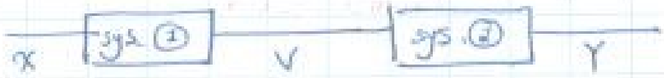
Diff eqn. $T[n] = \underbrace{Y[n] + V[n]}_{\text{both related to } x \text{ by other diff. eqn.s.}}$

sys. fn: $\dfrac{T}{X} = \dfrac{Y+V}{X} = \dfrac{Y}{X} + \dfrac{V}{X}$.

N.B: sys. fn. for the output of parallel systems is the sum of individual sys. fn.s for each system.

$$\boxed{\frac{Y}{X} = \sum_{i=1}^{n} (\text{sys fn. } i)}$$

Cascade of Systems



"Series".

eg.1

$v[n] = x[n] - x[n-1]$ $\qquad$ $y[n] = v[n] - v[n-1]$.

$\hookrightarrow y[n] = (x[n] - x[n-1]) - (x[n-1] - x[n-2])$

$\Rightarrow y[n] = x[n] - 2x[n-1] + x[n-2]$.

$\hookrightarrow$ operator: $\quad Y = V - DV$. $\quad , \quad V = X - DX$.

$$\hookrightarrow Y = (X - DX) - D(X - DX)$$

$$= X - 2DX + D^2X.$$

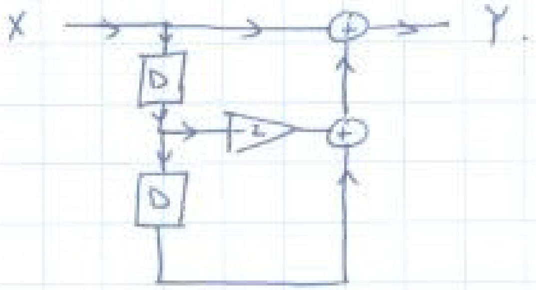
$\hookrightarrow$ sys. fn.: $\quad \frac{Y}{V} = \frac{P_1}{Q_1}$ $\qquad$ P_1 & Q_1 are polynomials.

$$\frac{V}{X} = \frac{P_2}{Q_2}.$$

$\Rightarrow \frac{Y}{X} = \frac{P_1 P_2}{Q_1 Q_2}$. in cascade $\boxed{\frac{Y}{X} = \prod_{i=1}^{n} (\text{sys. fn } i)}$

N.B// sys. fn. for a cascade of systems is the **product** of individual sys. fn.s for each system.

other Block Diagram for sys. above from the resultant Diff. eqn.



Cont. **Representation of sys.s :**

1) Difference Equations: function of $x[n]$, $x[n-1]$, ---
$y[n]$, $y[n-1]$, ---

2) Block Diagram :

- Adder


- multiplier


- Delay


3) Operator : "D" used as in polynomials for Delay.
i.e. $x[n-1] \iff DX$.

↳ eg on operator

① (from eg:1) $Y = X + DY + D^2 Y$.

↳ $(1 - D - D^2) Y = X$.

↳ $Y/X = \dfrac{1}{1 - D - D^2}$

② (from eg:2) $\dfrac{Y}{X} = \dfrac{1}{1 - (1+n)D}$.

③ (from eg:3) $\dfrac{Y}{X} = 1 - D$.

↳ N.B// $\dfrac{Y}{X}$ is the system function.

↳ order of the system $\iff$ degree of denominator of sys. fn.

Eg: ③ "Image Compression" , "Motion Detection".

looking at differences b/w inputs

$$\begin{cases} y[n] = x[n] - x[n-1] \\ \text{sys @ rest} \end{cases}$$

<u>0^{th} Order</u>

Diagram

* IR for $\delta[n]$: $y[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -1 & n = 1 \\ 0 & n > 1 \end{cases}$

⇓

FIR.

⇑

* If $x[n] = u[n]$. $y[n] = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 0 & x > 0 \end{cases}$

N.B $u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$

$\delta[n] = u[n] - u[n-1]$.

* for sys. with feedback, expect an IIR. (1st order diff equation & above).

Representation of a system:

Ⓘⓥ- Operator

1. Think of signals as objects (C++)

$x[n] \leftrightarrow X$, $y[n] \leftrightarrow Y$.

2- Operators:

i) addition: "$Z = X + Y$" $\leftrightarrow$ $Z[n] = X[n] + Y[n]$.

ii) Multiplication: "$Z = 2 \cdot X$" $\leftrightarrow$ $Z[n] = 2 \cdot x[n]$.

iii) Delay : "$Z = DX$" $\leftrightarrow$ $Z[n] = x[n-1]$.

eg ③ above : $Y = X - DX$.

eg ② before : $Y = (1+r)DY + X$.

eg ①: $Y = X + DY + D^2Y$.

3* "Impulse response"
↳ output of a system corresponding to an impulse input.

4* "Infinite Impulse Response" : IIR
↳ output is s.t.: $\exists n_0 : \forall n \geq n_0. \quad y[n] \neq 0$.

5* "Finite Impulse Response": FIR.
↳ output is s.t.: $\forall n_0, M_0 \quad \forall n \notin \{n_0, \dots, M_0\} \quad IR[n] = 0$.

* **Eg: ⓐ "BEMO"**

BEMO gives 6% interest rate computed yearly
on December 31st @ 3 pm, you deposit S dollars.

→ $y[n]$: wealth on Dec. 31st @ 3:30 pm after x years.

$y[0] = S$
$y[1] = S(1+r)$
$y[2] = S(1+r)^2$
⋮
$y[n] = S(1+r)^n$ . ....

1st Order

$$\begin{cases} y[n] = (1+r)\, y[n-1] \\ y[0] = S \quad \text{(initial condition)}. \end{cases}$$

Diagram

General

$$\begin{cases} y[n] = x[n] + (1+r)\, y[n-1]. \\ @ \text{ rest}. \end{cases}$$

with $x[n] = \delta[n]$. (impulse at "0")
(can add more inputs by defining $x[n]$).
(connect systems through inputs & outputs).

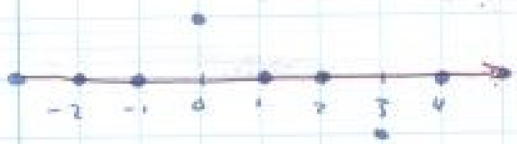
Papillon

Another Eg

$$x[n] = \begin{cases} 0 & x < 0. \\ 1 & x = 0 \\ -1 & x = 3 \\ 0 & \text{elsewhere} \end{cases}$$

* Unit Sample Signal
* unit impulse Signal
* impulse.

Signal:



Representation of Sys

I - Difference Equation:

Eg. ①
$$\begin{cases} y[n] = y[n-2] + y[n-1]. \\ y[0] = y[1] = 1 \quad \leadsto \text{initial conditions.} \end{cases}$$

OR

$$\begin{cases} y[n] = x[n] + y[n-2] + y[n-1] \\ \text{Sys. @rest.} \end{cases}$$

⇒ 2^{nd} order difference equation

i.e. output depends (possibly explicitly) on previous output.

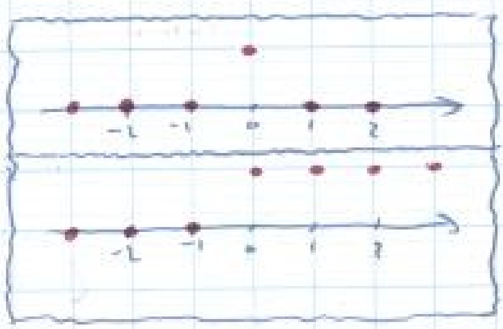
II - Block Diagram:

Eg: ①

* Signal Def

1 - "Impulse" or "unit sample"

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{elsewhere} \end{cases}$$



2 - "Unit step"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

* Let $b[n]$: # brand new restaurants (didn't break even yet)
* Let $o[n]$: # of old restaurants (broke even)

$\rightarrow$ $y[n] = b[n] + o[n]$

$$\begin{cases} b[n] = o[n-1] & ① \\ o[n] = o[n-1] + b[n-1] & ② \end{cases}$$

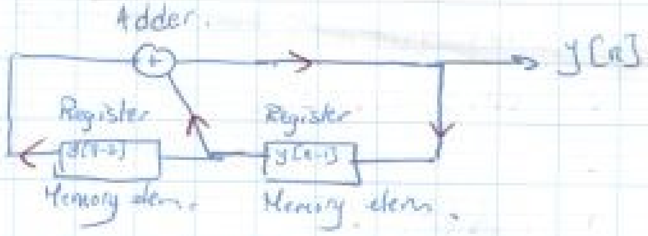
$\hookrightarrow$ $y[n] = \underbrace{o[n-1]} + y[n-1]$ (stores of last year + old of prev. year that broke even).

$y[n-2] = o[n-2] + b[n-2]$. (from ②)

2^{nd} Order Sys: $\hookrightarrow$ $\boxed{y[n] = y[n-1] + y[n-2]}$ $\quad y[0]=1$, $y[1]=1$

Fibonacci.

$\rightarrow$ **Diagram**



$\rightarrow$ Elements:

* Adder. $x[n] \rightarrow (+) \rightarrow$ out: $x[n] + y[n]$ (in: $x[n]$, in: $y[n]$)

* Register $x[n] \rightarrow \boxed{D} \rightarrow x[n-1]$ "delay".

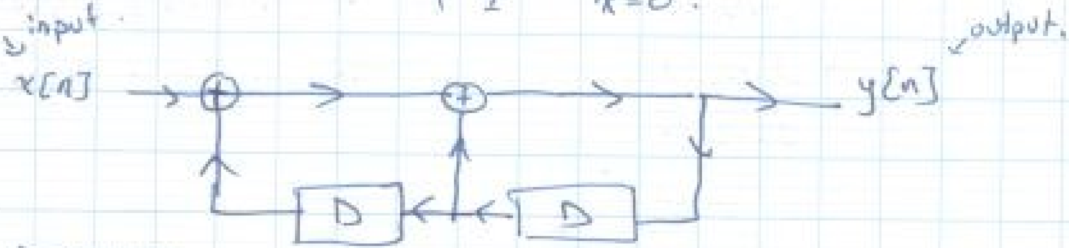
$\rightarrow$ initial conditions: * initial values in registers, needed.
* at rest $\Rightarrow$ NOTHING in registers.

$\Rightarrow$ Another Representation.

Assume: @ Rest.

$$\& \ x[n] = \begin{cases} 0 & x \neq 0. \\ 1 & x = 0. \end{cases}$$

Generalization



x	D	D	
1	0	0	$y[0]=1$
0	0	1	$y[1]=1$
0	1	1	$y[2]=2$ ...

Representations of DT systems (LTI's).

1- Difference Equations.

2- Block Diagrams.

3- System functions $\frac{Y}{X}$ → $\frac{Y(z)}{X(z)} = H(z) + ROC_H$. ; → $\frac{Y}{X} = \frac{P(D)}{Q(D)}$ $\boxed{D \leftrightarrow z^{-1}}$

4- Impulse Response $h[n] \xleftrightarrow{z} H(z) + ROC$

__illustration__ $x[n] \longrightarrow \underset{h[n]}{\boxed{\text{sys}}} \longrightarrow y[n] = x[n] * h[n]$.

$$\rightarrow Y(z) = X(z) \cdot H(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} + ROC_H.$$

__Def__ DT LTI systems

$\boxed{1.}$ __Causality__ a DT system is causal iff. $y[n] = fn\{x[n], \underbrace{y[n]}_{\text{previous}}\}$

i.e. iff. $y[n]$ is a function of previous values of $x[n]$.

i.e. $y[n] = fn\{x[n], x[n-1], ---\}$

__practically__ all sys. are effectively causal.

$\leftarrow y[n] = fn\{x[n], x[n-1], --$
$x[n+1], x[n+2], --, x[n+10]\}$.

Let $v[n] = x[n+10]$.

$\rightarrow y[n] = fn[v[n-10], ---$
$v[n-9], --, v[n]]$.

__Theorem__ LTI sys. is causal iff $h[n]$ is causal.

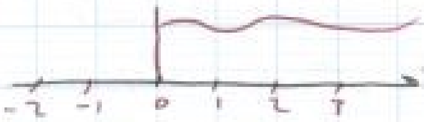
pf// $y[n] = (x * h)[n]$

$= --- + x[n+2]h[-2] + x[n+1]h[-1] + x[n]h[0] + x[n-1]h[1] + --$

"$h[n] = 0 \quad n < 0$"

causality:

signal: $h[n]$

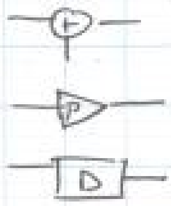


Diff. Eqn: $y[n] = fn\{x[n], y[n]\}$ previous.

$$\sum_{0}^{N} a_l\, y[n-l] = \sum_{0}^{M} b_k\, x[n-k]$$

$$a_0 y[n] + a_1 y[n-1] + \dots + a_N y[n-N]$$
$$= b_0 x[n] + \dots + b_M x[n-M].$$

Block Diagram:



causal operations.

System. fn: "∞" $\in$ ROC.

eg// non-causal: $h[n] = -a^n u[-n-1]$.

$y[n] = -\{a^{-1} x[n+1] + a^{-2} x[n+2] + \dots\}$

[2] Stability

absolute stability iff $\begin{cases} |h[n]| \to 0 & \text{as } n \to +\infty \\ |h[n]| \to 0 & \text{as } n \to -\infty \end{cases}$

$\longleftrightarrow$

"in Z-Domain" iff $\begin{cases} \text{"U.C"} \subset \text{ROC} \\ \text{U.C: unit circle set of } z \text{ s.t } |z| = 1. \end{cases}$

proof Analysis valid for rational sys. functions.

$\Longrightarrow$

$D \leftrightarrow z^{-1}$

$$H(z) = \frac{P_s(z^{-1})}{Q_s(z^{-1})} = z^{k}\,\frac{P_z(z)}{Q_z(z)}$$

// z^{k}: time-shift by k - doesn't affect.

$$= \text{poly}(z) + \boxed{\frac{P(z)}{Q(z)}}$$ proper

// poly(z): in t-domain: combination of δ's, No effect.

partial fraction expansion:

$$\sum_{l} a_l \frac{1}{(1-p_l z^{-1})} + \sum_{l}\left(\alpha_l \frac{1}{(1-p_l z^{-1})^2} + \beta_l \frac{1}{1-p_l z^{-1}}\right) + \cdots$$

simple poles

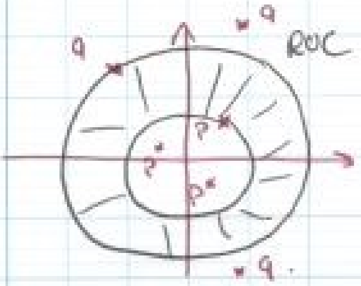
double poles

inverse Z-transform:

$$h[n] = \sum_{l} a_l \cdot \text{poly}[n] \times (p_l)^n u[n] + \sum_{l} \alpha_l (q_l)^n u[-n-1]$$

poles "inside" ROC

poles "outside" ROC.



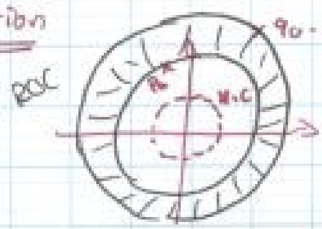
$|h[n]| \to 0$ as $n \to \pm\infty$.

$\Leftrightarrow$ $n>0 \to |p_l|<1$

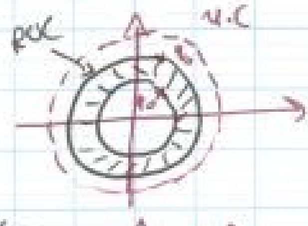
$n>0 \to |q_l|>1$

$\Leftrightarrow$ "U.C." $\subset$ ROC.

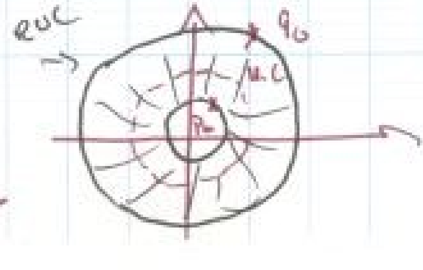
illustration



$\to |p_0|>1 \to$ diverges as $n \to \infty$



$\to |q_0|<1 \to$ diverges as $n \to -\infty$



$\to$ $|p_0|<1$, $|q_0|>1$ $\to$ converges.

Another Representation of DT LTI sys.

State-space Description

- highlight notion of state.
- useful when one has initial conditions.
- very popular in control community.

Def// **State** quanty(ies) that summarizes the past.

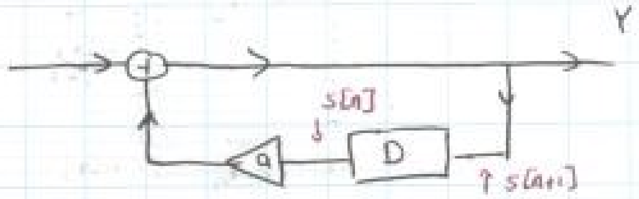
Implications To evaluate output of a system, need:

- state(s)
- current & future inputs.

Def let $s[n]$: state vector $s[n] \in \mathbb{R}^K$. $x[n] \in \mathbb{R}$.

$$\begin{cases} s[n+1] = A\, s[n] + B\, x[n] \\ y[n] = C\, s[n] + D\, x[n] \end{cases} \qquad A \in \mathbb{R}^{K\times K},\ B \in \mathbb{R}^K,\ C \in \mathbb{R}^{1\times K},\ D \in \mathbb{R}$$

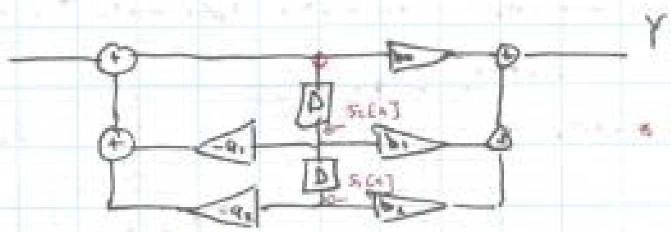
eg①:



$$y[n] = x[n] - a\, y[n-1]$$

$$\leftrightarrow \begin{cases} s[n+1] = a\, s[n] + x[n]. \\ y[n] = a\, s[n] + x[n] \end{cases}$$

eg//



$$\frac{Y}{X} = \frac{\sum_{k=0}^{2} b_k z^{-k}}{\sum_{l=0}^{2} a_l z^{-l}} = H(z) \qquad \frac{Y}{X} = \frac{\sum_{0}^{2} b_k D^k}{\sum_{0}^{2} a_l D^l}$$

$(a_0 = 1)$

$$\cancel{a_0}^{1}\, y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$\begin{array}{cccc} & & s_1[n] & s_2[n] \\ & & \downarrow & \downarrow \end{array}$$

$$S[n+1] = \begin{pmatrix} s_1[n+1] \\ s_2[n+1] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} s[n] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x[n]$$

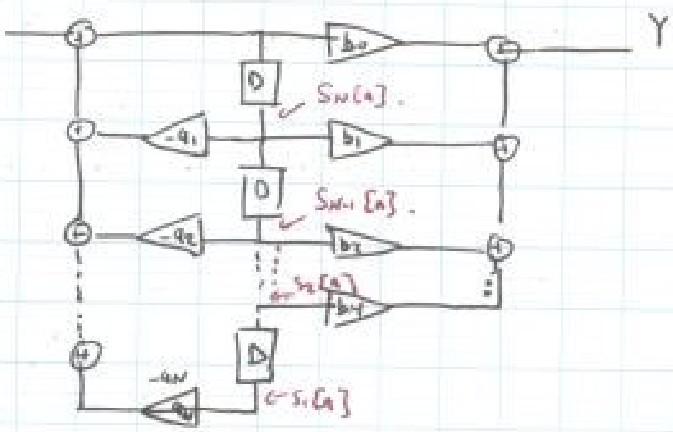
$$s_1[n+1] = s_2[n].$$

$$s_1[n] = -a_2 s_1[n] - a_1 s_2[n] + x[n].$$

$$\begin{aligned} y[n] &= b_0 s_2[n+1] + b_1 s_2[n] + b_2 s_1[n] \\ &= b_0(-a_2 s_1[n] - a_1 s_2[n] + x[n]) + b_1 s_2[n] + b_2 s_1[n] \\ &= (b_2 - a_2 b_0) s_1[n] + (b_1 - a_1 b_0) s_2[n] + b_0 x[n] \\ &= ((b_2 - a_2 b_0), (b_1 - a_1 b_0)) s[n] + b_0 x[n]. \end{aligned}$$

General Case $\sum_0^N a_l y[n-l] = \sum_0^M b_k x[n-k].$

DF II



$$\Rightarrow \quad s_1[n+1] = s_2[n].$$
$$s_2[n+1] = s_3[n]$$
$$\vdots$$
$$s_{N-1}[n+1] = s_N[n].$$
$$s_N[n+1] = -a_N s_1[n] - a_{N-1} s_2[n] - \dots - a_1 s_N[n] + x[n]$$

$$\Rightarrow S[n+1] = \begin{array}{r} s_1[n+1] \rightarrow \\ \\ \\ \\ s_N[n+1] \rightarrow \end{array} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ -a_N & -a_{N-1} & \cdots & \cdots & a_1 \end{pmatrix} s[n] + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} x[n].$$

N.B// if $M < N$ // $b_i = 0$ for all $M > i \geqslant N$.

$$y[n] = b_0 S_N[n+1] + S_N[n] b_1 + S_{N-1}[n] b_2 + \dots + S_1[n] b_N.$$

$$\Rightarrow y[n] = (b_N - b_0 a_N, b_{N-1} - b_0 a_{N-1}, \dots, b_1 - b_0 a_1) S[n] - b_0 x[n]$$

$$\begin{cases} s[n+1] = A s[n] + B x[n]. \\ y[n] = C s[n] + D x[n] \end{cases}$$

$\boxed{*}$ $\begin{cases} \text{"}x[n]\text{" is given from time '0' onward} \\ s[n]. \end{cases}$

$\rightarrow$ Need: $y[n]$ from '0' onward.

$$\begin{aligned} s[n+1] &= A s[n] + B x[n] \\ &= A(A s[n-1] + B x[n-1]) + B x[n] \\ &= A[A[A s[n-2] + B x[n-2]] + B x[n-1]] + B x[n]. \\ &= \dots \\ &= A^{n+1} s[0] + \sum_0^n A^k B x[n-k] \end{aligned}$$

$$\Rightarrow \begin{cases} s[n+1] = A^{n+1} s[0] + \sum_0^n A^k B\, x[n-k]. \\ y[n] = C A^n s[0] + \sum_0^{n-1} C A^k B\, x[n-1-k] + D x[n]. \end{cases}$$

Z-transform

Def// z-transform of a vector/matrix is the vector/matrix of the z-transform.

Z-trans

$$\begin{cases} z S(z) = A S(z) + B X(z) \\ Y(z) = C S(z) + D X(z) \end{cases}$$

$$\longleftrightarrow \begin{cases} (zI-A)\,S(z) = B\,X(z) \\ Y(z) = C\,S(z) + D\,X(z) \end{cases}$$

$$\Longleftrightarrow \begin{cases} S(z) = (zI-A)^{-1} B\,X(z) \\ Y(z) = \left[C\,(zI-A)^{-1} B + D \right] X(z). \end{cases}$$

$\rightarrow$ modes $\equiv$ poles of sys function??.

- $\frac{Y}{X} = C\,\underline{(zI-A)}^{-1} B + D$

N.B// $(zI-A)^{-1} = \frac{1}{Q(z)} \begin{pmatrix} poly(z) & poly_1(z) & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$

$$= \frac{1}{\det(zI-A)}\, C_o\,(zI-A)^T$$

<u>illustration</u>

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

s.t. $a_{11} = \det \begin{pmatrix} \cancel{A_{11}} & \cdots & \cancel{A_{1n}} \\ \cancel{A_{21}} & A_{22} & \cdots & A_{2n} \\ & & & \vdots \\ \cancel{A_{n1}} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$

$a_{21} = \det \begin{pmatrix} \cancel{A_{11}} & A_{12} & \cdots & A_{1n} \\ \cancel{A_{21}} & \cancel{A_{22}} & \cdots & \cancel{A_{2n}} \\ & & & \vdots \\ \cancel{A_{n1}} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$

if transpose: a_{12} is replaced by a_{21}.

$\Rightarrow$ Poles of sys. are the roots of $\boxed{\det(zI-A)=0}$

$\equiv$ Eigen Values of matrix A.

Recall $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_N & -a_{N-1} & \cdots & \cdots & -a_2 & -a_1 \end{pmatrix}$

$$\det(zI - A) = \sum a_i z^i .$$

Unilateral Z-transform:

appropriate for initial conditions.

Def $\tilde{X}(z) = \sum_0^\infty x[n] z^{-n}$ over ROC

observe:

1. Don't have $x[n]$ for $n<0$.
2. if $x[n] \neq y[n]$ with $x[n]u[n] = y[n]u[n]$
   $\rightarrow \tilde{X}(z) = \tilde{Y}(z)$

3. No uniqueness.
4. $\tilde{X}(z) \xrightarrow{\text{inv. } Z} x[n]$ for $n \geq 0$ only
5. $x[n] \rightarrow \tilde{X}(z)$, $x[n]u[n] \rightarrow \tilde{X}(z)$.

$\Rightarrow$ useful to work with causal functions only.

Properties ① linear ---

② $x[n] \rightarrow \tilde{X}(z)$ ROC_x.

$\Rightarrow n\,x[n] \rightarrow -z \dfrac{d\tilde{X}}{dz}(z)$ ROC_x.

③ $x[n] \rightarrow \tilde{X}(z)$

$x[n-1]u[n-1] \rightarrow z^{-1} \tilde{X}(z)$.

$$Z\{x[n-1]u[n-1]\} = \sum_0^\infty x[n-1]u[n-1] z^{-n}$$

$$= \sum_1^\infty x[n-1] z^{-n} \qquad n' = n-1$$

$$= z^{-1} \sum_0^\infty x[n'] z^{-n'} = z^{-1} \tilde{X}(z).$$

④ $x[n] \rightarrow \tilde{x}(z)$ ROC_x

$a^n x[n] \rightarrow \tilde{X}(a^{-1} z)$ ROC_x.

⑤ * $y[n] = x[n+1]$

$$\tilde{Y}(z) = \sum_0^{\infty} y[n] z^{-n} = \sum_0^{\infty} x[n+1] z^{-n}$$

$$= x[1] z^0 + x[2] z^{-1} + x[3] z^{-2} + \cdots$$

$$= z (x[1] z^{-1} + x[2] z^{-2} + \cdots)$$

$$= z (\underline{+x[0] z^0} + x[1] z^{-1} + x[2] z^{-2} + \cdots \underline{- x[0] z^0})$$

$$= z (\tilde{X}(z) - x[0]).$$

* $y[n] = x[n+2]$

$\rightarrow Y(z) = z^2 \tilde{X}(z) - z^2 x[0] - z x[1]$.

⋮

* $y[n] = x[n+N]$

$\tilde{Y}(z) = z^N \tilde{X}(z) - z^N x[0] - z^{N-1} x[1] - \cdots - z^1 x[N-1]$.

→ Useful for causal ⊕ initial condition.

$y[n] = 3y[n-1] - 2y[n-2] + x[n-2]$.

if we know x from '0' onward

Need initial conditions of $y[0]$ & $y[1]$.

Unilateral Z-trans.

eg// $y[n+2] - 3y[n+1] + 2y[n] = x[n]$

$\rightarrow (z^2 \tilde{Y}(z) - z^2 y[0] - z y[1]) - 3 (z \tilde{Y}(z) - z y[0]) + 2 \tilde{Y}(z) = \tilde{X}(z)$

- - - -

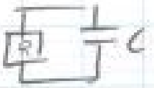
$$\tilde{Y}(z) = \underbrace{\frac{1}{z^2 - 3z + 2} \tilde{X}(z)}_{\text{Zero State Response (ZSR)}} + \underbrace{\frac{z y[1] - (-z^2 + 3z) y[0]}{z^2 - 3z + 2}}_{\text{Zero Input Response (ZIR)}}$$

Papillon

New

Continuous-Time (CT) Sys & Signals

eg// ① EM signals. velocity. RC circuits ---



$$\frac{dv}{dt} = -\frac{1}{RC} v = -\frac{v}{\tau} \qquad \tau: \text{time constant.}$$

$$\rightarrow v(t) = v(0)\, e^{-t/\tau}$$

Procedure * Discretize Time

i.e. Transform a CT sys. into a DT sys.

→ Sample @ every "T" // differential eqn → difference eqn.

eg// $v[n] = v[nT]$ (sample at every T)

Forward Euler's Approximation!

$$\frac{dv}{dt}(nT) = \frac{v((n+1)T) - v(nT)}{T} = \frac{v[n+1] - v[n]}{T}$$

$$\rightarrow \frac{dv[nT]}{dt} = -\frac{v[nT]}{\tau} \Leftrightarrow \frac{v[n+1] - v[n]}{T} = -\frac{v[n]}{\tau}.$$

so $v[n+1] = (1 - \frac{T}{\tau})\, v[n]. \quad = y[n+1] = \alpha\, y[n].$

→ solution is $v[n] = v[0](1 - T/\tau)^n$.

Comparison in CT: $v(t) = v_0\, e^{-t/\tau}$

$$v(nT) = v_0\, e^{-nT/\tau}$$

$$= v_0\, (e^{-T/\tau})^n$$

$$= v_0\, (1 - \tfrac{T}{\tau})^n \quad \text{for } T \ll \tau.$$

ECE340-"1"

Papeterie

eg/ ② $\frac{d^2 v(t)}{dt^2} + A v(t) = 0$

$\rightarrow v(t) = \alpha \cos\sqrt{A}t + \beta \sin\sqrt{A}t.$

for $\left.\begin{array}{l} v(0) = 0 \\ v'(0) = 0 \end{array}\right\} \rightarrow v(t) = \beta \sin(\sqrt{A}t)$

<u>Discretize</u> $v[nT] = v[n].$

$\frac{dv(nT)}{dt} = \frac{v[n+1] - v[n]}{T}.$

$\frac{dv^2(nT)}{dt^2} = \frac{d}{dt}\left(\frac{dv}{dt}(nT)\right)$

$= \dfrac{\frac{v[n+2]-v[n+1]}{T} - \frac{v[n+1]-v[n]}{T}}{T}$

$= \frac{v[n+2] - 2v[n+1] + v[n]}{T^2}$

$\Rightarrow v[n+2] - 2v[n+1] + v[n] + T^2 A v[n] = 0.$

$v[n+2] - 2v[n+1] + (1+T^2A)v[n] = 0.$

<u>z-trans</u>

$z^2 - 2z + (1+T^2A) = 0$

$\Delta = -4T^2A$

$p_1 = \frac{2 + 2j\sqrt{T^2A}}{2} = j\sqrt{T^2A} + 1 = \sqrt{1+T^2A}\, e^{j\theta}.$

$p_2 = \frac{2 - 2j\sqrt{T^2A}}{2} = 1 - j\sqrt{T^2A} = \sqrt{1+T^2A}\, e^{-j\theta}.$

<u>$|p| > 1$</u>

$\rightarrow v[n] = \alpha p_1^n u[n] + \beta p_2^n u[n]$

$= (\alpha(\sqrt{1+T^2A}\, e^{j\theta})^n + \beta(\sqrt{1+T^2A}\, e^{-j\theta})^n) u[n]$

$= \cdots = \alpha(\sqrt{1+T^2A})^n \cos n\theta \; u[n].$

$\rightarrow$ diverging $A > 0.$

⚠ Not Stable. Approximation
since real signal behaves differently.

— the first column of which is the upper right ... of the ...

Similarly for Backward Euler's Approx:,

$$\frac{d^2v}{dt^2}(nT) = \frac{d}{dt}\left(\frac{v[nT] - v[(n-1)T]}{T}\right) = \frac{v[n] - 2v[n+1] + v[n+2]}{T^2}.$$

$\rightarrow (1+T^2A) - 2z^{-1} + z^{-2} = 0.$

$$P_1 = \frac{1}{1+T^2A} e^{j\theta} \qquad P_2 = \frac{1}{1+T^2A} e^{-j\theta} \qquad \underline{\underline{|P|<1}}$$

$\rightarrow$ converges to zero.
Not similar to real case behavior.

N.B// we can use forward then backward approx. to get $|P| = 1$. so that it would be near real case.

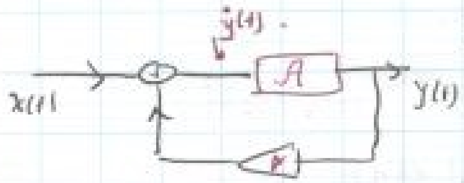
Representations of CT sys (LTI's)

$\boxed{\text{N.B// } \dot{y}(t) = \frac{dy(t)}{dt}}$

① - Differential equations $\quad \dot{y}(t) = \kappa y(t) + \beta x(t)$

② Block Diagrams $\quad \mathcal{A}: \mathcal{A}x = y.$

eg// $\Rightarrow \int x(t)\,d(t) = y(t).$



③ $\frac{Y}{X}$ fn. of $\mathcal{A}$:

$$y = \mathcal{A}(x + \kappa y)$$
$$= \mathcal{A}\frac{x}{1-\kappa A}.$$

$$\frac{Y}{X} = \frac{A}{1-\kappa A}. \qquad \text{Modes: roots of denominator}$$

$\rightarrow$ decompose systems to cascade & parallel fundamental Blocks.

④ ~~Im~~ Impulse Response - - - (coming soon)

⑤ Laplace Transform - - - (coming soon)

Dealing with $x(t)$

$\mathbb{R} \to \mathbb{C}$ (or $\mathbb{R}$) $\qquad x(t) \to \boxed{\text{sys.}} \to y(t)$ $\qquad$ LTI'S ONLY

$t \to x(t)$.

Def// Linear: A CT sys is Linear iff:

$$\left.\begin{array}{l} x_1(t) \to y_1(t). \\ x_2(t) \to y_2(t) \end{array}\right\} \to a_1 x_1(t) + b x_2(t) \to a_1 y_1(t) + b y_2(t)$$

Def// Time invariant: A CT sys is time invariant iff.

$$x(t) \to y(t) \Rightarrow x(t-\tau) \to y(t-\tau) \quad \forall \tau \in \mathbb{R}.$$

$\Rightarrow$ Representations

① cts. coeff differential equations:

"Generally" in form of:

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{N-1} \dot{y}(t) + a_N y(t)$$
$$= b_n x^{(n)}(t) + b_{n-1} x^{(n-1)}(t) + \cdots + b_1 \dot{x}_1(t) + b_0 x(t).$$

$1 \leftarrow a_0$

Assumption $a_0 = 1$.

② computing output for given input.

$\to$ solve differential eqn:

i) Tables.

ii) properties, separation.

iii) Particular / homogeneous solutions.

iv) Laplace Transformations.

eg// $x(t) \to \boxed{\text{sys}} \to y(t)$

$$\dot{y}(t) = x(t) + \alpha y(t)$$

1st order differential eqn.

??

Analogy

CT

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{"}\infty\text{"} & t = 0 \end{cases}$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$\dot{u}(t) = \delta(t)$$

DT

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

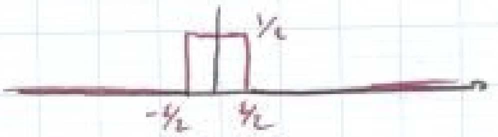
$$u[n] - u[n-1] = \delta[n]$$

Impulse $\delta(t) = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0 \end{cases}$

↳ is equivalent to all-zero fn.

$\Rightarrow$ " $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{"}\infty\text{"} & t = 0 \end{cases}$ "

$\Rightarrow$ * $\delta(t)$ is a limit for $g_\varepsilon(t)$.



$= \begin{cases} 0 & |t| > \varepsilon/2 \\ 1/\varepsilon & |t| \leq \varepsilon/2 \end{cases}$

$\rightarrow$ " $\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t)$ "

$\Rightarrow$ * $\delta(t)$ is a distribution /generalized function: Dirac Impulse

$$\delta: f(t) \rightarrow f(0)$$

$\therefore$ a dirac impulse is defined by its action on functions.

$$\int_{-\infty}^{+\infty} \delta(t) f(t)\, dt = f(0)$$

$$\boxed{\int_{0^-}^{0^+} \delta(t) = 1}$$

Properties

i) $\int_{-\infty}^{t} f(\tau)\,\delta(\tau)\,d\tau = \begin{cases} 0 & t<0 \\ f(0) & t\geq 0 \end{cases}$

ii) $\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases} = u(t)$

$\rightarrow \dot{u}(t) = \delta(t) \iff \int_{-\infty}^{+\infty} \delta(t)\,dt = u(t)$

iii) $g(t)\,\delta(t) = g(0)\,\delta(t)$

pf// $\int_{\mathbb{R}} (g(t)\,\delta(t))\,f(t)\,dt = \int_{\mathbb{R}} \delta(t)\,f(t)\,g(t)\,dt = f(0)\,g(0)$

$= g(0)\int_{\mathbb{R}} f(t)\,\delta(t)\,dt = g(0)\,f(0)$

iv) $\int_{-\infty}^{t} \delta(t) = u(t) \dashrightarrow \overbrace{\int \cdots \int}^{n} \delta(\tau)\,d\tau = \frac{t^{n-1}}{(n-1)!}\,u(t)$

$\iint \delta(t) = \int_{-\infty}^{t} u(t)\,dt = \begin{cases} 0 & \tau<0 \\ t & \tau\geq 0 \end{cases} \quad \left| = \int_0^t d\tau = t \right.$

--- same procedure.

$\boxed{\text{es//}}$ $\dot{y}(t) = x(t) + \alpha\, y(t)$

Solve for $x(t) = \delta(t)$

Impulse Response

$y(t) = C e^{\alpha t} u(t)$

$\dot{y}(t) = \alpha C e^{\alpha t} u(t) + C\,\delta(t)$

$\rightarrow \dot{y}(t) = \delta(t) + \alpha\, y(t)$

$\rightarrow \quad = \underline{\delta(t)} + \alpha C e^{\alpha t} u(t)$

But $\dot{y}(t) = \alpha C e^{\alpha t} u(t) + \underline{C\,\delta(t)}$

$\Rightarrow \underline{C=1}$ IR// $y(t) = e^{\alpha t} u(t)$

exponential in "t"

→ the first column of which is the [illegible] [illegible] [illegible] of 1st [illegible]

$\Rightarrow$ $\dot{y}(t) = x(t) + \kappa y(t)$ $\qquad$ $y[n+1] = \kappa y[n] + x[n]$

IR: $y(t) = e^{\kappa t} u(t)$ $\qquad$ IR: $y[n] = \kappa^n u[n]$.

$u(t) \leftrightarrow u[n]$.

$e^{\kappa} \leftrightarrow \kappa$.

$t \leftrightarrow n$.

IR $\to 0$ as $t \to \infty$

↳ need $Re\{\kappa\} < 0$.

$= e^{\kappa} < 1$

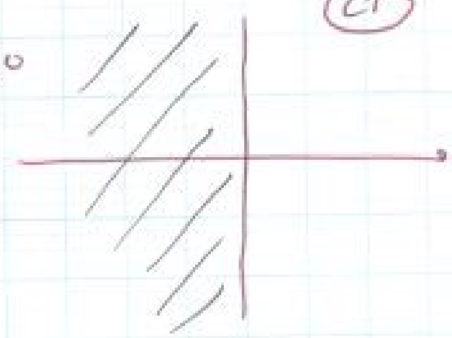
IR $\to 0$ $n \to \infty$

↳ need $|\kappa| < 1$

Pf// $e^{\kappa t} = e^{(\kappa_r + j\kappa_i)t} = \underbrace{e^{\kappa_r t}}_{\text{effecting}} + \underbrace{e^{j\kappa_i t}}_{|e^{j\kappa_i t}| = 1}$.

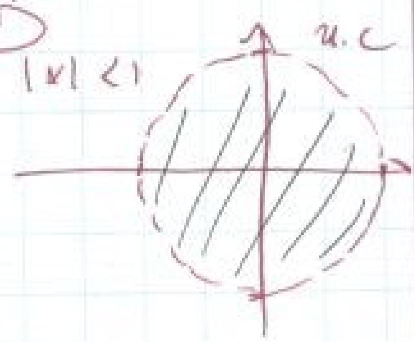
CT

$Re\{\kappa\} < 0$



DT

$|\kappa| < 1$



ECE3240

Representations of CT LTI systems:

i) Differential equations:

$$\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{l=0}^{n} b_l x^{(l)}(t)$$

eg/ $\dot{y}(t) = x(t) + \alpha y(t)$ 1st order system.

$\rightarrow$ if $x(t) = \delta(t)$ $\rightarrow$ IR: $y = e^{\alpha t} u(t)$

"Sign of $Re\{\alpha\}$ is important"

ii) OPERATORS: X & Y are objects, define:

$X + Y$: $x(t) + y(t)$.

$2X$: $2x(t)$.

AX : $\int_{-\infty}^{t} x(t)\,dt$ "use it as polynomials".

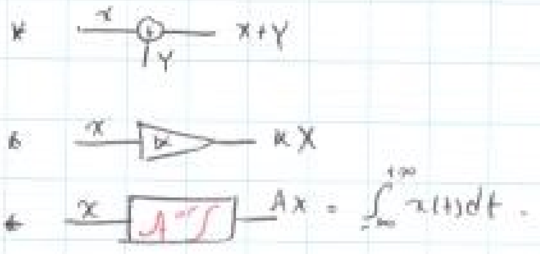
$\frac{1}{A}X$: differentiation

eg/ $\dot{y}(t) = x(t) + \alpha y(t)$

$\rightarrow \frac{1}{A} Y = X + \alpha Y \rightarrow \frac{Y}{X} = \frac{A}{1 - \alpha A}$.

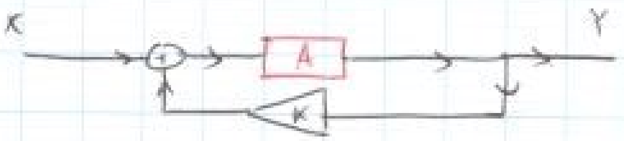
Def sys. fn : $\frac{Y}{X}$ = rational function of A.

iii) Block Diagrams:



eg/ $\dot{y}(t) = x(t) + \alpha y(t)$

$\rightarrow$



General $d/dt = 1/A$, $d^2/dt^2 = 1/A^2$.

$\rightarrow \sum_{j=0}^{n} a_j \frac{1}{A^j} Y = \sum_{l=0}^{n} b_l \frac{1}{A^l} X$

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$\rightarrow \sum_{j=0}^{n} a_j A^{n-j} Y = \sum_{l=0}^{n} b_l A^{n-l} X$.

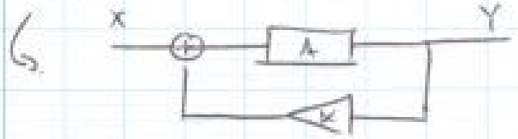
$\Rightarrow$ Sys. fn. $\frac{Y}{X} = \frac{\sum b_l A^{n-l}}{\sum a_j A^{n-j}}$

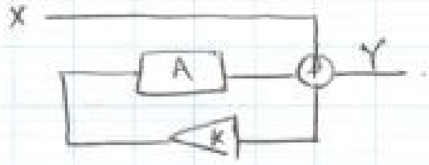
$\rightarrow a_n y(t) + a_{n-1}\int y(t)dt + a_{n-2}\iint y(t)dt + \dots + a_0 \iiint^{n} y(t)dt = b_0 x + b_1\int x(t)dt + \dots$

$a_n Y + a_{n-1} A Y + a_{n-2} A^2 Y + \dots + a_0 A^n Y = b_0 X + b_1 A X + \dots$

General. $\Rightarrow$ Block Diagram

$\Rightarrow \dot{y}(t) = x(t) + \alpha y(t)$ $\quad\longleftrightarrow\quad$ $\dot{y}(t) = \dot{x}(t) + \alpha y(t)$





Superposition

$Y = \left(\frac{A}{1-\alpha A}\right) X = A\left[1 + \alpha A + \alpha^2 A^2 + \dots\right] X$

IR: $u(t) + \alpha t\, u(t) + \alpha^2 \frac{t^2}{2!} u(t) + \dots + \alpha^l \frac{t^l}{l!} u(t) + \dots$

IR: $y = e^{\alpha t} u(t) = \sum_{l=0}^{\infty} \frac{(\alpha t)^l}{l!} u(t)$

eg// $\ddot{y}(t) = \omega_0^2 (x(t) - y(t))$

$\frac{1}{A^2} Y = \omega_0^2 (X - Y)$

$\rightarrow \frac{Y}{X} = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2}$. (cascade of 1st order systems)

ECE340 - "1"

Decomposition into fundemental Blocks:-

$$\frac{Y}{X} = \frac{w_0^2 A^2}{1 + w_0^2 A^2}$$

Cascade

Finding poles
i) $A \hookrightarrow s^{-1}$

ii) denominator in form! +ve powers of s.

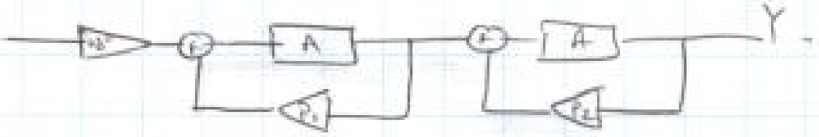
$\rightarrow 1 + w_0^2 s^{-2} \rightarrow s^{-2}(s^2 + w_0)$

iii) find roots of resultant form of denominator.

$p_1 = jw_0 \quad \& \; p_2 = -jw_0 .$

iv) $1 + w_0^2 A^2 = (1 - p_1 A)(1 - p_2 A) \; ---$

cf/ above $\Rightarrow \frac{Y}{X} = w_0^2 \cdot \frac{A}{1 - p_1 A} \cdot \frac{A}{1 - p_2 A}$.

$\Rightarrow$ 

Parallel

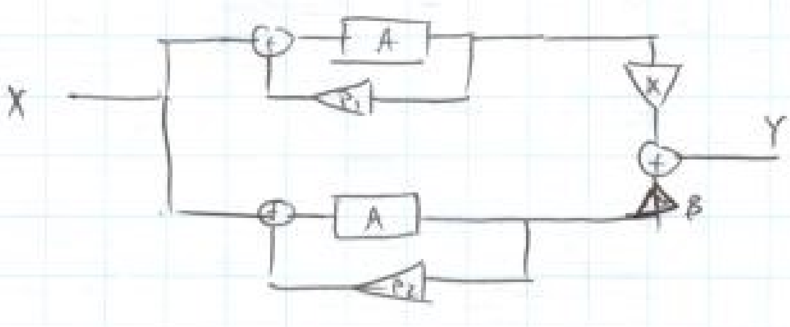
Partial fraction expansion

$$\frac{Y}{X} = \alpha \frac{A}{1 - p_1 A} + \beta \frac{A}{1 - p_2 A} + \cdots$$

$$\alpha = \frac{Y}{X} \frac{(1 - p_1 A)}{A} \Big|_{A \leftarrow 1/p_1} \qquad \alpha = \frac{w_0^2}{p_1 - p_2}$$

$$\beta = \frac{Y}{X} \frac{(1 - p_2 A)}{A} \Big|_{A \leftarrow 1/p_2} . \qquad \beta = \frac{w_0^2}{p_2 - p_1}$$

$$\Rightarrow \frac{Y}{X} = \frac{w_0}{2j} \frac{A}{1 - jw_0 A} - \frac{w_0}{2j} \frac{A}{1 + jw_0 A} .$$



NB// Parallel Systems are better at determining Impulse Responses (instead of convolution in Cascade)

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In previous eg: $g(t) = \frac{w_0}{2j} e^{jw_0 t} u(t) - \frac{w_0}{2j} e^{-jw_0 t} u(t)$

$$= w_0 \left[\frac{e^{jw_0 t} - e^{-jw_0 t}}{2j} \right] u(t)$$

$$= w_0 \cdot \sin(w_0 t) \cdot u(t).$$

⇒ Fundamental Blocks

1st Order: $\frac{A}{1 - \alpha A} \rightarrow e^{\alpha t} u(t)$.

Multiple Order: $\frac{A^k}{(1-\alpha A)^k}$ $k \geq 2 \rightarrow \underbrace{poly(t)}_{\text{doesn't affect convergence}} e^{\alpha t} u(t)$.

Summary on Impulses

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{"}\infty\text{"} & t = 0 \end{cases}$$

Def: Defined by its action on functions.

$f: \mathbb{R} \rightarrow \mathbb{R}$.

$$\int_a^b f(t)\,\delta(t)\,dt = \begin{cases} f(0) & a < 0 < b. \\ 0 & \text{otherwise} \end{cases}$$

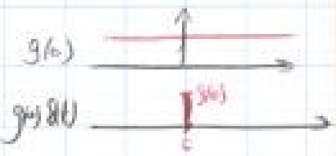
$$\Rightarrow \int_{t_1}^{t_2} f(t)\,\delta(t-t_0)\,dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2. \\ 0 & \text{o.w.} \end{cases}$$

Properties:

i) $\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t)$ $\qquad g_\varepsilon(t) = \begin{cases} 1/\varepsilon & -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2}. \\ 0 & \text{o.w.} \end{cases}$

ii) $g(t)\,\delta(t-t_0) = g(t_0)\,\delta(t-t_0)$

N.B.: $g(t)\,\delta(t) = g(0)\,\delta(t) \neq g(0)$.



iii) $\dot{u}(t-t_0) = \delta(t-t_0)$

iv) $\delta(at) = \frac{1}{|a|}\,\delta(t)$.

Proof: $\int_{-\infty}^{+\infty} f(t)\,\delta(at)\,dt = \int_{-\infty}^{+\infty} f\!\left(\frac{t'}{a}\right)\delta(t')\,\frac{dt'}{a} = \frac{1}{a} f(0) = \frac{1}{a}\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt$.

$$\Rightarrow \delta(-t) = \delta(t)$$

ECE3210 - 4

Properties

$$\Rightarrow f(t) = \int f(\tau)\,\delta(\tau - t)\,d\tau$$

$$\Leftrightarrow f(t) = \int f(\tau)\,\delta(t-\tau)\,d\tau$$

$\Rightarrow$ any signal is composed of a collection of "$\delta(t-t_0)$"'s

Representation using Impulse Response

$x(t) \rightarrow \boxed{sys} \rightarrow y(t)$

$\delta(t) \rightarrow \boxed{sys} \rightarrow h(t)$

$\int x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \boxed{sys} \rightarrow \int x(\tau)\,h(t-\tau)\,d\tau$

$\Rightarrow$ $x(t) \rightarrow \underset{h(t)}{\boxed{sys}} \rightarrow$ $\quad y(t) = (x * h)(t) = \displaystyle\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$

Properties of convolution

1) $x * y = y * x$.
2) $x * (y + z) = x * y + x * z$
3) $x * (y * z) = (x * y) * z$.

Computing convolution

1) Plug & do the Math.
2) Graphically : Flip & Slide.
3) Superposition (useful for signal = combination of δ's)

Eg// on superposition :

$x(t) = a\,\delta(t-t_0) + b\,\delta(t-t_1)$

$y(t) = \ldots$

$x * y(t) = a\,y(t-t_0) + b\,y(t-t_1)$.

on mathematical :

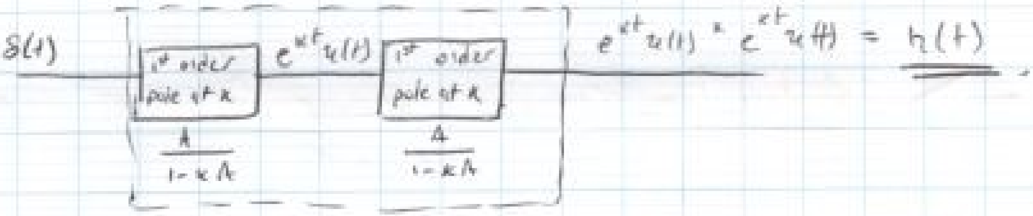
$e^{\alpha t} u(t) * e^{\alpha t} u(t)$

$= \displaystyle\int_{-\infty}^{+\infty} e^{\alpha\tau} u(\tau)\, e^{\alpha(t-\tau)} u(t-\tau)\,d\tau$ $\quad \left\{ \begin{array}{l} t < \tau \rightarrow 0 \\ t > \tau \rightarrow 1 \end{array} \right.$

$= \left\{ \begin{array}{ll} 0 & t < 0 \\ \displaystyle\int_0^t e^{\alpha\tau} e^{\alpha(t-\tau)}\,d\tau & t \geq 0 \end{array} \right.$

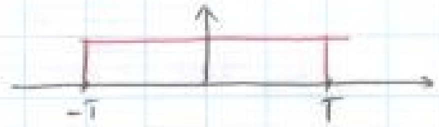
$$e^{\alpha t}u(t) * e^{\alpha t}u(t) = \begin{cases} 0 & t<0 \\ e^{\alpha t}\int_0^t d\tau = te^{\alpha t} & t\geq 0 \end{cases}$$

$\rightarrow e^{\alpha t}u(t) * e^{\alpha t}u(t) = te^{\alpha t}u(t)$

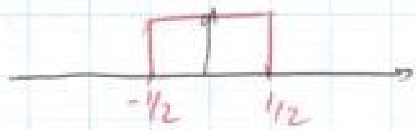


Eg on graphical convolution

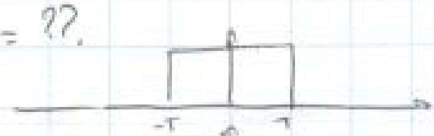
N.B// $x(t) = \text{rect}\left(\frac{t}{2T}\right) = u(t+T) - u(t-T)$



where $\text{rect}(t)$

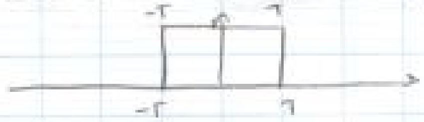


$\rightarrow x(t) * x(t) = ??$



$x(t)$

1) flip.



$x(-t)$

2) Slide

3) Mult. & integrate.

mult. → 0
integ. → 0
$\}\ t < -2T$ $\qquad t+T < -T$.

mult. → 0
integ → 0
$\}\ t > 2T$ $\qquad t-T > T$.

mult

integ $\quad t+2T$

$\}\ -2T < t < 0$.

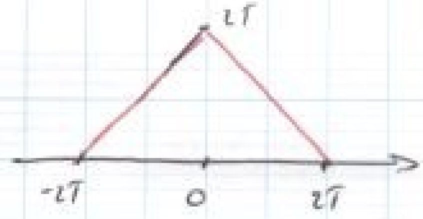
mult

integ. $\quad 2T-t$

$\}\ 0 < t < 2T$.

when area of overlap changes variation (inc.-dec) → cut the interval

N.B// take intervals according to shape of the functions + according to whether the area is increasing or decreasing
of overlap

$\Rightarrow$ 

$$\begin{cases} 0 & t < -2T \\ t+2T & -2T < t < 0 \\ 2T - t & 0 < t < 2T \\ 0 & t > 2T \end{cases}$$

$\Rightarrow$ **Laplace Transform** $\quad f: \mathbb{R} \to \mathbb{C} \xrightarrow{\mathcal{L}} F: \mathbb{C} \to \mathbb{C}$

$\quad x \to f(x) \qquad\qquad s \to F(s).$

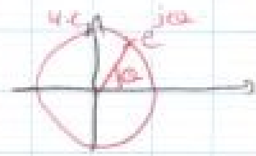
Def $F(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt \quad$ **+ ROC**

whenever the integral is finite.

i.e. $s \in ROC$ where $ROC \subset \{\mathbb{C} \cup \{\text{"}-\infty\text{"}\} \cup \{\text{"}+\infty\text{"}\}\}$

Def ROC $\left\{ \begin{array}{l} s \in \mathbb{C} \cup \{\text{"}+\infty\text{"}\} \cup \{\text{"}-\infty\text{"}\} \\ \text{s.t. } \int_{-\infty}^{+\infty} |f(t)||e^{-st}|\, dt < \infty \text{ "finite"} \end{array} \right\}$

$\to$ depends on real part of s. $Re\{s\}$.



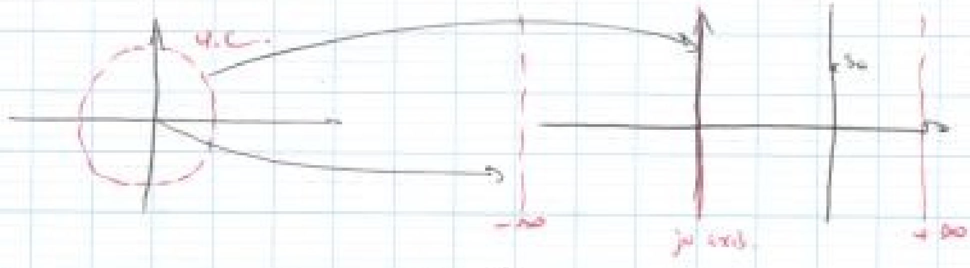
$\to \quad |e^{-st}| = |e^{-(x+jy)t}| = |e^{-xt}| \cdot \underbrace{|e^{-jyt}|}_{=1.\ \text{pt. on u.c}}$

$= |e^{-xt}|$

$\Rightarrow$ **Prop. of ROC**

1) ROC is a collection of vertical lines $\Rightarrow$ single vertical strip.

i.e. if $s_0 \in ROC$ any s s.t. $Re\{s\} = Re\{s_0\}$ then $s \in ROC$.



2) Note: checking $s \overset{?}{\in} ROC$ if $\int_{-\infty}^{+\infty}$ (big) converges i.e. limits $\pm\infty$

"$+\infty$" $\to$ if $\exists x(t) \neq 0$ for some $t < 0$ $\quad \{\infty\} \notin ROC$

"$-\infty$" $\to$ if $\exists x(t) \neq 0$ for some $t > 0$ $\quad \{-\infty\} \notin ROC$.

3) $x(t)$ right-sided $\to$ ROC right-sided
$x(t)$ left-sided $\to$ ROC left-sided.

for right-sided: i.e. $\exists t_0 \in \mathbb{R}$ s.t. $x(t) = 0 \ \forall t < t_0$.

if $s_0 \in ROC$ $\forall s$ s.t. $Re\{s\} \geq Re\{s_0\} \to s \in ROC$.

pf// let s be s.t. $Re\{s\} \geq Re\{s_0\}$.

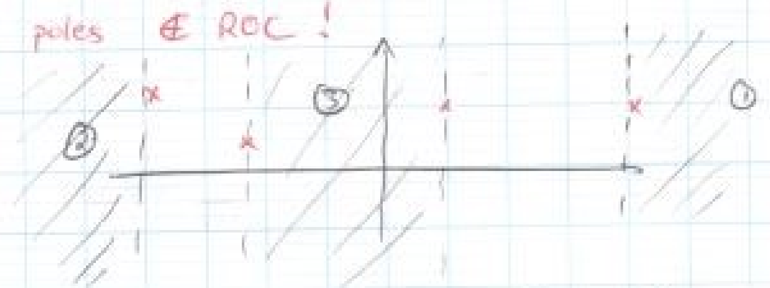
$$\int_{-\infty}^{\infty} |x(t)|\,|e^{-st}|\,dt = \int_{-\infty}^{\infty} |x(t)|\,|e^{-Re\{s\}t}|\,dt \qquad Re\{s\} \geq Re\{s_0\} \ \text{---}$$

4) $x(t)$ double sided $\to$ ROC one single vertical strip.

5) $x(t)$ finite duration $\to$ $\mathbb{C} \subset ROC$ check "$\pm\infty$".

6) $X(s)$ is rational $\to$ ROC determined by poles
& poles $\notin$ ROC!

eg//



if right sided $\to$ ROC right sided like 1.
if left sided $\to$ ROC left sided like 2.

7) $x(t)$ causal $\to$ ROC right-sided & includes "$+\infty$"
$x(t)$ anti-causal $\to$ ROC left sided & includes "$-\infty$"

Papillon

Prop. of Laplace

$x(t) \longrightarrow X(s)$ ROC$_x$

$y(t) \longrightarrow Y(s)$ ROC$_Y$

1) Linearity $\quad a x(t) + b y(t) \overset{\mathcal{L}}{\longleftrightarrow} a X(s) + b Y(s) \quad$ "ROC$_x \cap$ ROC$_Y \subset$ ROC"

2) Delay $\quad x(t-T) \overset{\mathcal{L}}{\longleftrightarrow} e^{-sT} X(s) \quad$ ROC$_x$ △ "$\pm\infty$"

3) Mult by exp. $\quad e^{-\alpha t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s+\alpha) \quad$ Shift ROC by "$-\alpha$"

4) Time Reverse $\quad x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s) \quad$ "$-$ROC$_x$"

5) Mult. by t. $\quad t\, x(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds} X(s) \quad$ ROC$_x$

6) Differentiation $\quad \frac{d}{dt} x(t) \overset{\mathcal{L}}{\longleftrightarrow} s X(s) \quad$ ROC$_x \subset$ ROC.

7) Integration $\quad \int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} X(s) \quad$ ROC $\supset \{$ROC$_x \cap (\mathrm{Re}\{s\} > 0)\}$

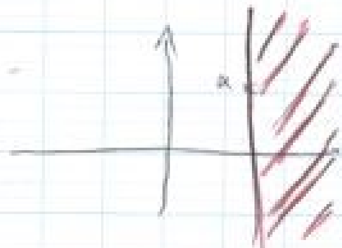
8) Convolution $\quad x * y \overset{\mathcal{L}}{\longleftrightarrow} X(s) \times Y(s) \quad \{ROC_x \cap$ ROC$_Y\} \subset$ ROC.

9) Time Scale $\quad x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad$ "$a \times$ ROC$_x$".

Ex/ ① $x(t) = e^{\alpha t} u(t)$.

$$X(s) = \int_{-\infty}^{+\infty} e^{\alpha t} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s-\alpha)t}\, dt$$

$$= \frac{1}{s-\alpha} e^{-(s-\alpha)t} \Big|_{\infty}^{0}$$

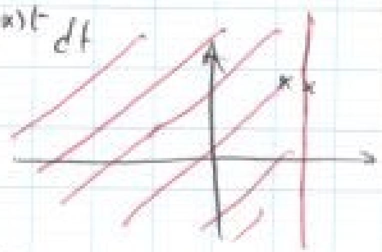


$$X(s) = \frac{1}{s-\alpha} \qquad \text{ROC} = \{\mathrm{Re}\{s\} > \mathrm{Re}\{\alpha\}\} \cup \{\text{"}\infty\text{"}\}$$

② $x(t) = -e^{\alpha t} u(-t)$.

$$X(s) = -\int_{-\infty}^{+\infty} e^{\alpha t} u(-t)\, e^{-st}\, dt = -\int_{-\infty}^{0} e^{-(s-\alpha)t}\, dt$$

$$X(s) = \frac{1}{s-\alpha} e^{-(s-\alpha)t} \Big|_{0}^{-\infty}$$



$$X(s) = \frac{1}{s-\alpha} \qquad \text{ROC} = \{\mathrm{Re}\{s\} < \mathrm{Re}\{\alpha\}\} \cup \{\text{"}-\infty\text{"}\}$$

Unilateral Laplace

$x(t) \longleftrightarrow \tilde{X}(s) = \int_0^\infty x(t)\, e^{-st}\, dt.$

$ROC = \{ s \in \mathbb{C} \cup \{-\infty\} \cup \{+\infty\} \text{ s.t. } \int_0^\infty |x(t)||e^{-st}|\, dt \text{ finite} \}$

→ uniqueness only in causal systems.

→ $\begin{cases} \text{one-to-one} \\ \text{"}+\infty\text{"} \in ROC \\ \text{"}-\infty\text{"} \in ROC \text{ iff } x(0) \neq 0 \text{ only} \Rightarrow x(t) = k\,\delta(t). \end{cases}$

Theorem Initial & final Values.

Let $x(t) = 0 \quad \forall t < 0$ "causal"

① if $x(t)$ has no impulses at zero

$\rightarrow \quad x(0^+) = \lim_{s \to \infty} s X(s).$

② if $x(t)$ has a limit as $t \to \infty$.

$x(\infty) = \lim_{s \to 0} s X(s).$

pf idea: $sX(s) = \int_0^\infty x(t)\, s\, e^{-st}\, dt.$

$\lim_{s \to \infty} s e^{-st} = \delta(t).$

Prop. of Unilateral

1) linearity --

2) delay --

3) mult. by t --

4) mult. by exp. --

5) $\dot{x}(t) \rightsquigarrow s\tilde{X}(s) - x(0).$

$\ddot{x}(t) \rightsquigarrow s^2 \tilde{X}(s) - s\,x(0) - \dot{x}(0).$

$\vdots$

$\Rightarrow x^{(n)}(t) \overset{\mathcal{L}}{\rightsquigarrow} s^n \tilde{X}(s) - s^{n-1} x(0) - s^{n-2} \dot{x}(0) - \dots - s^0 x^{(n-1)}(0).$

useful for initial conditions

c// know $x(t)$ $t \geq 0$.
find $y(t)$ $t \geq 0$? + initial conditions

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \dots + a_0 y(t) = b_{n-1} x^{(n-1)}(t) + b_{n-2} x^{(n-2)}(t) + \dots + b_0 x(t)$$

$\mathcal{L}$:

$$\left[s^n \tilde{Y}(s) - s^{n-1} y(0) - s^{n-2} \dot{y}(0) - \dots - y^{(n-1)}(0)\right]$$
$$+ a_{n-1} \left[s^{n-1} \tilde{Y}(s) - s^{n-2} y(0) - s^{n-3} \dot{y}(0) - \dots - y^{(n-2)}(0)\right]$$
$$+ \dots$$
$$+ a_1 \left[s \tilde{Y}(s) - y(0)\right]$$
$$+ a_0 \left[\tilde{Y}(s)\right]$$

$$= b_{n-1} \left[s^{n-1} \tilde{X}(s) - s^{n-2} x(0) - s^{n-3} \dot{x}(0) - \dots - x^{(n-2)}(0)\right].$$
$$+ \dots$$
$$+ b_1 \left[s \tilde{X}(s) - x(0)\right]$$
$$+ b_0 \left[\tilde{X}(s)\right]$$

$$\Rightarrow \tilde{Y}(s)\left[s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0\right] = \tilde{X}(s)\left[b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \dots + b_1 s + b_0\right] + \boxed{\text{initial conditions}}$$

$$\Rightarrow \tilde{Y}(s) = \underbrace{\frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}}_{\text{ZSR: zero-state-response}} \tilde{X}(s) + \underbrace{\frac{\boxed{\text{initial conditions}}}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}}_{\text{ZIR: zero-input response}}$$

ZSR: zero-state-response
H(s) for causal system.

ZIR: zero-input response

N.B// ZSR & ZIR
have same denominators ⇒ Same Poles.

Eigen functions & Values

CT

$x(t) = e^{\alpha t}$ → LTI $h(t)$ → $y(t) = e^{\alpha t} H(\alpha)$

$x(t)$: eigen function; $H(\alpha)$: scalar eigen value

DT

$x[n] = \alpha^n$ → $h[n]$ → $y[n] = H(\alpha)\,\alpha^n$

$x[n]$: eigen fn.; $H(\alpha)$: eigen value (scalar)

pf
$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{\alpha(t-\tau)}\, dt$$
$$= e^{\alpha t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-\alpha\tau}\, d\tau}$$
$$= e^{\alpha t} \cdot H(\alpha)$$

N.B// * $e^{\alpha t}$ is eigen function & not $e^{\alpha t}u(t)$.
* applicable in t-domain. $\triangle$ $e^{\alpha t} \xrightarrow{\mathcal{L}} \not\exists$ (no Laplace trans.)

State-Space description

multiple inputs multiple outputs.

$$\begin{cases} \underset{n\times 1}{\dot{s}(t)} = \underset{n\times n}{A}\, \underset{1\times n}{s(t)} + \underset{n\times 1}{B}\, \underset{1\times 1}{x(t)} \\ \underset{1\times 1}{y(t)} = \underset{n\times 1}{C}\, \underset{1\times n}{s(t)} + \underset{1\times 1}{D}\, \underset{1\times 1}{x(t)} \end{cases} \qquad s(t) \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}$$

eg// $y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + a_{n-2} y^{(n-2)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t)$
$= b_{n-1} x^{(n-1)}(t) + b_{n-2} x^{(n-2)}(t) + \cdots + b_1 \dot{x}(t) + b_0 x(t)$.

* $y(t) = s_1(t) = (1\ 0 \cdots 0)\, S(t)$.
* $$\dot{S}(t) = \begin{pmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -a_1 & \vdots & \vdots & & 1 \\ a_0 & 0 & 0 & \cdots & 0 \end{pmatrix} S(t) + \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} x(t)$$

$\dot{s}_1(t) = s_2(t) - a_{n-1} s_1(t) + b_{n-1} x(t)$.

$\dot{s}_2(t) = s_3(t) - a_{n-2} s_1(t) + b_{n-2} x(t)$.

$\vdots$

$\dot{s}_{n-1}(t) = s_n(t) - a_1 s_1(t) + b_1 x(t)$.

$\dot{s}_n(t) = - a_0 s_1(t) + b_0 x(t)$

$$\dot{s}(t) = \begin{pmatrix} -a_{N-1} & 1 & 0 & \cdots & 0 \\ -a_{N-2} & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_1 & \vdots & & & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{pmatrix} s(t) + \begin{pmatrix} b_{N-1} \\ b_{N-2} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} x(t).$$

Stability of CT LTI.

1- Absolute Stability:

def: an LTI sys with IR: $h(t)$. is abs iff:

$$\left\{ \begin{array}{l} * \; h(t) \xrightarrow[t \to \infty]{} 0 \\ * \; \dot{h}(t) \xrightarrow[t \to \infty]{} 0 \\ \vdots \\ * \; h^{(n)}(t) \xrightarrow[t \to \infty]{} 0 \end{array} \right.$$

(=) output goes to zero for no input & any initial condition.

2- BIBO Stability:

def for any $x(t)$ s.t. $\exists M_x$

$|x(t)| < M_x \quad \forall t$

then $\exists M_y$ s.t. $|y(t)| \leq M_y \quad \forall t$.

BIBO Stability $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| \, d\tau < \infty$

Proof $\rightarrow$

Papillon

$\boxed{\Leftarrow}$ let $x(t)$ be s.t. $|x(t)| \leq M_x$.

$$|y(t)| = \int |h(\tau)\, x(t-\tau)|\, d\tau$$

$$\leq \int |h(\tau)|\, |x(t-\tau)|\, d\tau$$

$$\leq M_x \underbrace{\int |h(\tau)|\, d\tau}_{<\infty}$$

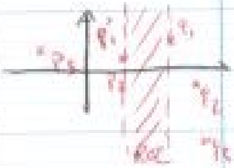
$\boxed{\Rightarrow}$ Assume $\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau = \infty$

then $x(t) = \dfrac{h^*(-t)}{|h(-t)|}$ bounded by $|x(t)| \leq 1 \ \forall t$.

$$y(0) = \int h(\tau)\, x(0-\tau)\, d\tau$$

$$= \int \frac{|h(\tau)|^2}{|h(\tau)|}\, d\tau = \int |h(\tau)| = \infty$$

$\equiv$ Eq. IN Laplace Domain:



$H(s)$ rational $= \dfrac{P(s)}{Q(s)}$ ROC given.

1- Long division $H(s) = \text{poly}(s) + \dfrac{P(s)}{Q(s)} \rightarrow$ proper fn.

2- partial fraction expansion: $H(s) = \sum \dfrac{A_l}{s-p_l} + \sum \dfrac{B_k}{s-q_k}$.

$\rightarrow h(t) = \sum A_l e^{p_l t} u(t) + \sum B_k e^{q_k t} u(-t)$

$+$ higher orders: $\sum A_l P(t) e^{p_l t} u(t) + \sum B_k Q(t) e^{q_k t} u(-t)$

$t > 0$ p's affect limit at ∞

$t < 0$ q's affect limit at $-\infty$

Limit at $+\infty$: $|\text{poly}(t) e^{p_l t} u(t)| \xrightarrow[t\to\infty]{} 0$ $\rightarrow \text{Re}\{p_l\} < 0$.

Limit at $-\infty$: $|\text{poly}(t) e^{q_k t} u(-t)| \xrightarrow[t\to-\infty]{} 0$ $\rightarrow \text{Re}\{q_k\} > 0$.

$\rightarrow$ $j\omega$-axis $\subset$ ROC. "Rational Functions"

$\equiv$ absolute stability and BIBO stability:

N.B.// integ. of sum $\leq$ sum of integrals

since exponentials are converging $\rightarrow$ integral will converge.

Q: "causal System" ; $H(s)$ rational $= \frac{P(s)}{Q(s)}$. Is it stable?

i.e. $\Rightarrow$ are all poles in left half plane LHP?

i.e. $\Rightarrow$ are all roots of $Q(s)$ in LHP?

Routh-Hurwitz (RH) method:

Idea: ⊛ $as^2 + bs + c$

$$= a(s - p_1)(s - p_2)$$

$$= s^2 - (p_0 + p_1)s + p_0 p_1 \qquad \boxed{a = 1}$$

if $p_0 \,\&\, p_1 < 0 \xrightarrow[\text{(not } \Leftrightarrow)]{} b > 0 \,\&\, c > 0$. "works for complex"

⊛ $s^3 + bs^2 + cs + d$.

$$= (s - p_0)(s - p_1)(s - p_2)$$

$$= (s^2 - (p_0 + p_1)s + p_0 p_1)(s - p_2)$$

$$= s^3 - (p_0 + p_1 + p_2)s^2 + (p_0 p_2 + p_1 p_2 + p_0 p_1)s - p_0 p_1 p_2.$$

if all roots have -ve Re{part} $\xrightarrow[\text{(not } \Leftrightarrow)]{}$ all coefficient are positive.

$\Rightarrow$ if one of coeff. < 0 $\Rightarrow$ system not stable (Not all poles are in LHS)

Routh's Table

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

$\Rightarrow$ if $a_0 = 0$ $\Rightarrow$ system not stable

columns $= \lceil \frac{n}{2} \rceil$ $(n+1)$ Rows

s^n	a_n	a_{n-2}	a_{n-4}	---
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	---
s^{n-2}	b_1	b_2	b_3	---
s^{n-3}	c_1	c_2	c_3	---
⋮				
s_1				
1.				

// fill zeroes for unified row-length

Stable:
1st Column has **no**:
- zero entries
- sign changes.

where; $b_l = \dfrac{a_{n-1} a_{n-2l} - a_n a_{n-2l-1}}{a_{n-1}}$

$c_l = \dfrac{b_1 a_{n-2l-1} - a_{n-1} b_{l+1}}{b_1}$.

N.B// # sign changes = # poles ∉ LHP
zero entries = poles on jω-axis

Similarly each parameter depends on the previous two rows the first column of which & the upper right normalized by 1st element of 1st col.

Papillon

In 1^{st} Column:

observation $\Rightarrow$ $\begin{cases} \text{zero elements} \Rightarrow \text{poles on } j\omega\text{-axis.} \\ \#\text{ of sign changes} = \#\text{ of poles not in LHP.} \end{cases}$

Stable: no sign changes & no zero entries in the 1^{st} Column

eg// $as^3 + bs^2 + cs + d$

s^3	a	c
s^2	b	d
s	$\frac{bc-ad}{b}$	0
1	d	

" Frequency Domain: "

Fourier

"Frequency Response"

Start with CT LTI sys:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

LTI, REAL

eigen functions & values

$$e^{\alpha t} \longrightarrow \underbrace{H(\alpha)}_{\text{scalar}} e^{\alpha t} \quad (\alpha \in ROC)$$

$$\hookrightarrow e^{j\omega t} \longrightarrow H(j\omega)\, e^{j\omega t} \quad \omega \in \mathbb{R}$$

$$\cos(\omega t) \longrightarrow \frac{1}{2}\left[H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[H(j\omega)e^{j\omega t} + H^*(j\omega)e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[r e^{j\theta} e^{j\omega t} + r e^{-j\theta} e^{-j\omega t}\right]$$

$$= \frac{r}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

$$\cos(\omega t) \longrightarrow r \cos(\omega t + \theta)$$

cosine $\longrightarrow \boxed{LTI} \longrightarrow$ "cosine"

$$\cos(\omega t) \longrightarrow |H(j\omega)| \cos(\omega t + \angle H(j\omega))$$

↳ Bode Plots.

⇓

$H(j\omega)$ is of specific INTEREST.

Bode Plots:

causal, stable, & real sys.

→ study of: $|H(j\omega)|$ & $\angle H(j\omega)$.

$$H(s) = \kappa \frac{(s-z_0)(s-z_1)\cdots}{(s-p_0)(s-p_1)\cdots}$$

$$\rightarrow |H(j\omega)| = |\kappa| \cdot \frac{\prod |j\omega - z_i|}{\prod |j\omega - p_j|}$$

$$\rightarrow 20\log|H(j\omega)| = 20\log|\kappa| + \sum 20\log|j\omega - z_i| - \sum 20\log|j\omega - p_j| \ \text{dB}$$

$$\rightarrow \angle H(j\omega) = \angle\kappa + \sum \angle(j\omega - z_i) - \sum \angle(j\omega - p_j).$$

Bode Plots:

1) plot $20\log_{10}|H(j\omega)|$

fn. of $\log_{10}\omega$ $\quad \omega > 0$

2) plot of $\angle H(j\omega)$

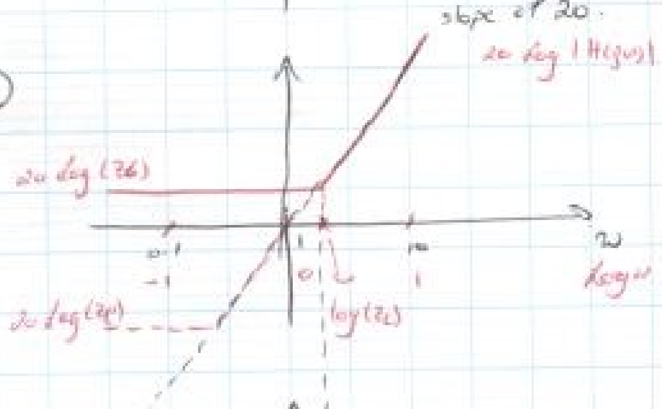
fn. of $\log_{10}\omega$ $\quad \omega > 0$.

Linear Approximations :

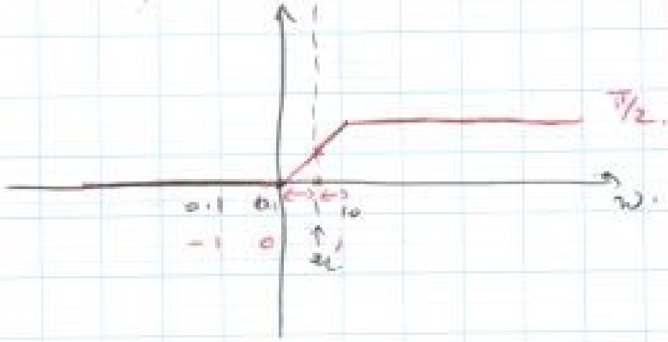
1- one "zero" $(j\omega - z_0)$

↳ Bode plot :

1

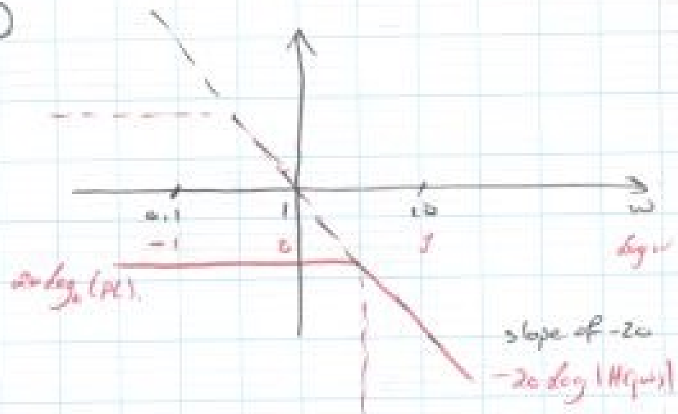


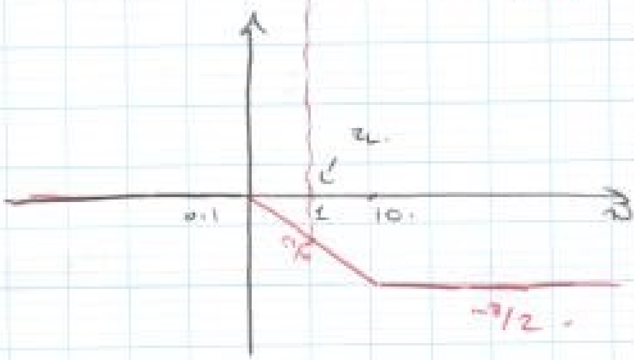
2



2- one "pole" $\frac{1}{j\omega - p_0}$

1





$x e^x$ x c^x

1 e^x

0 c^x

(c) Fourier Transform:

Let $x(t) : \mathbb{R} \to \mathbb{C}$

Its FT : $\mathbb{R} \to \mathbb{C}$.

$$\omega \longrightarrow X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt. \quad \text{"}X(\omega)\text{"}.$$

⚠ Not Unique Definition.

alternative def: $\omega \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.$

$$\omega \longrightarrow \int_{-\infty}^{+\infty} x(t) e^{-j 2\pi f t} dt.$$

N.B: $X(j\omega) = X(s) \big|_{s=j\omega}$ True if ROC of $X(s)$ includes the $j\omega$-axis.

N.B// There are cases where $X(s)$: $j\omega$-axis $\not\subset$ ROC.
so $X(s)|_{s=j\omega}$ $\nexists$ "doesn't exist"
However, we can still make sense of $X(j\omega)$.

→ FT is nothing but the $\mathcal{L}T$ on $j\omega$: special case of $\mathcal{L}T$!
→ FT is a "generalization" of the $\mathcal{L}T$.

Theorem It is one-to-one:

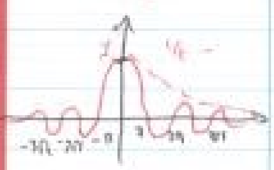
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.$$

IFT: $\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega}$ = inverse Fourier transform.

eg// ① $x(t) = \delta(t)$ $\quad X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1 \quad \forall \omega \in \mathbb{R}.$

$$\delta(t) \xrightarrow{FT} 1.$$

N.B: $\text{sinc}(t) = \frac{\sin t}{t}$.



② $x(t) = \text{rect}(t)$ $\quad X(j\omega) = \int_{-1/2}^{1/2} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-1/2}^{1/2} = \frac{\sin \omega/2}{\omega/2}.$

$$\text{rect}(t) \xrightarrow{FT} \text{sinc}\left(\frac{\omega}{2}\right).$$

$$\Rightarrow \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} d\omega$$

Properties of FT :

① **Linearity :** $a\,x(t) + b\,y(t) \xrightarrow{FT} a\,X(j\omega) + b\,Y(j\omega)$

② **Time Shifts :** $x(t-t_0) \xrightarrow{FT} e^{-j\omega t_0} X(j\omega)$.

Eg: $x(t) = \cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$ .... later.

③ **Mult by exp :** $x(t)\,e^{j\omega_0 t} \xrightarrow{FT} X(j(\omega-\omega_0))$.

④ **Time reverse :** $x(-t) \xrightarrow{FT} X(-j\omega)$.

pf: $\int_{-\infty}^{+\infty} x(-t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{+\infty} x(t')\,e^{j\omega t'}\,dt'$

⑤ **Conjugation :** $x^*(t) \xrightarrow{F.T} X^*(-j\omega)$

pf: $\int_{-\infty}^{+\infty} x^*(t)\,e^{-j\omega t}\,dt = \left[\int x(t)\,e^{j\omega t}\,dt\right]^*$

⑥ **Initial values :**
* $x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega$
* $X(j0) = \int_{-\infty}^{+\infty} x(t)\,dt$.

N.B
$x(t)$ even:
i.e $x^*(-t) = x(t)$.
$X^*(j\omega)$ $X(j\omega)$

⑦ **Property: even :**

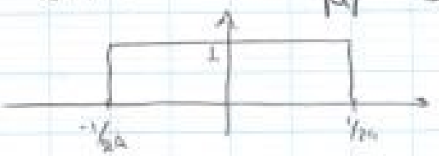
$x(t)$ is real $\rightarrow X(j\omega) = X^*(-j\omega)$: Even
hermitian Symmetric

$x(t)$ is even $\rightarrow X(j\omega)$ is Real.

$x(t)$ real & even $\rightarrow X(j\omega)$ real & even. $X(j\omega) = X(-j\omega)$.

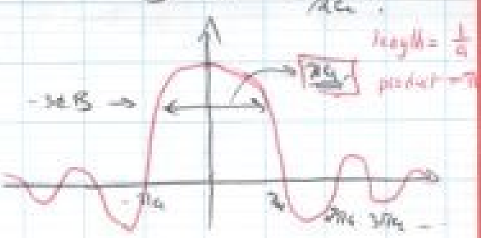
⑧ **Time Scaling :** $x(at) \xrightarrow{F.T} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$] uncertainty Principle

eg// rect(at)



$$\int_{-\infty}^{+\infty} e^{-j\omega t}\,dt = \int_{-1/2a}^{1/2a} e^{-j\omega t}\,dt = \frac{1}{j\omega}\left[e^{j\omega/2a} - e^{-j\omega/2a}\right] = \frac{1}{a}\,\frac{\sin(\omega/2a)}{\omega/2a}$$

$= \underbrace{\frac{1}{a}}_{\text{width of rectangle}} \underbrace{\sin\left(\frac{\omega}{2a}\right)}$ $\hookrightarrow$ sinc (half width)



Observation * ~~Shift~~ i. Stretching in one domain → compression in the other.
* width (time) · width (freq) ≈ π.

9 Derivation: $\frac{dx(t)}{dt} \xrightarrow{FT} j\omega X(j\omega)$

10 Mult. by t: $t\,x(t) \xrightarrow{F.T.} \frac{1}{j\omega} d \frac{X(j\omega)}{d\omega}$

11 Property: $g(t) \xrightarrow{FT} F(\omega)$

$F(t) \xrightarrow{FT} 2\pi\, g(-j\omega)$

pf// $\int_{-\infty}^{\infty} F(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(\tau) e^{-jt\tau} d\tau \right) e^{-j\omega t} dt$

$= \iint g(\tau) e^{-j\omega t - jt\tau}\, dt\, d\tau$

$= \int g(\tau) \left(\int_{-\infty}^{\infty} e^{-j(\omega+\tau)t} dt \right) d\tau$

$= 2\pi f(-\omega)$

ex// ~~x cos ω₀t~~ $= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \xrightarrow{FT} \frac{1}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

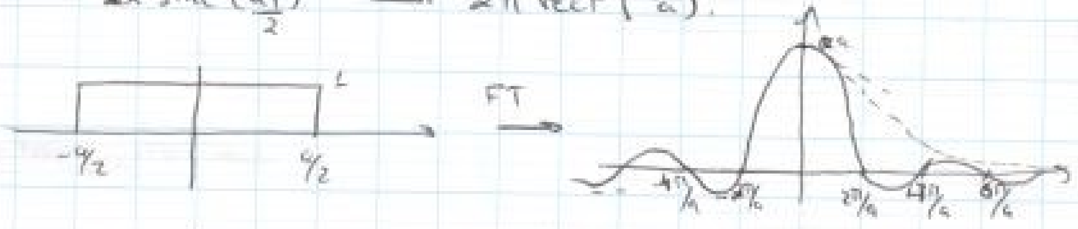
~~x rect(at) → a sinc(at/2)~~

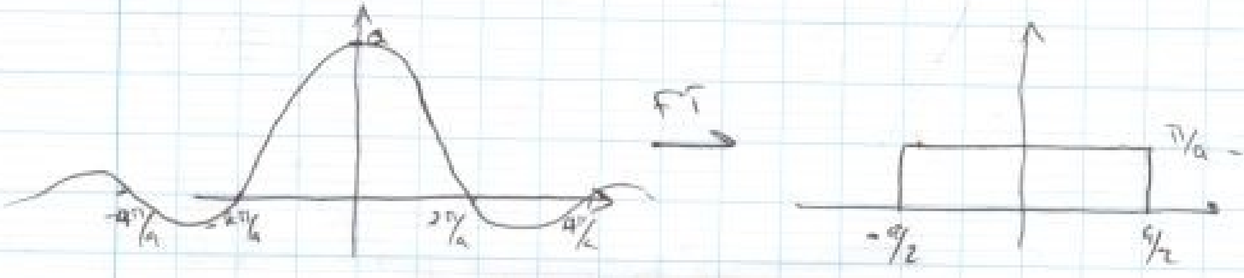
→ $\underbrace{\text{rect}(t/a)}_{\text{width}} \longrightarrow \underbrace{a}_{\text{width}}\ \text{sinc}\left(\frac{a\omega}{2}\right)$ — $\frac{1}{a/2}$ width

eg(1) $\sin(at)$?? $\quad \text{sinc}(at) \xrightarrow{FT} \frac{\pi}{a}\,\text{rect}\left(\frac{\omega}{a}\right)$

$\text{rect}(t/a) \xrightarrow{FT} a\,\text{sinc}\left(\frac{a\omega}{2}\right)$

$a\,\text{sinc}\left(\frac{at}{2}\right) \longrightarrow 2\pi\,\text{rect}\left(\frac{\omega}{a}\right)$



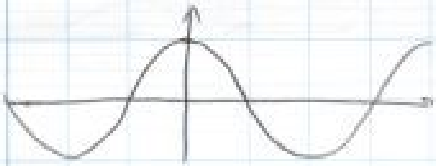


uncertaintily: width (t) × width (ω) ≃ 2π eg/ $\delta(t) \longrightarrow 1$.

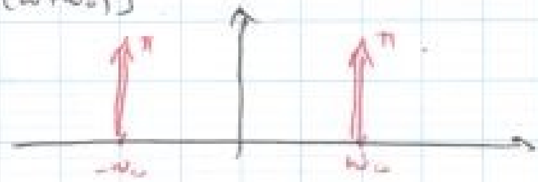
$1 \longrightarrow 2\pi\,\delta(-j\omega) = 2\pi\,\delta(j\omega)$.

$\rightarrow \cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$

$= \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$



$\xrightarrow{F.T}$



$\boxed{12}$ Convolution: $(x*y)(t) \longrightarrow X(j\omega)\cdot Y(j\omega)$.

$\boxed{13}$ Product: $x(t)\cdot y(t) \longrightarrow \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$

$\longrightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\,Y(j(\omega-\theta))\,d\theta$.

$x(t) * y(t) \xrightarrow{FT} X(j\omega)\,Y(j\omega) \xrightarrow{FT} 2\pi\,(x*y)(-t)$.

$x(t) \longrightarrow X(j\omega) \longrightarrow 2\pi x(-j\omega)$

$y(t) \longrightarrow Y(j\omega) \longrightarrow 2\pi\, y(-j\omega)$.

$X(j\omega)Y(j\omega) \longrightarrow 2\pi\,(x*y)(-j\omega)$

$2\pi\int x(-j\tau)\, y(j(-\omega-\tau))\,d\tau$ -

$2\pi\int x(-j\theta)\, y(j(-\omega+\theta))\,d\theta$

$2\pi\int_{-\infty}^{+\infty} x(-j\theta)\, y(-j(\omega-\theta))\,d\theta$

$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\cdot Y(j(\omega-\theta))\,d\theta$.

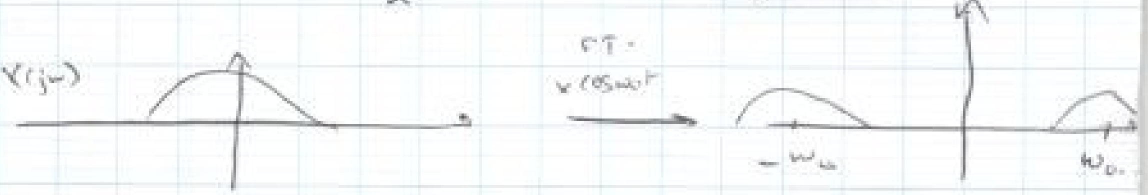
$= \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$

$x_e = \frac{1}{2}(x(t) + x(-t))$

eg// $x(t) \longrightarrow X(j\omega)$

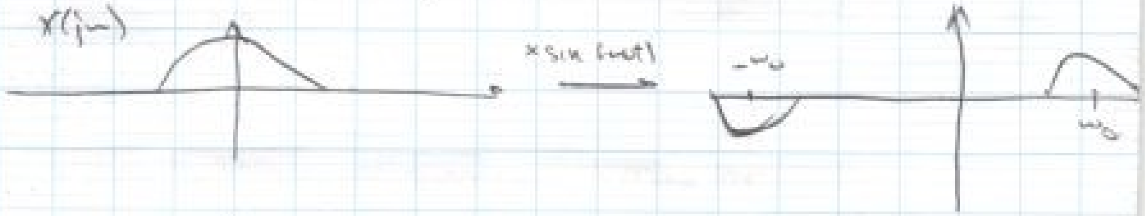
* $x(t)\cos\omega_0 t \longrightarrow \frac{1}{2\pi}\left(X(j\omega) * \left[\pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)\right]\right)$

$$= \frac{1}{2}\quad X(j(\omega-\omega_0)) + X(j(\omega+\omega_0)).$$



* $x(t)\sin(\omega_0 t) \longrightarrow \frac{1}{2\pi}\left(X(j\omega) * \left[\frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)\right]\right)$

$$= \frac{1}{2j}\left[X(j(\omega-\omega_0)) - X(j(\omega+\omega_0))\right]$$



eg//

rect(t) * rect(t)

~~Usage~~ generated signals are bandlimited.

***

eg// $\mathcal{F}\{u(t)\} = ??$ $\quad u(t) \xrightarrow{F.T} \frac{1}{j\omega} + \pi\delta(\omega)$

why this procedure?? not regular ↳ laplace tran. evaluated at $j\omega$ // since $(j\omega)$ is not in ROC of $\mathcal{L}\{u(t)\}$ as it has a pole at zero ($\frac{1}{s}$) so can't evaluate $\mathcal{F}\{u(t)\}$ using the integral of Laplace.

$u(t) \xrightarrow{F.T} U(j\omega) = U_e(j\omega) + U_o(j\omega)$

1) $u(t) + u(-t) = 1$ (almost everywhere at $t=0 \to 2$)

$U(j\omega) + U(-j\omega) = 2\pi\delta(\omega).$

$= U_e(j\omega) + U_e(-j\omega) = 2\pi\delta(\omega).$

$\to U_e(j\omega) = \pi\delta(\omega)$

2) $u'(t) = \delta(t).$

$j\omega U(j\omega) = 1 \quad \to j\omega(U_e(j\omega) + U_o(j\omega)) = 1.$

$\to j\omega(\pi\delta(\omega) + U_o(j\omega)) = 1.$

$\to U_o = \frac{1}{j\omega}.$

Eg $\mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\} \longrightarrow \frac{X(j\omega)}{j\omega} + \pi X(j0)\, \delta(\omega).$

pf $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = (x * u)(t).$

$Y(j\omega) = X(j\omega) \cdot U(j\omega).$

G $\mathcal{F}\left\{ t\, x(t) \right\} \longrightarrow -\frac{1}{j} \frac{d}{d\omega}(j\omega).$

IMP !

t		f
real	$\rightarrow$	even
even	$\rightarrow$	real

even: $x(t) = x^*(-t)$.

periodic $\rightarrow$ impulsive

impulsive $\rightarrow$ periodic.

"width" $\longleftrightarrow$ " $\frac{1}{\text{width}}$ "

width go in opp. variation : stretch $\rightarrow$ shrink.

shrink $\rightarrow$ stretch.

NB

periodic — period "T"

$\rightarrow$ FT : "impulsive"

↕ relation ??

$\rightarrow$ Fourier Series $\sum_{-\infty}^{+\infty} C_n e^{j\frac{2\pi}{T}nt}.$

$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt.$

Papillon

Discrete Time Fourier Transform

Discrete Time Fourier trans

DTFT defined for DT signals.

eigen fns & values:

idea: $x[n] \to \boxed{DT\,LTI}_{h[n]} \to y[n]$

$z^n \to H(z)\, z^n$

$e^{j\omega n} \to H(e^{j\omega})\, e^{j\omega n}$

$\cos \omega n = \frac{1}{2}[e^{j\omega n} + e^{-j\omega n}] \to \boxed{\ } \to \frac{1}{2}[H(e^{j\omega})e^{j\omega n} + H(e^{-j\omega})e^{-j\omega n}]$

$= |H(e^{j\omega})| \cos(\omega n + \angle H(e^{j\omega}))$

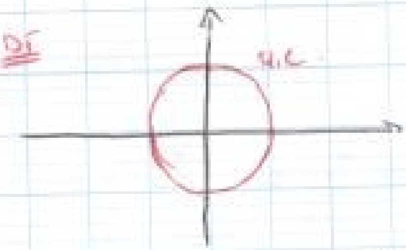
*** ROC of Laplace trans contains $j\omega$-axis → for CT.

for DT: ROC of Z-trans contains unit circle

Def $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n}$.

Note $X(e^{j\omega}) = X(z)\Big|_{z \leftarrow e^{j\omega}}$ if UC $\subset$ ROC

DT



CT

$X(z)\Big|_{z \leftarrow e^{j\omega}}$ = DTFT (at u.c) $X(s)\Big|_{s \leftarrow j\omega}$ = FT (at $j\omega$

u.c $\subset$ ROC "for Z-trans" $j\omega$-axis $\subset$ ROC "for L-trans"

Inv $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$.

Unique for DTFT. "Not in FT"

↳ It is Periodic! "period 2π".

↳ integration over one period.

~~Note max oscillation in DT~~

Note $X(e^{j\omega})$; freq cts in DT - <u>max freq =</u> with step 1 ~~max st~~

max period = 2.

max freq $= \frac{1}{2}$ Hz

$= \frac{1}{2}\, 2\pi$ r/s

$= \underline{\pi}$.

so the freq Domain.

period 2π

Final Note on (CT) Fourier Trans:

Parseval
$$\int_{-\infty}^{+\infty} f(t)\, g^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\, G^*(j\omega)\, d\omega$$

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(j\omega)|^2\, d\omega .$$

Pf

$f(t) \longrightarrow F(j\omega)$

$g(t) \longrightarrow G(j\omega)$

$g^*(t) \longrightarrow G^*(-j\omega)$.

$$f(t)\, g^*(t) \longrightarrow \frac{1}{2\pi} F(j\omega) * G^*(-j\omega).$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\nu)\, G^*(j(\nu-\omega))\, d\nu$$

$$= \frac{1}{2\pi}\int F(j\nu)\, G^*(j(\nu-\omega))\, d\nu$$

$$\Rightarrow \int f(t)\, g^*(t)\, dt = FT(0).$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\nu)\, G^*(j\nu)\, d\nu$$

DTFT

def $X(e^{j\Omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\Omega n}$.

Prop $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$.

!

$\Rightarrow$ If $X(z)$ s.t UC CROC $\Rightarrow X(e^{j\Omega}) = X(z)\Big|_{z = e^{j\Omega}}$.

UC CROC : Take C = U.C.

$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$

on C : $z = e^{j\Omega}$. $\Omega = -\pi \longrightarrow \pi$

$X(z) = X(e^{j\Omega})$

$z^{n-1} = e^{j\Omega(n-1)}$

$\frac{dz}{d\Omega} = je^{j\Omega}$; $dz = je^{j\Omega} d\Omega$.

$\Rightarrow x[n] = \frac{1}{2\pi j} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega(n-1)} \cdot j e^{j\Omega} d\Omega$.

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$.

NOTE $X(e^{j\Omega})$ periodic $T = 2\pi$.

Real $\Leftrightarrow$ Even
Even $\Leftrightarrow$ Real
periodic $\Leftrightarrow$ Impulsive
Impulsive $\Leftrightarrow$ periodic
"width" ~ "$\frac{1}{\text{width}}$"

Pairs $\delta[n] \longrightarrow 1.$

$1 \longrightarrow 2\pi \sum_{k\in\mathbb{Z}} \delta(\Omega - 2k\pi)$ // $2\pi\delta[n]$ $[-\pi,\pi)$ periodic

Properties

① Linear $\quad a x[n] + b y[n] \longrightarrow a X(e^{j\Omega}) + b Y(e^{j\Omega})$

② Time Reversal $\quad x[-n] \longrightarrow X(e^{-j\Omega})$

③ Conjugation $\quad x^*[n] \longrightarrow X^*(e^{-j\Omega})$

$\Rightarrow$ $x[n]$ real $\rightarrow X(e^{j\Omega}) = X^*(e^{-j\Omega})$: "even" // hermitian Symm

$\Rightarrow$ $x[n]$ imaginary $\rightarrow X(e^{j\Omega}) = -X^*(e^{-j\Omega})$: "odd" // hermitian anti-Sym

$\Rightarrow$ $x[n]$ even $x[-n] = x^*[n] \rightarrow X(e^{j\Omega})$ is real.

③ Time Shifts $\quad x[n-n_0] \Rightarrow e^{-j\Omega n_0} X(e^{j\Omega})$

④ $e^{j\Omega_0 n}$ mult. $\quad e^{j\Omega_0 n} x[n] \longrightarrow X(e^{j(\Omega-\Omega_0)})$

⑤ $n\,x[n]$ $\quad n\,x[n] \longrightarrow -\frac{1}{j}\frac{d X(e^{j\Omega})}{d\Omega}.$

⑥ Convolution $\quad (x*y)[n] \longrightarrow X(e^{j\Omega}) \cdot Y(e^{j\Omega}).$

Duality $\times$.

not convolution over $-\infty$ to $+\infty$

⑦ Product $\quad x[n]\cdot y[n] \longrightarrow \frac{1}{2\pi}\int_{\boxed{-\pi}}^{\boxed{\pi}} X(e^{j\vartheta})\, Y(e^{j(\Omega-\vartheta)})\, d\vartheta.$

Pf// let $V(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\vartheta})\, Y(e^{j(\Omega-\vartheta)})\, d\vartheta.$

$\Rightarrow v[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{j\Omega})\, e^{j\Omega n}\, d\Omega.$ $\quad e^{j(\Omega-\vartheta)n}\cdot e^{j\vartheta n}$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\vartheta}) Y(e^{j(\Omega-\vartheta)})\, d\vartheta \;\; e^{j\Omega n}\, d\Omega.$

$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\vartheta})\, e^{j\vartheta n}}_{x[n]} \underbrace{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j(\Omega-\vartheta)})\, e^{j(\Omega-\vartheta)n}\, d\Omega\right]}_{y[n]} d\vartheta$

$= x[n]\cdot y[n].$

⑧ Initial & Final Values

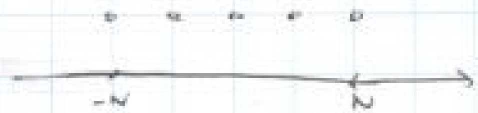
* $x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega.$

* $X(e^{j0}) = \sum_{n\in\mathbb{Z}} x[n]$

⑨ Parseval $\quad \sum_{n\in\mathbb{N}} x[n]\, y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, Y^*(e^{j\Omega})\, d\Omega.$

$\hookrightarrow \sum_{n\in\mathbb{N}} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega.$

Eg//



$$x[n] = \sum_{-N}^{N} \delta[n-k]$$

$$X(e^{j\Omega}) = \sum_{-\infty}^{+\infty} x[n]\, e^{-jn\Omega} = \sum_{-N}^{N} e^{-j\Omega n}$$

$$= e^{j\Omega N}\, \frac{1 - e^{-j\Omega(2N+1)}}{1 - e^{-j\Omega}}$$

$$= e^{j\Omega N}\, \frac{e^{-j\Omega\left(\frac{2N+1}{2}\right)}\left[e^{j\Omega\frac{2N+1}{2}} - e^{-j\Omega\frac{2N+1}{2}}\right]}{e^{-j\Omega/2}\left[e^{j\Omega/2} - e^{-j\Omega/2}\right]}$$

$$= \left[e^{-j\Omega\left[-\frac{2N+1}{2} + \frac{1}{2} + N\right]}\right] \cdot \left[\frac{\sin \Omega\left(\frac{2N+1}{2}\right)}{\sin(\Omega/2)}\right]$$

pairs

$$\delta[n] \xrightarrow{DTFT} 1$$

$$\delta[n-n_0] \xrightarrow{DTFT} e^{-j\Omega n_0}$$

$$\sum_{-N}^{N} \delta[n-k] \xrightarrow{DTFT} \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin(\Omega/2)}.$$

period "T" $\quad$ special: $\quad \sum \delta[t-nT] \xrightarrow{\boxed{F.T}} \frac{2\pi}{T}\sum_{n\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi}{T}n\right)$

period "N" $\quad$ special: $\quad \sum \delta[t-kN] \xrightarrow{\boxed{DTFT}} \frac{2\pi}{N}\sum_{-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{N}k\right).$

Fourier	CT	DT
$\boxed{FT}$	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$ $x(t) = \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$	$\boxed{DTFT}$ $X(e^{j\Omega}) = \sum_{-\infty}^{+\infty} x[n]\, e^{-j\Omega n}$ $x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$
$\boxed{\text{Fourier Series}}$	$x(t) = \frac{1}{2\pi}\sum C_n\, e^{j\frac{2\pi}{T}nt}$ $C_n = \frac{2\pi}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt$	$x[n] = \frac{1}{2\pi}\sum_{0}^{N-1} C_k\, e^{j(\frac{2\pi}{N}k)n}$ $C_k = \frac{2\pi}{N}\sum_{0}^{N-1} x[n]\, e^{-j(\frac{2\pi}{N}k)n}$

have to be periodic $\Rightarrow$

FT $\xleftrightarrow{??}$ Fourier Series

Def Dirac "Comb"

" $\sum_{k\in\mathbb{Z}} \delta(t-kT)$ "

Fourier Series Decomposition: period: T.

$$x(t) = \frac{1}{2\pi}\sum_{-\infty}^{+\infty} C_n\, e^{j\frac{2\pi}{T}nt}$$

$$C_n = \frac{2\pi}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt.$$

$$\Rightarrow C_n = \frac{2\pi}{T}\int_{-T/2}^{T/2} \sum_{k\in\mathbb{Z}} \delta(t-kT)\, e^{-j\frac{2\pi}{T}nt}\, dt.$$

$$= \frac{2\pi}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-j\frac{2\pi}{T}nt}\, dt = \cancel{1}\, \frac{2\pi}{T}$$

$$\Rightarrow x(t) = \frac{1}{T}\sum_{-\infty}^{+\infty} e^{j\frac{2\pi}{T}nt}$$: Fourier Series.

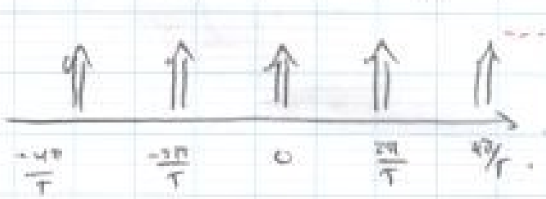
F.T

$$X(j\omega) = \frac{2\pi}{T}\sum_{-\infty}^{+\infty} \delta\left(j\left(\omega - \frac{2\pi}{T}n\right)\right)$$

$x(t) = \text{"} \sum_{k \in \mathbb{Z}} \delta(t - kT) \text{"}$

FT ↳

$X(j\omega)$.



Prop

Let $x(t)$ be periodic (T)

procedure ⇒ ① $x_T(t) = x(t)$ over $[-T/2, T/2)$

$$x(t) = x_T(t) + x_T(t-T) + x_T(t-2T) + \cdots + x_T(t+T) + x_T(t+2T) + \cdots$$

$$= x_T(t) * \left[\sum \delta(t - kT)\right].$$

$x(t) = x_T(t) * \left(\sum \delta(t - kT)\right)$

$$x_T(t) = \begin{cases} x(t) & [-T/2, T/2) \\ 0 & \text{e.w.} \end{cases} \xrightarrow{\text{F.T.}} X_T(j\omega).$$

⇒ F.T ② $x(t) \xrightarrow{\text{F.T}} X_T(j\omega) \cdot \sum \left[\frac{2\pi}{T} \delta\left(j\left(\omega - \frac{2\pi}{T} n\right)\right)\right]$

$$= \frac{2\pi}{T} \sum_{-\infty}^{+\infty} \boxed{X_T\left(j \frac{2\pi}{T} n\right)} \delta\left(\omega - \frac{2\pi}{T} n\right).$$

Fourier Series

$$x(t) = \frac{1}{2\pi} \sum_{-\infty}^{+\infty} c_n e^{j \frac{2\pi}{T} n t}.$$

$$↳ X(j\omega) = \frac{1}{2\pi} \sum_{-\infty}^{+\infty} \boxed{c_n} \delta\left(\omega - \frac{2\pi}{T} n\right).$$

Fourier Series Coeff

$$\boxed{c_n = \frac{(2\pi)^2}{T} X_T\left(j \frac{2\pi}{T} n\right)}$$

9// $x(t)$ periodic $\quad x_T(t) = \mathrm{rect}(t)$.

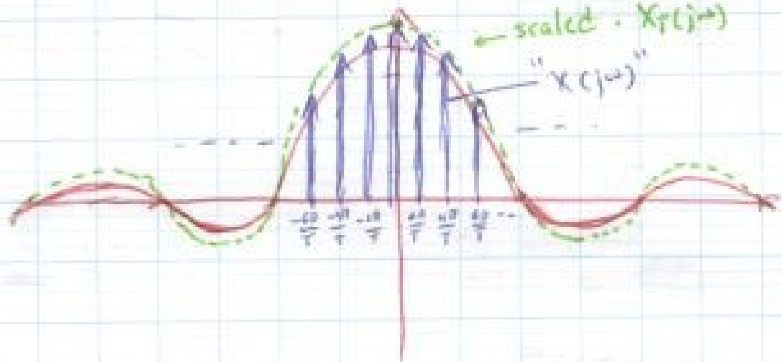
① Look at $x_T(t)$ $\quad$: $x_T(t) = \mathrm{rect}(t)$

② Find $X_T(j\omega)$ $\quad$: $X_T(j\omega) = \mathrm{sinc}(\omega/2)$.

$X_T(0) \times \frac{2\pi}{T} = \frac{C_0}{2\pi}$ $\quad \ldots \quad$ $X_T(\frac{2\pi}{T}) \times \frac{2\pi}{T} = \frac{C_1}{2\pi}$ $\quad \ldots$

$\Rightarrow$ Fourier Series Coeff $\left(\frac{C_n}{2\pi}\right)$ are Fourier transform of 1 period

Sampled @ $\frac{2\pi}{T}$. & scaled by $\frac{2\pi}{T}$.



$x_T(t)$ was impulsive. $\Rightarrow X_T(j\omega)$ is periodic

$x(t)$ was periodic $\Rightarrow X(j\omega)$ is impulsive

Conclusion

① $X(j\omega)$ is impulsive

② scaled (by $\frac{2\pi}{T}$) "sampled" Fourier Transform of 1 window (period)

↳ "Repetition" $\xrightarrow{F.T}$ "Sampling" in F. domain.

$\Rightarrow$ Similar of $X(j\omega)$ = F.T "Fourier Series" or F.T. of $x(t) = \sum C_n e^{j\frac{2\pi}{T}nt}$

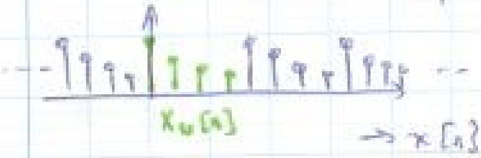
$$\mathcal{F}\{\sum C_n e^{j\frac{2\pi}{T}nt}\} = 2\pi \sum_{-\infty}^{+\infty} C_n \,\delta(\omega - \tfrac{2\pi}{T}n)$$

↳ $$\boxed{C_n = \frac{1}{T} X_T\left(j\frac{2\pi}{T}n\right)}$$

Relations in (DT) $x[n]$ is periodic "N"

① Define $x_N[n] = x[n]$ over one period

$$= \begin{cases} x[n] & n \in \{0, N-1\} \\ 0 & \text{o.w.} \end{cases}$$



$$\Rightarrow x[n] = x_N[n] + x_N[n-N] + x_N[n-2N] + \cdots + x_N[n+N] + x_N[n-2N] + \cdots$$

$$\Rightarrow x[n] = x_N[n] * \left[\sum_{-\infty}^{+\infty} \delta[n - kN] \right]$$

② DT.F.T $X(e^{j\Omega}) = ??$

N.B $\overset{DTFT}{\mathcal{F}} \left\{ \sum_{-\infty}^{+\infty} \delta[n-kN] \right\} = \mathcal{F} \left\{ \frac{1}{N} \sum_{0}^{N-1} e^{j\frac{2\pi}{N}kn} \right\}$

$$= \frac{2\pi}{N} \sum_{0}^{N-1} \delta\left(\Omega - \frac{2\pi}{N}k\right)$$

$$\Rightarrow \overset{DTFT}{\mathcal{F}} \{x[n]\} = X(e^{j\Omega}) = X_N(e^{j\Omega}) \cdot \left[\frac{2\pi}{N} \sum_{-\infty}^{+\infty} \delta\left(\Omega - \frac{2\pi}{N}k\right) \right]$$

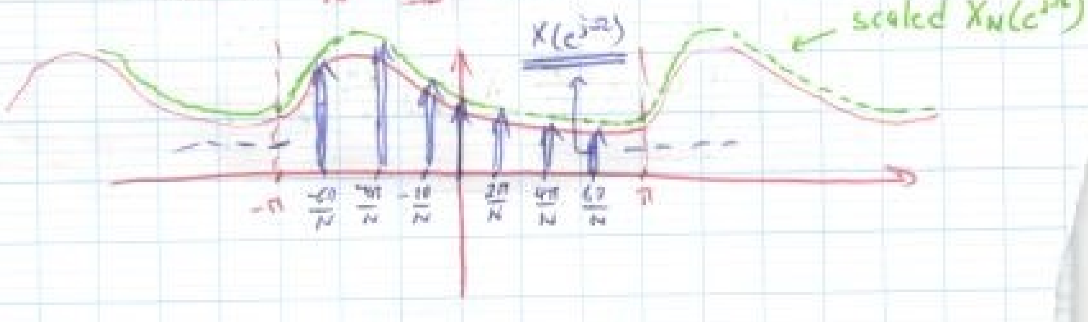
$$= \frac{2\pi}{N} \sum_{-\infty}^{+\infty} X_N(e^{j\Omega}) \, \delta\left(\Omega - \frac{2\pi}{N}k\right)$$

$$= \frac{2\pi}{N} \sum_{-\infty}^{+\infty} X_N\left(e^{j\frac{2\pi}{N}k}\right) \delta\left(\Omega - \frac{2\pi}{N}k\right).$$

$$\Rightarrow X(e^{j\Omega}) = \frac{2\pi}{N} \sum_{-\infty}^{+\infty} X_N\left(e^{j\frac{2\pi}{N}k}\right) \delta\left(\Omega - \frac{2\pi}{N}k\right).$$

$x_N[n]$ is impulsive $\Rightarrow X_N(e^{j\Omega})$ is periodic

$x[n]$ was periodic $\Rightarrow X_N(e^{j\Omega})$ is impulsive



Conclusion ① $X(e^{j\Omega})$ is impulsive.

② scaled (by $\frac{2\pi}{N}$) "sampled" Fourier trans. of x

$$\Rightarrow \quad x[n] = \sum_{0}^{N-1} C_k e^{j\frac{2\pi}{N}kn}.$$

DTFT ↳ $$X(e^{j\omega}) = 2\pi \sum_{0}^{N-1} C_k \, \delta(\omega - \frac{2\pi}{N}k).$$

$$\Rightarrow \boxed{C_n = \frac{1}{N} X_N(e^{j\frac{2\pi}{N}n})}$$

Final Comment on periodic Signals:

→ LTI processing of periodic signals $\quad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ (LTI)

period "T" $\longrightarrow \boxed{} \longrightarrow$ period "T"

FT → $X(j\omega)$ $\qquad Y(j\omega) = X(j\omega) H(j\omega)$.

Fourier Series → $\sum_{n\in \mathbb{Z}} C_n e^{j\frac{2\pi}{T}nt}$ $\quad \frac{??}{??} \quad$ $y(t) = \sum_{n\in \mathbb{Z}} b_n e^{j\frac{2\pi}{T}nt} = \sum_{n\in \mathbb{Z}} C_n H(j\frac{2\pi}{T}n) e^{j\frac{2\pi}{T}nt}$

N.B // Given $e^{j\frac{2\pi}{T}nt} \longrightarrow \boxed{} \longrightarrow H(e^{j\frac{2\pi}{T}n}) e^{j\frac{2\pi}{T}nt}$

⇒ ✱ CT — ?? — DT

(CT) $\qquad$ (DT)

object $\quad x(t) \qquad y[n]$

Sampling

$x(t) \longrightarrow \times \longrightarrow y[n]$ ... "$x(0)$" "$x(T_s)$" "$x(2T_s)$" ---

T_s $\qquad$ --- $y[0]$ $\quad y[1]$ $\quad y[2]$ ---

↓ FT ... $\qquad$ ↓ DTFT

$$a(t) = \sum_{n\in \mathbb{Z}} x(nT_s)\, \delta(t - nT_s) \quad : \text{auxiliary signal.}$$

$\curvearrowright = \sum_{\mathbb{Z}} x(t)\,\delta(t-nT_s)$

$a(t) = \sum_{\mathbb{Z}} x(nT_s) \sum \delta(t-nT_s)$

① FT $\quad a(t) = \sum_{\mathbb{Z}} x(t)\,\delta(t-nT_s)$

$a(t) = x(t) \cdot \text{comb}(t)$

$\rightarrow A(j\omega) = \frac{1}{2\pi} X(j\omega) * \text{comb}(j\omega)$

$A(j\omega) = \frac{1}{T_s} \sum_{n\in\mathbb{Z}} X\left(j\left(\omega - \frac{2\pi}{T_s}n\right)\right)$

② $\quad y[n] = x(nT_s)$

$a(t) = \sum y[n]\,\delta(t-nT_s)$

$A(j\omega) = \sum_{n\in\mathbb{Z}} y(n)\, e^{-j(\omega T_s)n}$

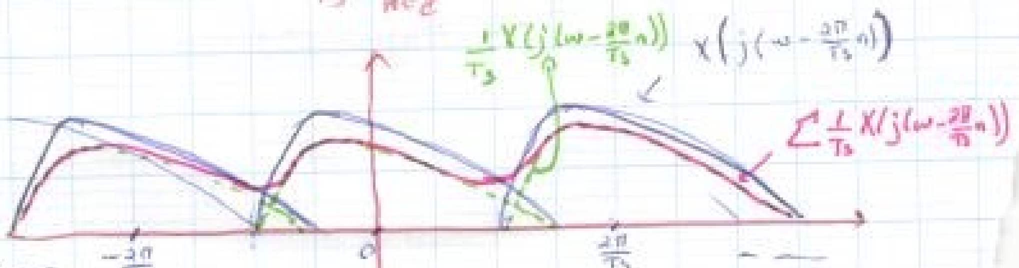
$= \sum y[n]\, e^{-j\Omega n} = Y(e^{j\omega T_s})$ DTFT of $y[n]$

$\therefore \quad x(t) \underset{T_s}{\times} y[n]$

$* \; y[n] = x(nT_s)$

$Y(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{\mathbb{Z}} X\left(j\left(\omega - \frac{2\pi}{T_s}n\right)\right)$

$\Longrightarrow Y(e^{j\Omega}) = \frac{1}{T_s} \sum_{n\in\mathbb{Z}} X\left(j\left(\frac{\Omega - 2\pi n}{T_s}\right)\right) \qquad \Omega = \omega T_s$



$A(j\omega) \longrightarrow \dots \quad -\frac{2\pi}{T_s} \quad 0 \quad \frac{2\pi}{T_s} \quad \dots$

$Y(e^{j\Omega}) \longrightarrow \dots \quad -2\pi \quad 0 \quad 2\pi \quad \dots$

$A(j\omega) = \frac{1}{T_s}\left(\dots + X\left(j\left(\omega + \frac{2\pi}{T_s}\right)\right) + X(j\omega) + X(j(\omega - \dots\right.$

$Y(e^{j\Omega}) = \frac{1}{T_s}\left(\dots + X\left(j\left(\frac{\Omega + 2\pi}{T_s}\right)\right) + X(j\omega) + X(j(\dots\right.$

$\Rightarrow$ ① copies shifted by $2\pi/T_s$.
② Scale by $1/T_s$ & Sum
③ Change label $\frac{2\pi}{T_s} \longleftrightarrow 2\pi$.
$w \qquad \Omega := wT_s$.

$\boxed{Q}$: Going Back ??.

$x(t) \overset{?}{\longleftrightarrow} y[n]$.

$\Updownarrow$

$X(jw) \overset{?}{\longleftrightarrow} Y(e^{jwT_s})$ } ⊕ Sum
$\Omega = \frac{w}{T_s}$

related to the overlap b/w shifted $X(j(w - \frac{2\pi}{T_s}n))$ ... "T_s"

Sampling

$x(t)$ ——— $\underset{T}{\times}$ ——— $y(t) = x(nT_s)$

$\Updownarrow$ FT $\qquad$ $\Updownarrow$ DTFT

$X(jw) \qquad Y(e^{j\Omega}) = \frac{1}{T_s} \sum X\left(j\left(\frac{\Omega}{T_s} - \frac{2\pi n}{T_s}\right)\right)$

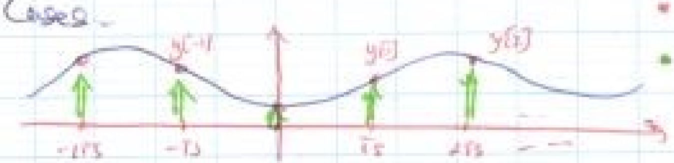
Aux signal $\qquad a(t) = \sum_{\mathbb{Z}} x(nT_s)\,\delta(t - nT_s)$

$$A(jw) = \frac{1}{T_s} \sum_{\mathbb{Z}} X\left(j\left(w - \frac{2\pi n}{T_s}\right)\right)$$

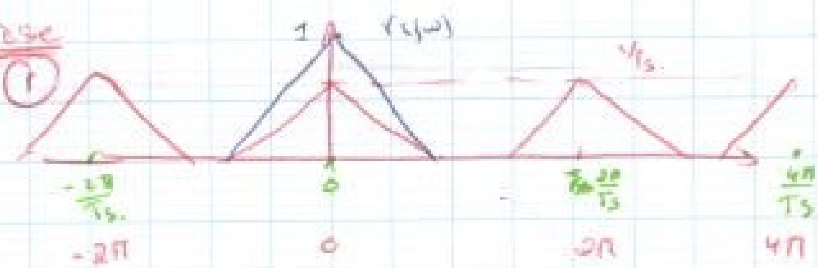
Drawing $\qquad$ Two Cases.

• : $y[n]$
• : $a(t)$.

Time Domain



Freq Domain: $\quad$ case ①



No overlap
No Aliasing

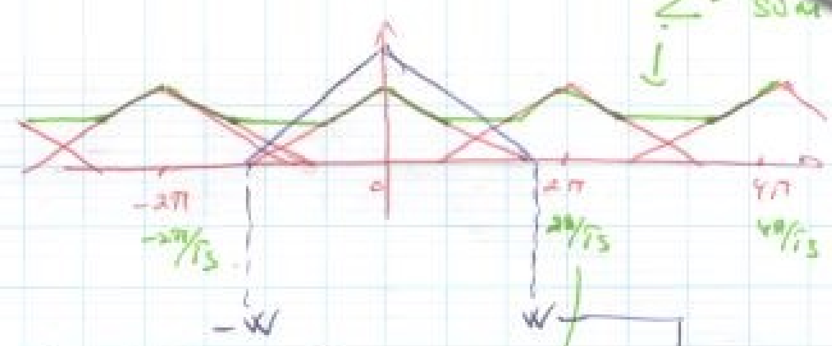
red curve is for $Y(e^{j\Omega})$ & $A(jw)$

But: $\quad$ red scale $\rightarrow Y(e^{j\Omega})$
green scale $\rightarrow A(jw)$

overlap

Case ②

Aliasing



Shanon-Nyquist Sampling Theorem:

Let $x(t)$ be a band limited signal $\omega \in [-W, W]$ rad/sec

$f \in [-B, B]$ Hz.

Sampling $x(t)$ every T_s seconds

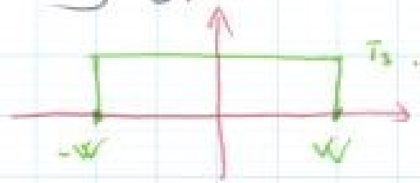
such that sampling is information lossless $\stackrel{\text{if}}{=}$ "No overlap"

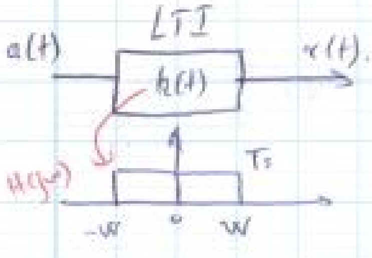
$\Rightarrow \frac{2\pi}{T_s} \geq 2W$ i.e $\boxed{\frac{1}{T_s} \geq 2B \text{ (Hz)}}$ $\frac{1}{2B} > T_s$.

sampling rate : # samples/sec.

Going Back from DT to CT "assuming Nyquist holds"

$\Rightarrow$ multiply $Y(e^{j\Omega})$ by





$x(t) = \sum y[n]\, h(t - nT_s)$

$x(t) = \sum y[n] \frac{WT_s}{\pi} \operatorname{sinc}(W(t - nT_s))$.

Sampling Summary

$x(t)$ —— $\underset{T_s}{\times}$ —— $y[n] \triangleq x(nT_s)$.

$a(t) = \sum y[n]\, \delta(t - nT_s)$.

$A(j\omega) = \frac{1}{T_s} \sum_{k \in \mathbb{Z}} X_{c}\left(\omega - \frac{2\pi k}{T_s}\right)$.

$Y(e^{j\Omega}) = \frac{1}{T_s} \sum X_{c}\left(\frac{\Omega}{T_s} - \frac{2\pi k}{T_s}\right)$.

Interesting prop. of DT signals:

« Upsampling »

$$x[n] \longrightarrow X(e^{j\Omega})$$

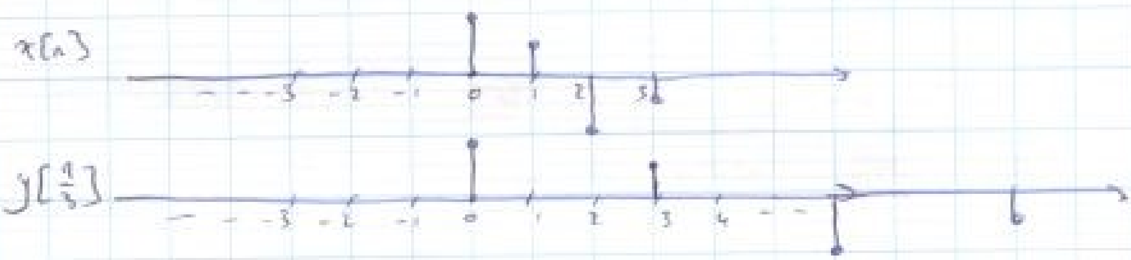
$$y[n] \hat{=} \begin{cases} x[\frac{n}{k}] & n \in k\cdot\mathbb{Z} \\ 0 & o.w. \end{cases} \quad k \in \mathbb{N}^*$$

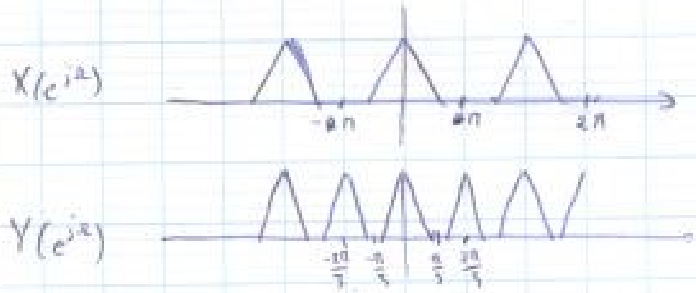
$$Y(e^{j\Omega}) = \sum_{n\in\mathbb{Z}} y[n] e^{-j\Omega n} = \sum_{n\in k\mathbb{Z}} y[n] e^{-j\Omega n}$$ ← $=0$ for n not multiple of k.

$$= \sum_{\ell=-\infty}^{+\infty} y[k\ell]\, e^{-j\Omega k\ell}$$

$$= \sum_{\ell} x[\ell]\, e^{-j(\Omega k)\ell}$$

stretch





« Down Sampling »

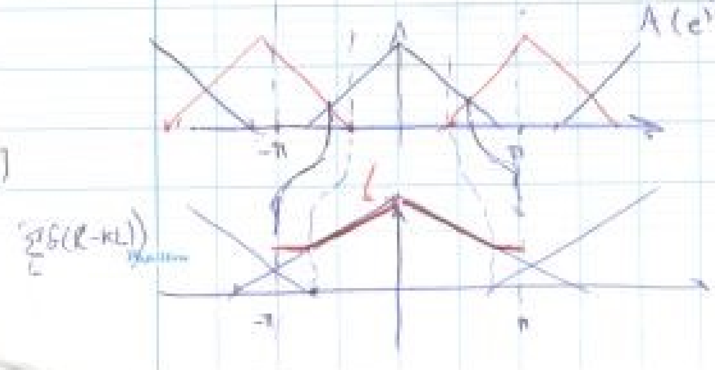
$$x[n] \longrightarrow X(e^{j\Omega})$$

$$y[n] = x[k.n] \quad k \in \mathbb{N}^*$$

⇒ compress in time & stretch in frequency.

$$\Rightarrow Y(e^{j\Omega}) = \frac{1}{kT_s} \sum_{\ell} X\left(\frac{\Omega}{T_s} - \frac{2\pi\ell}{kT_s}\right)$$

$$\Lambda(e^{j\Omega}) = \frac{1}{k}\frac{1}{T_s} \sum_{\ell} X\left(\omega - \frac{2\pi\ell}{kT_s}\right)$$



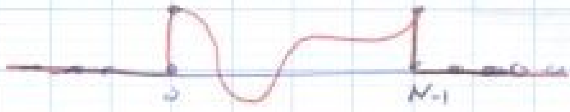
$$Y(e^{j\Omega}) = \frac{1}{k} \sum_{\ell} X\left(e^{j\left(\frac{\Omega}{k} + \frac{2\pi\ell}{k}\right)}\right)$$

overlaps

over a period three terms of $X(e^{j\Omega})$ might contribute to $Y(e^{j\Omega})$

Discrete Fourier Transform (DFT)

Focus finite length discrete-time signal.



$x[n] = 0$ for $n \notin \{0, 1, \ldots\}$

DTFT but search for Simplicity !?

1. Similar to periodic DT with period N.
2. Bandlimited $\longleftrightarrow$ enough to have $x(t)$ "sampled". $X(e^{j\omega})$.
   Freq — Time
   Time — Freq.

$\rightarrow$ Need only few samples from DTFT.

Discrete Fourier Trans DFT

$x[n] \longleftrightarrow x_p[n]$ periodic with "N"

Def $$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}.$$

summation over the limited-time interval $[0 \rightarrow N-1]$ start zero ends N-1

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

"Fourier series of $x_p[n]$"

N.B// $X[k]$ is the DFT of $x[n]$.

$\equiv$ is the Fourier Series Coeff. of $x_p[n]$.

$\equiv$ Scaled samples of DTFT of $x_N[n]$. @ $2\pi/N$.

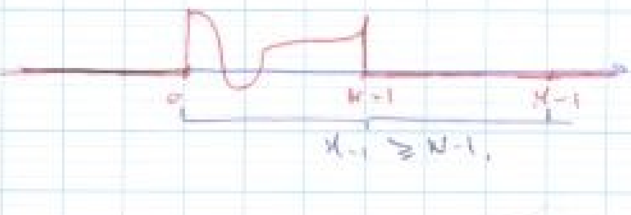
Note $x[n]$ is a sequence of "N-values"

$X[k]$ is defined for "N-values"

⚠ Some texts use different scaling. $(\frac{1}{N})$

Properties **Notation** $((l))_N = l$ (modulo operation) $l \in [0, 1, \ldots, N-1]$ $x = l\ [N]$

0. Can be defined for any "M" bigger than duration.



$M-1 \geq N-1$.

Papillon

$N \geq M$ take bigger

① **Linear!**

$x[n]$ N-pts N-pts $\xrightarrow{DFT}$ $X[k]$

$y[n]$ M-pts N-pts $\longrightarrow$ $Y[k]$

$\rightarrow a\,x[n] + y[n] \longleftrightarrow a\,X[k] + b\,Y[k]$

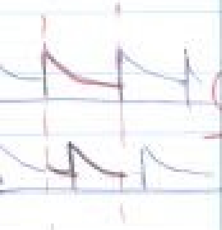
$k = 0, 1, \dots, N-1$

② **Time Reversal**

$x[((-n))_N]$

consider the periodic signal & apply modulo N (take window of $0 \to N-1$)

$x[((-n))_N] \xleftrightarrow{DFT} X[((-k))_N]$



③ **Time Shift**

$x[((n-n_0))_N] \xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}n_0 k}\, X[k]$

consider the periodic signal $\rightarrow$ shift it $\rightarrow$ then take window $(0 \to N-1)$

④ **Mult. by exp**

$e^{j(\frac{2\pi}{N}k_0)n}\, x[n] \xleftrightarrow{DFT} X[((k-k_0))_N]$

⑤ **Conjugation:**

$x^*[n] \xleftrightarrow{DFT} X^*[((-k))_N]$

⑥ **Convolution:**

$(x \circledast_N y)[n] \xleftrightarrow{DFT} \frac{1}{N} X[k]\,Y[k]$

$\hookrightarrow \sum_{0}^{N-1} x[k]\, y[((n-k))_N]$

circular convolution

$x[n]$ $n = 0, \dots, N-1$

$y[n]$ $n = 0, \dots, N-1$

$(x \circledast_N y)_{[n]} = \sum_{0}^{N-1} x[k]\, y[((n-k))_N]$

⑦ $x[n]\,y[n]$ **Multiplication.**

$x[n] \cdot y[n] \xleftrightarrow{DFT} (X \circledast_N Y)[k]$

⑧ **Duality**

$x[n] \xrightarrow{DFT} X[k]$

$X[n] \xrightarrow{DFT} \frac{1}{N}\, x[((-k))_N]$

⑨ **Parseval**

$$\frac{1}{N} \sum_{0}^{N-1} |x[n]|^2 = \sum_{0}^{N-1} |X[k]|^2$$

Papillon

Discrete Fourier Transform:

$x[n]$ finite length.

i.e. $x[n]=0 \begin{cases} n<0 \\ n \geq N. \end{cases}$

$\boxed{\text{N-pts DFT.}}$

$$\begin{cases} X[k] = \frac{1}{N} \sum_{0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}. \\ x[n] = \sum_{0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}. \end{cases} \qquad k=0,1,\dots,N-1.$$

$$\begin{matrix} \text{N-pts.} & & \text{N-pts.} \\ x[n] \in \mathbb{C} & \longrightarrow & X[k] \in \mathbb{C}. \end{matrix}$$

⚠ Reminder : circular operations:

$((n-n_0))_N \; , \; ((-n))_N \; , \; \dots$

Computation of N-pts DFT:

- straightforward: $N \times \begin{cases} N \text{ complex } \otimes \\ N-1 \text{ complex } \oplus \\ 1 \; \otimes \end{cases}$

$$\sim \Theta(N^2) \text{ operations.}$$

more efficiently →

- Fast Fourier Transform:

def FFT: a collection of efficient algorithms to compute N-DFT. (N-pts DFT).

e.g. Assume $N = 2^m$ (power of 2).

focus: $X'[k] = \sum_{0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$ $\quad k=0,1,\dots,N-1.$

$$X'[k] = \sum_{\substack{n \text{ even} \\ n=2l \\ l=0,\dots,\frac{N}{2}-1}}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} + \sum_{\substack{n \text{ odd} \\ n=2l+1 \\ l=0,\dots,\frac{N}{2}-1}} x[n] e^{-j\frac{2\pi}{N}nk}.$$

even indices | odd indices

$$X'[k] = \underbrace{\sum_{l=0}^{\frac{N}{2}-1} x[2l]\, e^{-j\frac{2\pi}{N/2}kl}}_{\frac{N}{2}\text{-pts DFT of } x[2l]} + \left(e^{-j\frac{2\pi}{N}k}\right)\underbrace{\sum_{l=0}^{\frac{N}{2}-1} x[2l+1]\, e^{-j\frac{2\pi}{N/2}kl}}_{\frac{N}{2}\text{-pts DFT of } x[2l+1]}$$

$\Rightarrow X'[k] = E[k] + \left(e^{-j\frac{2\pi}{N}k}\right) O[k]$. $\sim \Theta(N \log N)$

→ Divide & Conquer approach.

→ operations ≡ Time.

let $T(N)$ be time to compute N-DFT.

$$\boxed{T(N) = 2 \times T\left(\frac{N}{2}\right) + N}$$ "Merge-Sort".

→ $T(N) \sim \Theta(N \log N)$.

Recap

(CT)

FT
$$\begin{cases} X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt & \omega \in \mathbb{R} \\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t}\, d\omega & t \in \mathbb{R} \end{cases}$$

(DT)

DTFT
$$\begin{cases} X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} & \Omega \in \mathbb{R} \\ x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n}\, d\Omega & n \in \mathbb{R} \end{cases}$$

* periodic: "T" $\Updownarrow C_k = \frac{1}{T} X_T\left(j\frac{2\pi}{T}k\right)$

* Periodic: "N" $\Updownarrow C_k = \frac{1}{N} X_N\left(e^{j\frac{2\pi}{N}k}\right)$

Fourier Series
$$\begin{cases} C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi}{T}kt}\, dt & k \in \mathbb{R} \\ x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{j\frac{2\pi}{T}kt} & t \in \mathbb{R} \end{cases}$$

DTFS
$$\begin{cases} C_k = \frac{1}{N}\sum_{1\text{ period}} x[n] e^{-j\frac{2\pi}{N}kn} & k = 0, \dots, N-1 \\ x[n] = \sum_{0}^{N-1} C_k e^{j\frac{2\pi}{N}kn} & n \in \mathbb{Z} \end{cases}$$

* Finite length: $0 \to N-1$

DFT
$$\begin{cases} X[k] = \frac{1}{N}\sum_{0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} & k = 0, \dots, N-1 \\ x[n] = \sum_{0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \end{cases}$$

DFT ≡ F.S. coeff. if $x[n]$ periodic & take one period

~~PROPS:~~

FS coeff are a scaled DTFT of 1 period "Sampled"

DTFT $\leftarrow$ $x[n]$ periodic.

FS $\leftarrow$ $x[n]$ periodic

$\equiv$

DFT $\leftarrow$ take one period

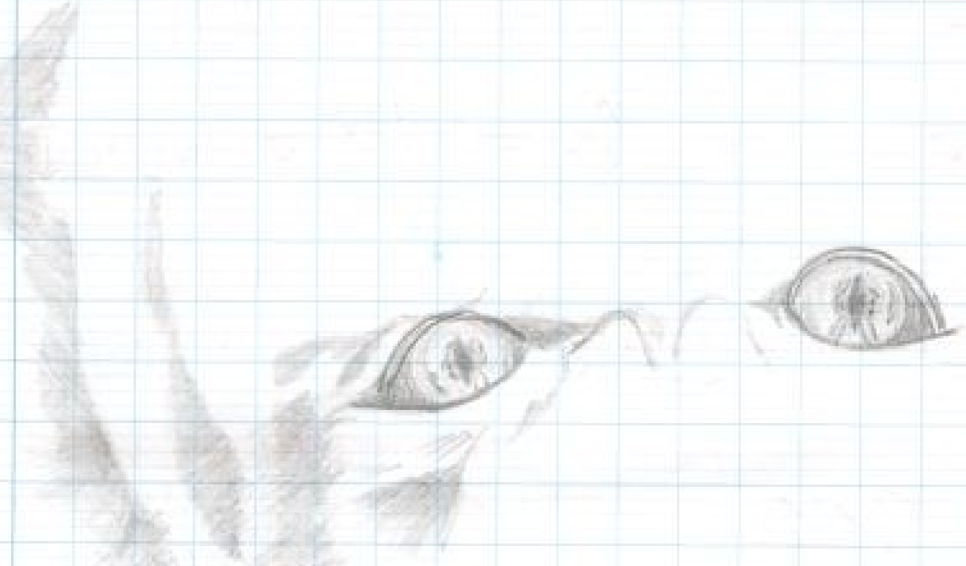
$x[n]$ finite duration $\rightarrow X(e^{j\Omega})$

$$\rightarrow X[k] = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right)$$

$k = 0, \ldots, N-1$.

CT $\leftrightarrow$ DT

$$X(e^{j\Omega}) = \frac{1}{T_s} \sum_{-\infty}^{+\infty} X\left(\frac{\Omega}{T_s} - \frac{2\pi}{T_s}k\right)$$



ECE 340 Cheat-sheet Midterm

* cascade: $\prod_{i=1} (Y_i/X_i)$
* Parallel: $\sum_{i=1} (Y_i/X_i)$ → better for IR.

* **General**:
  * feed forward: $\frac{Y}{X} = (1+p\mathcal{R})$ FIR. $y[n] \begin{cases} 1 & n=0 \\ p & n=1 \\ 0 & o.w. \end{cases}$
  * feed backward: $\frac{Y}{X} = \frac{1}{1-p\mathcal{R}}$ IIR $y[n] = p^n u[n]$ {$|p|<1$ → converge, $|p|>1$ → diverge, $|p|=1$ step, ...}

* **odd/even** $x_e[n] = \frac{1}{2}[x[n]+x[-n]]$, $x_o[n] = \frac{1}{2}[x[n]-x[-n]]$. // prove uniqueness, assume two exist → prove them equal

N.B// All derived expressions by inspection must be proved by induction (diff. eqn.s)

* **Taylor**
  * $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ $|x|<1$ $\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha-k-1}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ (Try to find recurrence).
  * $(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)\, x^n$.
  * $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ • $\log(1+x) = \sum_{1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ $|x|<1$.
  * $\sin x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ • $\cos x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

* **Impulse Response** • feed forward likely to have FIR. else:
  * ① diff. eqn: plug $\delta[n]$ for $x[n]$ & find it.
  * system function → ② Block Diagram; $\left(\frac{1}{(1-\alpha\mathcal{R})^2}\right)$ → i. use Taylor expansion. $H(\mathcal{R}) = \sum_{0}^{\infty} h[n]\, \mathcal{R}^n$ (coeff of $\mathcal{R}^n$ is $h[n]$)
    * ii. partial fraction expansion to change into parallel structures. (use known blocks + Taylor for repeated roots).
    * iii. convolution of fundamental series blocks.

$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$.
$\delta[n] = u[n+1] - u[n]$.

* **sin & cos outputs**: sys. fn. must have two conjugate complex roots $\frac{A}{(1-\alpha e^{-j\omega}\mathcal{R})(1-\alpha e^{j\omega}\mathcal{R})}$

Euler's formulas: $\cos x = \frac{e^{j\omega}+e^{-j\omega}}{2}$, $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$.
↳ partial fractions + known blocks to find $h[n]$.

* **finding Poles** $\mathcal{R} \leftrightarrow z^{-1}$ → find roots ↔ poles. But careful for multipliers.

N.B// eg// $\frac{\mathcal{R}}{\boxed{2}-10\mathcal{R}+12\mathcal{R}^2}$ → $p_1 = 3$ & $p_2 = 2$. $\frac{\mathcal{R}}{\boxed{2}(1-3\mathcal{R})(1-2\mathcal{R})}$

* **Complex Signals** $y[n] = y_r[n] + j\, y_i[n]$. // Derive expressions for real & imaginary parts.

* **Partial fraction expansion** $\frac{Y}{X} = \frac{2-10\mathcal{R}+12\mathcal{R}^2}{4-4\mathcal{R}-\mathcal{R}^2+\mathcal{R}^3} = \frac{2(1-3\mathcal{R})(1-2\mathcal{R})}{4(1+\frac{1}{2}\mathcal{R})(1-\mathcal{R})(1-\frac{1}{2}\mathcal{R})} = \frac{2}{4}\left[\frac{A}{1+\frac{1}{2}\mathcal{R}} + \frac{B}{(1-\mathcal{R})} + \frac{C}{(1-\frac{1}{2}\mathcal{R})}\right] = \frac{2}{4}\frac{Y'}{X'}$

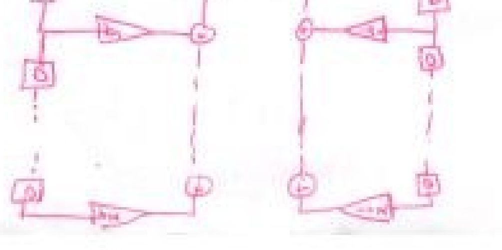
$A = \frac{Y'}{X'}(1+\frac{1}{2}\mathcal{R})\big|_{\mathcal{R}\to 1/p_A = 1/(-\frac{1}{2}) = -2}$ & $B = \frac{Y'}{X'}(1-\mathcal{R})\big|_{\mathcal{R}\to 1/p_B = 1}$ & $C = \frac{Y'}{X'}(1-\frac{1}{2}\mathcal{R})\big|_{\mathcal{R}\to 1/p_C = 1/\frac{1}{2} = 2}$.

IR for degree ≥ 2 poles
1) recurrence → use diff eqn. to get general form → prove by induction. $y[0], y[1], \ldots \to y[n]$ for $x[n] = \delta[n]$.
2) Taylor series → $\frac{Y}{X} = H(\mathcal{R}) = \sum_{n=0}^{\infty} h[n]\, \mathcal{R}^n$ → find $h[n]$ by coeff $\mathcal{R}^n$ in expanded form
3) Perturbation method; $\frac{1}{(1-p\mathcal{R})^3} = \frac{1}{(1-p\mathcal{R})} \cdot \frac{1}{1-(p+\epsilon)\mathcal{R}} \cdot \frac{1}{1-(p+\kappa)\mathcal{R}}$.
→ partial fraction → use fundamental blocks → expand using Taylor if necess. Take $\epsilon \to 0$ & $\kappa \to 0$.

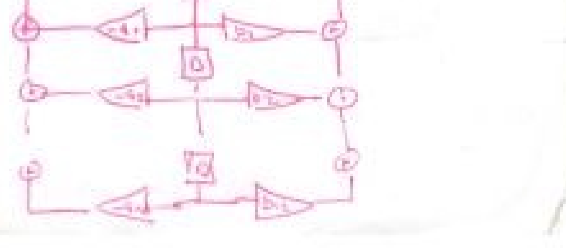
2 $(n+1)\, \alpha^n\, u[n]$
3 $\frac{(n+2)(n+1)}{2}\, \alpha^n\, u[n]$

LTI $\frac{Y}{X} = \frac{\sum_{k=0}^{M} b_k \mathcal{R}^k}{\sum_{l=0}^{N} a_l \mathcal{R}^l}$ // $a_0 = 1$. → $y[n] = \sum_{l=0}^{M} b_l\, x[n-l] - \sum_{k=1}^{N} a_k\, y[n-k]$.

DF I



DF II



Convolution $(x*y)[n] = \sum_{k\in\mathbb{Z}} x[k]\, y[n-k]$. commutative, associative, distributive

→ computing: 1) plug & Do the Math (step functions change bounds of Σ).
2) Graphically: flip → slide → mult & add. $x*y(0) = x(0)y(0)$

$x[n] \neq 0$	$N_1 \leq n \leq N_2$
$y[n] \neq 0$	$N_3 \leq n \leq N_4$
$(x*y)[n] \neq 0$	$N_1+N_3 \leq n \leq N_2+N_4$

Z-transform $X(z) = \sum_{n\in\mathbb{Z}} x[n] z^{-n}$. ROC: $\{z \in \mathbb{C}$ s.t. $\sum_{-\infty}^{+\infty} |x[n]|\,|z|^{-n}$ is finite $\}$

ROC: check '0' & '∞' then other values for finite sum.

⇒ for '0': $\exists n>0$ s.t. $x[n] \neq 0$ → $0 \notin$ ROC. ⇒ for '∞': $\exists n<0$ s.t. $x[n]\neq 0$ → $\infty \notin$ ROC

$X(z) = \underbrace{\cdots + x(-2)z^2 + x(-1)z^1}_{\text{check } \infty} + x(0)z^0 + \underbrace{x[1]z^{-1} + x[2]z^{-2} + \cdots}_{\text{check zero}}$

⇒ finite duration: $\mathbb{C}$ & check 0 & ∞.

⇒ right-sided: extends outward. if rational $X(z)$: outermost pole extending outward. if $\boxed{\text{causal}: \infty \in \text{ROC}}$.

⇒ left-sided: extends inward: if rational $X(z)$: innermost pole extending inward. if $\boxed{\text{anti-causal}: 0 \in \text{ROC}}$

Properties

$X(z)\big|_{z\in e^{j\Omega}}$

- real shifting: $x[n-N] \rightsquigarrow X(z)z^{-N}$ ROC$_x$: check 0 & ∞.
- Linearity: $a_1 x[n] + a_2 y[n] \to a_1 X(z) + a_2 Y(z)$ ROC$_x \cap$ ROC$_y \subseteq$ ROC
- complex shift: $a^n x[n] \rightsquigarrow X(z/a)$ ROC $= |a|\cdot$ ROC$_x$.
- Time scale: $x[n/k] \rightsquigarrow X(z^k)$ $k>0$
- Time reverse: $x[-n] \rightsquigarrow X(z^{-1})$ ROC $= \frac{1}{\text{ROC}_x}$ $z \in \text{ROC}_x \to \frac{1}{z} \in$ ROC
- mult. by n: $n\,x[n] \rightsquigarrow -z\frac{dX}{dz}(z)$ ROC$_x$.
- convolution: $x*y \rightsquigarrow X(z)\cdot Y(z)$ ROC$_x \cap$ ROC$_y \subset$ ROC
- conjugation: $x^*[n] \rightsquigarrow X^*(z)$ ROC$_x$.
- Summation: $\sum_{-\infty}^{n} x[k] \rightsquigarrow \frac{1}{1-z^{-1}} X(z)$

$$\frac{x[n]}{n} \longleftrightarrow -\int_0^z \frac{X(z)}{z}\,dz$$

⇒ + initial Value: $x[0] = \lim_{z\to\infty} X(z)$

⇒ + final Value: $x[\infty] = \lim_{z\to 1} (z-1)X(z)$.

Table

- $\delta[n] \to 1$ $\quad \mathbb{C}$
- $\delta[n-N] \to z^{-N}$ $\quad \mathbb{C} - \{0\}$
- $\delta[n+N] \to z^{N}$ $\quad \mathbb{C}$
- $a^n u[n] \to \frac{1}{1-az^{-1}}$ $\quad \mathbb{C}$ $|z| > |a|$.
- $-a^n u[-n-1] \to \frac{1}{1-az^{-1}}$ $\quad \mathbb{C}$ $|z| < |a|$.
- $b^{|n|} \to \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}$ $\quad$ $|b|<1 \to \mathbb{C}$ $|b| < |z| < 1/|b|$; $|b|>1 \to$ No Z-trans.
- $\sin bn \rightsquigarrow \frac{z\sin b}{z^2 - 2z\cos b + 1}$ $\quad |z|>1$
- $\cos bn \rightsquigarrow \frac{z(z-\cos b)}{z^2 - 2z\cos b + 1}$ $\quad |z|>1$.

Inverse Z-trans * careful for $u[n]$ at the end → specify the signal for all values of n.

1) use pairs & prop.

2) Taylor ⟶ $\sum x[n]z^{-n}$ → $x[n]u[\cdots]$ (determined by bounds of Σ).

N.B// $\sum_{n_1}^{n_2} x_1[n]z^{-n} + \sum_{n_2}^{n_3} x_2[n]z^{-n} \to x[n] = \begin{cases} x_1[n] & n_1 \leq n \leq n_2 - 1 \\ x_2[n] & n_2+1 \leq n \leq n_3. \\ x_1[n]+x_2[n] & n = n_2. \end{cases}$

Unilateral Z-trans. $\tilde{X}(z) = \sum_{0}^{\infty} x[n] z^{-n}$, ROC : causal + initial conditions.

+ same properties (+). $y[n] = x[n+N]$.

⇒ $\tilde{Y}(z) = z^N \tilde{X}(z) - z^N x[0] - z^{N-1}x[1] - \cdots - z\,x[N-1]$.

pf $\tilde{Y}(z) = \sum_0^{\infty} x[n+N]z^{-n} = x[N]z^0 + x[N+1]z^{-1} + x[N+2]z^{-2} + \cdots$

$= z^N[x[N]z^{-N} + x[N+1]z^{-N-1} + \cdots] = z^N[x[0]z^0 + x[1]z^{-1} + \cdots + x[N-1]z^{-N+1} + x[N]z^{-N} + \cdots - x[0]z^0 - x[1]z^{-1} - \cdots - x[N-1]z^{-N+1}]$

$= z^N(\tilde{X}(z) - x[0]z^0 - x[1]z^{-1} - \cdots - x[N-1]z^{-N+1}) =$ the form above.

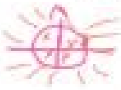
Causality system is causal if:

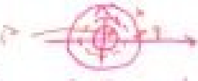
1- $h[n]$ is causal $\forall n < 0 \; h[n] = 0$.

2- diff. eqn: $y[n]$ depends on previous x & y.

3- sys. fn: "∞" $\in$ ROC "got no positive powers of z plugging ∞ won't diverge".

Stability: $|h[n]| \to 0$ as $n \to \pm\infty \iff$ U.C $\subset$ ROC.

if causal: all poles in ROC  & all poles must have $|p| < 1$

if double sided: p's in ROC & q's outside  $|p| < 1$, $|q| > 1$

$\hookrightarrow h[n] = \sum K_i \, (p_i)^n u[n] + \sum K_L (q_L)^n u[-n-1]$ (p's inside ROC & q's outside).

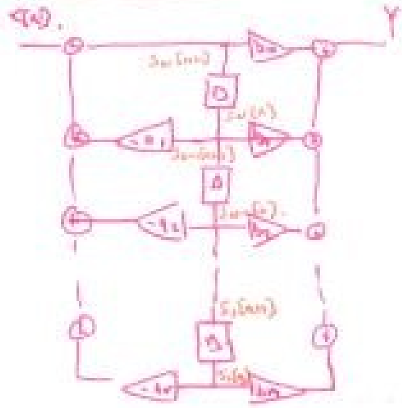
BIBO Stable when sys. fn $\frac{Y}{X}$ is rational $\iff$ stable.

Bounded input bounded output prove $\sum_{-\infty}^{\infty} |h[n]|$ is finite

$y[n] = \sum h[k] \, x[n-k]$ but bounded input $x[n-k] \leq Max$.

$y[n] \leq Max \sum_{-\infty}^{+\infty} h[k] \to \sum_{-\infty}^{+\infty} h[k]$ must be finite.

State-Space to evaluate output: - states
- current & future inputs



$$S[n+1] = \begin{pmatrix} S_1[n+1] \\ \vdots \\ S_N[n+1] \end{pmatrix} = \begin{array}{c} S_1[n+1] \to \\ \vdots \\ S_N[n+1] \to \end{array} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & & 0 \\ & & \cdots & 0 & 1 \\ -a_N & -a_{N-1} & \cdots & -a_2 & -a_1 \end{pmatrix} S[n] + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} x[n].$$

$S_1[n]$ ... $S_{N-1}[n]$ $S_N[n]$

$y[n] = (b_N - b_0 a_N, \; b_{N-1} - b_0 a_{N-1}, \; \cdots, \; b_1 - b_0 a_1) \, S[n] - b_0 x[n]$.

// $S_{N-1}[n+1] = S_N[n]$ --- $S_1[n+1] = S_2[n]$.

$S_N[n+1]$ = by inspection $= -a_N S_1[n] - a_{N-1} S_2[n] \cdots - a_1 S_N[n]$.

* $y[n]$ = by inspection $= b_N S_1[n] + \cdots + b_1 S_N[n] + b_0 S_N[n+1] \to$ substitute exp. of $S_N[n+1]$ to get $y[n]$.

$$\begin{cases} S[n+1] = A S[n] + B x[n] \\ y[n] = C S[n] + D x[n] \end{cases} \to \begin{cases} S[n+1] = A^{n+1} S[0] + \sum_{0}^{n} A^k B x[n-k] \\ y[n] = C A^n S[0] + \sum_{0}^{n-1} C A^k B x[n-1-k] + D x[n] \end{cases}$$

Z-transform $\begin{cases} S(z) = (zI - A)^{-1} B X(z) \\ Y(z) = [C (zI - A)^{-1} B + D] X(z) \end{cases}$ to find poles check $(zI-A)^{-1}$??

$\to (zI - A)^{-1} = \frac{1}{\det(zI - A)} \, Co(zI - A)^T \to$ roots of $\det(zI - A)$ are poles.

$\det(zI - A) = \sum a_i z^i \to$ roots are poles. for causal sys

$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ ... $|r| < 1$

$x[0] = \lim_{z \to \infty} X(z)$

$x[\infty] = \lim_{z \to 1} (z-1) X(z)$

Unilateral Note $y[n+2] - 3y[n+1] + 2y[n] = x[n]$.

$= (z^2 \tilde{Y}(z) - z^2 y[0] - z y[1]) - 3(z \tilde{Y}(z) - z y[0]) + 2\tilde{Y}(z) = X(z)$.

$$\underbrace{\tilde{Y}(z) = \frac{1}{z^2 - 3z + 2} \tilde{X}(z)}_{ZSR} + \underbrace{\frac{z y[1] - (-z^2 + 3z) y[0]}{z^2 - 3z + 2}}_{ZIR}$$

Contour Integration $x[n] = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1}\, dz = \sum_{a_i} \text{Res}\,[X(z) z^{n-1}, a_i]$

inside ROC

poles of $f(z) = X(z) z^{n-1}$ that are inside C.

Find Res [f, a]

1) $\text{Res}[f,a] = a_{-1}$ $\quad f(z) = a_{-2}(z-a)^{-2} + a_{-1}(z-a)^{-1} + a_0 + a_1(z-a) + \dots$

2) $\text{Res}[f,a] = \lim_{z\to a} (z-a) f(z)$. $\quad$ * ① $z^{-1} \longleftrightarrow z$

3) $\text{Res}[f,a] = \frac{1}{(m-1)!} \frac{d^{m-1}[f(z)(z-a)^m]}{dz^{m-1}}\Big|_{z\to a}$

if a is pole of order m :
$f(z) = \frac{g(z)}{(z-a)^m}$
$g(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ analytic
No -ve powers of z.

Procedure : - find $f(z) = X(z) z^{n-1}$

- devide intervals of interest - $f(z) = \frac{z^n}{z-a}$: $|z|>|a|$ $\quad n\geq 0 \to$ no poles
$n<0 \to 0$ & a but a not in C. change $m=-n$.

---

e.g. $f(z) = \frac{z^{n+1}}{(z-a)^2}$ ← denominator always in this form.

* for $n \geq 0$ → a is pole of degree 2 $\quad x[n] = \lim_{z\to a}\left(\frac{d z^{n+1}}{dz} = (n+1)a^n\right)$ $\quad x[n] = (n+1)a^n u[n]$.

$n=0$ $y[0]=0$
$n=-1$ $y[-1]=0$

for $n<0$ → 0 is pole of degree $(-n-1) = m$

$y[n] = \text{Res}(f,0) = \frac{1}{(m-1)!} \lim \frac{d^{m-1}}{dz^{m-1}} z^m f(z) = \frac{1}{(-n-2)!} \lim_{z\to 0} \frac{d^{(-n-2)}}{dz^{-n-2}} \frac{1}{(z-a)^2}$.

$= \begin{cases} a^{-2} & n=-2. \\ a^n(-n-1) & n\leq -2. \end{cases}$

$= (-n-1)\, a^n u[-n-2] = -(n+1)\, a^n u[-n-2]$.

+ take values of n & find derivation with limit
& find a recurrence relation.

→ * $f(z) = \frac{z^{n-1}}{(1-az^{-1})^k} = \frac{z^{n+k-1}}{(z-a)^k}$.

$n\geq 0 \to k>0$. $\quad x[n] = \text{Res}(f,a) = \frac{1}{(k-1)!}\lim_{z\to a}\left(\frac{d^{k-1}}{dz^{k-1}} z^{n+k-1}\right)$

$= \begin{cases} a^n & k=1 \\ (n+k-1)\cdots(n+1)\frac{1}{(k-1)!}a^n. & k>1. \end{cases} = \begin{cases} a^n u[n] & k=1 \\ \binom{n+k-1}{n} a^n u[n] & k\geq 2 \end{cases}$

→ $f(z) = \frac{z^{n+k-1}}{(z-a)^k}$ $\quad n<0.$ → if $n+k-1 \geq 0$ → $n \geq 1-k$ no other than a.
if $n+k-1 < 0$ → pole at 0 of order $m = -(n+k-1)$.

* $\text{Res}(f,0) = \frac{1}{(1-m)!}\frac{d^{m-1}}{dz^{m-1}}(z-a)^{k}\Big|_{z=0}$.

$= \begin{cases} (-1)^k a^k = (-1)^n a^n. & m=1 \quad n=-k. \\ (-1)^k k(k+1)\cdots(-n-1)\,a^n & n<-k. \end{cases}$

$= (-1)^k \binom{-n-1}{k-1} a^n u[-n-k]$.

→ $f(z) = z^{n-1}\frac{z^2-\frac{1}{4}}{z^2-1/9}$. $\quad$ causal.

$n\geq 0$.

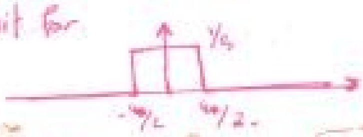
$x[n] = \begin{cases} \text{Res}(f, 1/3) + \text{Res}(f, -1/3) & n\geq 0 \\ 0 & n<0 \end{cases}$

$\text{Res}(f, 1/3) = \lim_{z\to 1/3} z^{n-1} \frac{z^2 - 1/4}{z+1/3} = \left(\frac{1}{3}\right)^{n-1}\frac{1}{24}$,

$\text{Res}(f, -1/3) = -\left(-\frac{1}{3}\right)^{n+1} 7/24$.

Continuous Time discritization isn't always stable.

Impulse "$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{"}\infty\text{"} & t=0 \end{cases}$ >> is a limit for



$= \begin{cases} 1/\epsilon & |t| \leq \epsilon/2 \\ 0 & |t| > \epsilon/2 \end{cases}$

* $\delta(t)$ is a distribution $\delta: f(t) \rightarrow f(0)$. $\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$. $\boxed{\dot{u}(t) = \delta(t)}$

Prop.

- $\int_{-\infty}^{t} f(t)\delta(t)\, dt = \begin{cases} 0 & t<0 \\ f(0) & t \geq 0 \end{cases} \rightarrow \int_{-\infty}^{t} \delta(t)\, dt = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases} = u(t)$.
- $g(t)\delta(t) = g(0)\delta(t)$.
- $\int \overbrace{\cdots}^{n} \int \delta(t)\, dt = \frac{t^{n-1}}{(n-1)!} u(t)$.
- $\int_{t_1}^{t_2} f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & o.w. \end{cases}$
- $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$

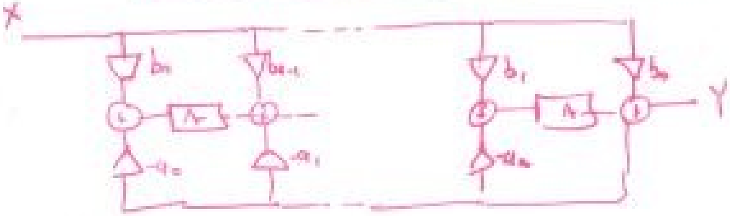
$\dot{u}(t-t_0) = \delta(t-t_0)$
$\delta(at) = \frac{1}{|a|}\delta(t)$
$\delta(t) = \delta(-t)$

Rep • differential Eqns: $\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{l=0}^{n} b_l x^{(l)}(t)$ // $a_0 = 1$.

N.B.// superpositioning: IR//
$\frac{Y}{X} = \frac{A}{1-\alpha A} = A(1+\alpha A+\alpha^2 A^2+ \cdots)$
$h(t) = \sum_{0}^{\infty} \frac{(\alpha t)^L}{L!} u(t) = e^{\alpha t} u(t)$

- Operators Ax: integration $\rightarrow \int_{-\infty}^{t} x(t)\,dt$.
  $\frac{1}{A}x$: differentiation.
- Block Diag.

$\longrightarrow$



- Sys. fn: $\frac{Y}{X} = \frac{\sum b_L A^{n-L}}{\sum a_j A^{n-j}}$

Fundament Blocks

Better Parallel decomposition for IR.

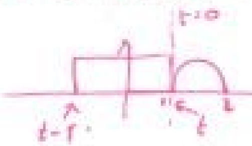
1st Order $\frac{A}{1-\alpha A} \longrightarrow$ IR: $e^{\alpha t} u(t)$

K^{th} Order $\frac{A^K}{(1-\alpha A)^K} \longrightarrow poly(t)\, e^{\alpha t} u(t)$.

Find poles:
- $A \rightarrow s^{-1}$
- poly(s)
- find roots

Convolution

- superposition $y(t) * \delta(t-t_0) = y(t-t_0)$ "for a signal that is combination of δ's"
- $\int x(\tau) h(t-\tau)\, d\tau$.
- graphical: - flip one of the curves. "take reference t=0".



  - slide back till there is no overlap.
  - take intervals the account for changes in shapes or variation of overlap area
  - limits of the integral are: - fixed from fixed curve.
  - variable from sliding curve.

N.B. $rect(\frac{t}{2T}) = u(t+T) - u(t-T)$

Laplace Transform $F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$ + ROC: $s \in \mathbb{C}$ s.t. $\int_{-\infty}^{\infty} |f(t)e^{-st}| dt < \infty \rightarrow$ depends on $Re\{s\}$

ROC prop
- strip.
- $\exists x(t) \neq 0$ for $t<0 \rightarrow \{\infty\} \notin ROC$.
- $\exists x(t) \neq 0$ for $t>0 \rightarrow \{-\infty\} \notin ROC$.
- $x(t)$ right-sided $\rightarrow$ ROC right-sided.
- double sided $\rightarrow$ ROC one strip.
- $x(t)$ finite duration $\rightarrow \mathbb{C} \subset ROC$ N.B.$\{\pm\infty\}$.
- $X(s)$ rational $\rightarrow$ ROC determined by poles.
- $x(t)$ causal $\rightarrow$ ROC right-sided $\{\infty\} \in ROC$
- $x(t)$ anti-causal $\rightarrow$ ROC left-sided $\{-\infty\} \in ROC$

prop. Laplace
- $x(t-T) \rightsquigarrow e^{-sT} X(s)$ ROC_x N.B. "$\pm\infty$"
- $e^{-\alpha t} x(t) \rightsquigarrow X(s+\alpha)$ ROC shift by "$-\alpha$"
- $x(-t) \rightsquigarrow X(-s)$ $-ROC$.
- $t\,x(t) \rightsquigarrow -\frac{d}{ds} X(s)$ ROC.
- $\frac{d}{dt} x(t) \rightsquigarrow s X(s)$. $ROC_x \subset ROC$
- $\int_{-\infty}^{t} x(t)\, dt \rightsquigarrow \frac{1}{s} X(s)$ $\{ROC_x \cap \{Re\{s\}>0\}\} \subset ROC$
- $(x * y)(t) \rightsquigarrow X(s) \cdot Y(s)$ $\{ROC_x \cap ROC_y\} \subset ROC$
- $x(at) \rightsquigarrow \frac{1}{|a|} X(\frac{s}{a})$ $a \times ROC_x$.

Unilateral $\tilde{X}(s) = \int_{0}^{\infty} x(t) e^{-st} dt$ + ROC
$\rightarrow$ uniqueness only in causal systems.

$x^{(n)}(t) \rightsquigarrow s^n \tilde{X}(s) - s^{n-1} x(0) - s^{n-2} \dot{x}(0) - \cdots - s^0 x^{(n-1)}(0)$.

Initial & Final Values

① if $x(t)$ has no impulse at zero.
$x(0^+) = \lim_{s \to \infty} s X(s)$

② if $x(t)$ has a limit at $t \to \infty$
$x(\infty) = \lim_{s \to 0} s X(s)$.

Notes:

* $\frac{Y}{X} = \frac{\sum a_i \lambda^{n-i}}{\sum b_i \lambda^{n-k}} \dashrightarrow \tilde{Y}(s) = \underbrace{\frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}}_{ZSR\;:\;H(s)\text{ for causal sys.}} \tilde{X}(s) + \underbrace{\frac{\text{initial conditions}}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}}_{ZIR}$

* **Eigen fns & Values**
  DT: $x[n] = \alpha^n \to \boxed{h[n]} \to H(\alpha)\,\alpha^n$; CT: $x(t) = e^{\alpha t} \to \boxed{h(t)} \to e^{\alpha t} H(\alpha)$

* **State-Space**
$$\begin{cases} \dot{s}(t) = A s(t) + B x(t) \\ y(t) = C s(t) + D x(t) \end{cases} \dashrightarrow$$

$y(t) = S_1(t) = (1\; 0\; 0 \cdots 0)\, S(t)$.
$$\dot{s}(t) = \begin{pmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -a_1 & & & & 0 \\ a_0 & 0 & \cdots & & 0 \end{pmatrix} s(t) + \begin{pmatrix} b_{n-1} \\ \vdots \\ b_0 \end{pmatrix} x(t).$$

* **Stability**
  absolute: $-\; h(t) \xrightarrow[t\to\infty]{} 0$
  $\vdots$
  $-\; h^{(n)}(t) \xrightarrow[t\to\infty]{} 0$
  ] output goes to zero for no input & for any initial condition.

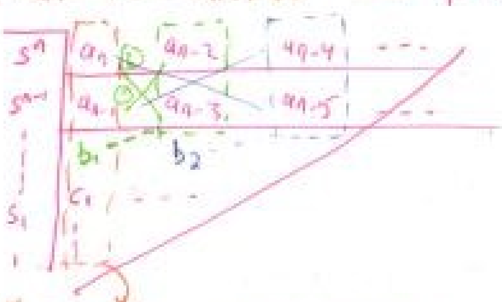
  BIBO: $\exists M_x$ s.t. $|x(t)| \le M_x\ \forall t$.
  $\Rightarrow \exists M_y$ s.t. $|y(t)| \le M_y\ \forall t$.
  $\Leftrightarrow$ absolute.

In Laplace Domain: $j\omega$-axis $\subset$ ROC. $Re\{p_i\} < 0$; $Re\{q_k\} > 0$.
$t > 0$: $e^{p_i t}u(t)$; $t < 0$: $-e^{q_k t}u(-t)$.

Routh-Hurwitz Method for Causal Systems.

$Q(s) = a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$. $\Leftrightarrow$ if one of coeff. $\le 0 \Rightarrow$ sys. not stable. if $a_0 = 0 \to$ not stable.

n+1 Rows



// fill zeroes for unified row-length.
$$b_i = \frac{a_{n-1} a_{n-2i} - a_n a_{n-2i-1}}{a_{n-1}}$$
$$c_i = \frac{b_1 a_{n-2i-1} - a_{n-1} b_{i+1}}{b_1}$$

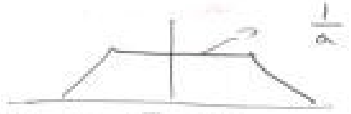
Look on 1st Column
zero elements $\to$ poles on $j\omega$-axis.
sign changes $\to$ # poles not in LHP.

$\Rightarrow$ Stable: no sign changes nor zero elements in 1st Column.

Laplace Table:

$\delta(t)$	1	all s.
$u(t)$	$1/s$	$Re\{s\}>0$
t^n	$\frac{n!}{s^{n+1}}$	$Re\{s\}>0$
e^{-at}	$\frac{1}{s+a}$	$Re\{s\}>-a$.
$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$	$Re\{s\}>0$
$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$	$Re\{s\}>0$

$\text{rect}(at) * \text{rect}(bt)$ $\quad a < b$

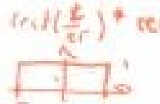


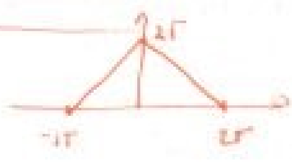
Creepy Fourier
$e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2}$
$e^{j\pi t^2} \longleftrightarrow e^{j\pi(\frac{1}{4} - f^2)}$
$e^{-a|t|} \xleftrightarrow{a>0} \frac{2a}{a^2 + 4\pi^2 f^2}$
$\text{sgn}(t) \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases} \longleftrightarrow \frac{1}{j\pi f}$

$\frac{1}{1+t^2} \longleftrightarrow \pi e^{-2\pi|f|}$
$\sin^2(t) \leftrightarrow \frac{1}{4}[2\delta(f) - \delta(f-\frac{1}{\pi}) - \delta(f+\frac{1}{\pi})]$
$\cos^2(t) \leftrightarrow \frac{1}{4}[2\delta(f) + \delta(f-\frac{1}{\pi}) + \delta(f+\frac{1}{\pi})]$
$\sin(2\pi f_0 t + \phi) \leftrightarrow \frac{j}{2}[e^{-j\phi}\delta(f+f_0) - e^{j\phi}\delta(f-f_0)]$
$\cos(2\pi f_0 t + \phi) \leftrightarrow \frac{1}{2}[e^{-j\phi}\delta(f+f_0) + e^{j\phi}\delta(f-f_0)]$
$\sum_{n\in\mathbb{Z}} F(n) = 2\pi \sum_{k\in\mathbb{Z}} f(2\pi k)$

Other Notes
$x(t) = x_T(t) * \text{comb}_T(t)$
$X(j\omega) = \frac{2\pi}{T}\sum X_T(j\frac{2\pi}{T}n)\,\delta(\omega - \frac{2\pi}{T}n)$
$C_n = \frac{1}{T} X_T(j\frac{2\pi}{T}n)$.

$\text{rect}(\frac{t}{2T}) * \text{rect}(\frac{t}{2T}) = \begin{cases} 0 & t \le -2T \\ t+2T & -2T < t \le 0 \\ 2T - t & 0 < t \le 2T \\ 0 & t > 2T \end{cases}$





Frequency Response

$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$. $H(j\omega)$ is of interest. It contributes to:

$e^{j\omega t} \longrightarrow H(j\omega)e^{j\omega t}$.

- $|H(j\omega)|$: mag
- $\angle H(j\omega)$: phase.

Bode Plots

$H(s) = K \dfrac{(s-z_0)(s-z_1)\ldots}{(s-p_0)(s-p_1)\ldots}$

$|H(j\omega)| = |K| \cdot \dfrac{\prod |j\omega - z_i|}{\prod |j\omega - p_i|}$

$\angle H(j\omega) = \angle K + \angle \prod (j\omega - z_i) - \angle \prod (j\omega - p_i)$

$\Rightarrow$ 1) $20\log|H(j\omega)| = 20\log|K| + \sum 20\log|j\omega - z_i| - \sum 20\log|j\omega - p_i|$ dB

2) $\angle H(j\omega) = \angle K + \sum \angle(j\omega - z_i) - \sum \angle(j\omega - p_i)$.

Drawing

start with $90 \times$ power of s-term $\angle K$.

for the zeroes:
- mag: at z_i, slope increase by 20 dB/decade
- phase: at $0.1 z_i$, slope increase by 45°/decade; at $10 z_i$, slop decrease by 45°/decade. } effect lasts $[0.1 z_i, 10 z_i]$ the change of 90°.

for the poles:
- mag: at p_i, slope decrease by 20 dB/decade
- phase: at $0.1 p_i$, slope decrease by 45°/decade; at $10 p_i$, slope increase by 45°/decade } effect lasts $[0.1 p_i, 10 p_i]$ change of 90°.

Fourier Transform

$x \longrightarrow X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt \longleftrightarrow X(s)|_{s=j\omega}$ if ROC of $X(s)$ includes $j\omega$-axis.

inv: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$.

Prop

- $x(t-t_0) \rightarrow e^{-j\omega t_0}X(j\omega)$
- $x(t)e^{j\omega_0 t} \rightarrow X(j(\omega-\omega_0))$
- $x(-t) \rightarrow X(-j\omega)$
- $x^*(t) \rightarrow X^*(-j\omega)$
- $x(at) \rightarrow \frac{1}{|a|}X(j\frac{\omega}{a})$
- $\frac{dx}{dt}(t) \rightarrow j\omega X(j\omega)$
- $t\,x(t) \rightarrow -\frac{1}{j}\frac{dX(j\omega)}{d\omega}$
- $x * y(t) \rightarrow X(j\omega)\cdot Y(j\omega)$.
- $x(t)\cdot y(t) \rightarrow \frac{1}{2\pi}[X(j\omega) * Y(j\omega)]$

// Initial Values
- $x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega$
- $X(j0) = \int_{-\infty}^{+\infty} x(t)\,dt$.

// real / even:

$x^*(-t) = x(t) \rightarrow X(j\omega)$ is Real.

$x(t)$ real $\rightarrow X(j\omega) = X^*(-j\omega)$

// Duality ***

$g(t) \longrightarrow G(\omega)$.

$G(t) \longrightarrow 2\pi\, g(-j\omega)$.

uncertainty:

width$(t) \times$ width$(\omega) \simeq 2\pi$.

periodicity: impulsive $\rightarrow$ periodic. periodic $\rightarrow$ impulsive.

1. $\delta(t) \rightarrow 1$.
2. $1 \rightarrow 2\pi\delta(j\omega)$ *** $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t}\,d\omega$
3. rect$(t) \rightarrow$ sinc$(\frac{\omega}{2})$
4. $u(t) \rightarrow \frac{1}{j\omega} + \pi\delta(\omega)$.

↳ $\frac{1}{s}$: $j\omega \notin$ ROC. so need $\neq$ definition

$\rightarrow u(t) = u_e + u_o$.

use $u(t) + u(-t) = 1 \dashrightarrow U_e(\omega) = \pi\delta(\omega)$

use $\dot{u}(t) = \delta(t) \dashrightarrow U_o(\omega) = \frac{1}{j\omega}$.

$\dashrightarrow \int_{-\infty}^{t} x(\tau)\,d\tau \Longrightarrow \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$

($X(j\omega)\cdot U(j\omega)$.)

Final Note

Parseval: $\int_{-\infty}^{+\infty} f(t)g^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)G^*(j\omega)\,d\omega$

$\dashrightarrow \int_{-\infty}^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(j\omega)|^2\,d\omega$.

5. $\cos(\omega_0 t) \rightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$
6. $\sin(\omega_0 t) \rightarrow \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

$x(t)\cos(\omega_0 t) \xrightarrow{\text{convolve}} \frac{1}{2}[X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))]$

Similar for $x(t)\sin(\omega_0 t)$.

$\Rightarrow$ as if making 2 instances of the fn.

Connections

CT

FT: $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$. $\omega \in \mathbb{R}$.

$x(t) = \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$. $t \in \mathbb{R}$

$c_k = \frac{1}{T}X_T(j\frac{2\pi}{T}k)$.

FS: $c_k = \int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}kt}\,dt$. $k \in \mathbb{R}$.

$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi}{T}kt}$ $t \in \mathbb{R}$.

DT

DTFT: $X(e^{j\Omega}) = \sum_{-\infty}^{+\infty} x[n]e^{-j\Omega n}$. $\Omega \in \mathbb{R}$.

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$. $n \in \mathbb{Z}$.

$c_k = \frac{1}{N}X_N(e^{j\frac{2\pi}{N}k})$

DTFS: $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$ // over 1 period

$x[n] = \sum_{0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$. $n \in \mathbb{Z}$

Dirac Comb

$\sum_{k \in \mathbb{Z}} \delta(t - kT)$

$\xrightarrow{FT} \frac{2\pi}{T}\sum_{n \in \mathbb{Z}} \delta(\omega - \frac{2\pi}{T}n)$

$\xrightarrow{DTFT} \frac{2\pi}{N}\sum_{-\infty}^{+\infty} \delta(\omega - \frac{2\pi}{N}n)$

FS: $x(t) = \frac{1}{T}\sum_{-\infty}^{+\infty} e^{j\frac{2\pi}{T}nt}$ / $c_n = \frac{1}{T}$

Discrete-Time Fourier Transform $X(e^{j\Omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\Omega n}$.

$X(z)|_{z=e^{j\Omega}}$ if U.C ⊂ ROC ⟺ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$.

Eigen fn. $e^{j\Omega n} \xrightarrow{h[n]} H(e^{j\Omega}) e^{j\Omega n}$.

(Periodic with period $= 2\pi$ min $T = 2$ max $f = \frac{1}{2}$ Hz $= \frac{1}{2} 2\pi$ r/s $= \pi$ r/s $f \in [-\pi, \pi]$)

Prop
- $x[-n] \to X(e^{-j\Omega})$.
- $x^*[n] \to X^*(e^{-j\Omega})$
  - real → even.
  - img. → odd.
  - even → real.
- $x[n-n_0] \to e^{-j\Omega n_0} X(e^{j\Omega})$
- $e^{j\Omega_0 n} x[n] \to X(e^{j(\Omega-\Omega_0)})$
- $n\,x[n] \to -\frac{1}{j}\frac{d X(e^{j\Omega})}{d\Omega}$.
- $(x * y)[n] \to X(e^{j\Omega}) \cdot Y(e^{j\Omega})$
- $x[n] \cdot y[n] \to \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\Omega-\theta)}) d\theta$.

Initial & Final Values

$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) d\Omega$.

$X(e^{j0}) = \sum_{n\in\mathbb{Z}} x[n]$.

Parseval

$\sum_{\mathbb{Z}} x[n] \cdot y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) Y^*(e^{j\Omega}) d\Omega$.

$\sum_{\mathbb{Z}} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$.

Pairs
- $\delta[n] \to 1$.
- $\delta[n-n_0] \to e^{-j\Omega n_0}$.
- $\sum_{-N}^{N} \delta[n-k] \to \frac{\sin(\frac{2N+1}{2}\Omega)}{\sin(\frac{\Omega}{2})}$
- $\cos\omega_0 n \to \pi(\sum \delta(\omega-\omega_0-2\pi k) + \delta(\omega+\omega_0-2\pi k))$
- $a^n u[n] \to \frac{1}{1-ae^{-j\omega}}$ $|a|<1$

Discrete Fourier Transform $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$.

Finite-length Signals $0 \to N-1$ can be extended with zero entries. $x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}$.

It similar to FS with period N. → Need Samples of DTFT of $x_p[n]$ "periodic with N".

Note: DFT of $x[n]$ ≡ FS coeff. of $x_p[n]$ in one period ≡ Scaled samples of DTFT of $x_p[n]$ on a period.

Note: $x[n]$ sequence of "N-values" → $X[k]$ defined for "N-values"

Prop
- $x[((-n))_N] \to X[((-k))_N]$
- $x[((n-n_0))_N] \to e^{-j\frac{2\pi}{N}n_0 k} X[k]$.
- $e^{j(\frac{2\pi}{N}k_0)n} \to X[((k-k_0))_N]$.
- $x^*[n] \to X^*[((-k))_N]$.

Convolution: $(x \circledast_N y)[n] \to \frac{1}{N} X[k] \cdot Y[k]$.
$\hookrightarrow \sum_{0}^{N-1} x[\ell] y[((n-\ell))_N]$

Multiplication: $x[n] \cdot y[n] \to (X \circledast_N Y)[k]$.

Parseval: $\frac{1}{N}\sum_{0}^{N-1} |x[n]|^2 = \sum_{0}^{N-1} |X[k]|^2$.

Duality $x[n] \to X[k]$
$X[n] \to \frac{1}{N} x[((-k))_N]$.

Note on FS $e^{j\frac{2\pi}{T}nt} \xrightarrow{h(t)} H(e^{j\frac{2\pi}{T}n}) e^{j\frac{2\pi}{T}nt}$. ⟷ $x(t) = \sum_{\mathbb{Z}} c_n e^{j\frac{2\pi}{T}nt} \to y(t) = \sum \underbrace{c_n H(e^{j\frac{2\pi}{T}n})}_{B_n} e^{j\frac{2\pi}{T}nt}$.

Sampling CT ⊂ DT

auxiliary: $a(t) = \sum_{n\in\mathbb{Z}} x(nT_s)\delta(t-nT_s)$.

① $a(t) = \sum x(t)\delta(t-nT_s)$
$= x(t) \cdot \text{comb}(t)$

$\Rightarrow A(j\omega) = \frac{1}{T_s}\sum_{\mathbb{Z}} X(j(\omega - \frac{2\pi}{T_s}n))$

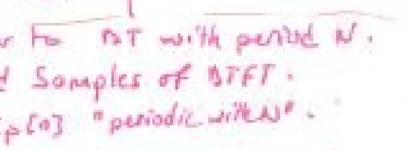
② $y[n] = x(nT_s)$.
$a(t) = \sum y[n]\delta(t-nT_s)$
$\Rightarrow A(j\omega) = \sum y[n] e^{-j\omega T_s n}$
$= \sum y[n] e^{-j\Omega n} = Y(e^{j\omega T_s})$ DTFT of $y[n]$.

$\Rightarrow y[n] = x(nT_s)$.
$Y(e^{j\omega T_s}) = \frac{1}{T_s}\sum_{\mathbb{Z}} X(j(\omega - \frac{2\pi}{T_s}n))$.
$Y(e^{j\Omega}) = \frac{1}{T_s}\sum_{\mathbb{Z}} X(j(\frac{\Omega}{T_s} - \frac{2\pi}{T_s}n))$.

$Y(e^{j\Omega}) = \frac{1}{T_s}(\dots + X(j\frac{\Omega}{T_s} - \frac{2\pi}{T_s}) + X(j\frac{\Omega}{T_s}) + X(j(\frac{\Omega}{T_s} + \frac{2\pi}{T_s})) + \dots)$

⟹ copies shifted by $2\pi/T_s$.
scale by $1/T_s$ & sum.
change label $\frac{2\pi}{T_s} \to 2\pi$.

Going back: depend on overlap of shifted copies.



let $x(t)$ be bandlimited
$\omega \in [-W, W]$ r/s. $f \in [-B, B]$ Hz.
Sample $x(t)$ every T_s s.t. $\frac{2\pi}{T_s} \geq 2W$. ⟹ $T_s \leq \frac{1}{2B}$

Shannon-Nyquist: multiply $Y(e^{j\Omega})$ by rect$(\frac{\Omega}{2W})$ scale by T_s.
equivalently $x(t) = \sum_{\mathbb{Z}} y[n] \frac{WT_s}{\pi} \text{sinc}(W(t-nT_s))$

Up Sampling $y[n] \to Y(e^{j\Omega})$ stretch in time ⟹ shrink in freq.
$y_1[n] = \begin{cases} y[\frac{n}{k}] & n \in k\mathbb{Z} \\ 0 & \text{o.w.} \end{cases} \xrightarrow{Y(e^{j\Omega k})} Y_1(e^{j\Omega}) = \sum_{\ell\in\mathbb{Z}} y[\ell] e^{-j(\Omega k)\ell}$.

Down Sampling $y_2[n] = \begin{cases} y[nk] & n \in k\mathbb{Z} \\ 0 & \text{o.w.} \end{cases} \to Y_2(e^{j\Omega}) = \frac{1}{k}\sum Y(e^{j(\frac{\Omega}{k} - \frac{2\pi}{k}\ell)})$

shrink time ⟹ stretch freq.

overlaps.. more terms of $Y(e^{j\Omega})$ contribute to $Y_2(e^{j\Omega})$