Problem Set 3 – Solutions

Problem 3.1

(a) Using the definition,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} \\ &= \frac{1}{9} z^{-2} \frac{1}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

whenever $|z| > (1/3)$. Also the signal is causal and therefore "∞" is also in the ROC.

(b) $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = -z + z^{-1} + 2z^{-2}$, which exists whenever z is neither "0" nor "∞".

Problem 3.2

(a)

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} u[k]u[n-k] = \sum_{k=0}^{\infty} u[n-k] \\ &= \begin{cases} \sum_{k=0}^{n} 1 = (n+1) & n \geq 0 \\ 0 & \text{o.w.} \end{cases} \\ &= (n+1)u[n]. \end{aligned}$$

(b)

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} u[k] \left(\frac{1}{2}\right)^{n-k} u[n-k] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n-k} u[n-k] \\ &= \begin{cases} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n-k} = \sum_{l=0}^{n} \left(\frac{1}{2}\right)^{l} = \frac{1-(1/2)^{n+1}}{1-(1/2)} & n \geq 0 \\ 0 & \text{o.w.} \end{cases} \\ &= 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] u[n]. \end{aligned}$$

(c)

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u[k] \left(\frac{1}{2}\right)^{n-k} u[n-k] = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{2}\right)^{n-k} u[n-k] \\
&= \begin{cases} \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k & n \geq 0 \\ 0 & \text{o.w.} \end{cases} \\
&= 3\left(\frac{1}{2}\right)^n \left[1 - \left(\frac{2}{3}\right)^{n+1}\right] u[n].
\end{aligned}
$$

(d)

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] \left(\frac{1}{2}\right)^{n-k} u[n-k] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} u[n-k] \\
&= \begin{cases} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n = (n+1)\left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & \text{o.w.} \end{cases} \\
&= (n+1)\left(\frac{1}{2}\right)^n u[n].
\end{aligned}
$$

Problem 3.3

(a) $X(z) = \dfrac{1}{z-1} = z^{-1}\dfrac{1}{1-z^{-1}}$

$X(z)$ is a rational function that has a single pole at "1". Since the ROC is a unique ring, we have *two* possible choices for the ROC:

i) $\{|z| > 1\} \cup \{\infty\}$, and $x[n]$ is a delayed –by one– unit step, i.e.,

$$x[n] = u[n-1].$$

ii) $\{|z| < 1\}$, and $x[n]$ is a delayed –by one– "$-u[-n-1]$", i.e.,

$$x[n] = -u[-n].$$

(b) $Y(z) = z^2 + 3 + z^{-1} + z^{-3}$, which has a pole of degree 3 at "0", and a pole of degree 2 at "∞". We have only *one* possible region of convergence with is $\mathbb{C} \setminus \{0\}$.

$$
y[n] = \begin{cases} 1 & n = -2 \\ 3 & n = 0 \\ 1 & n = 1 \\ 1 & n = 3 \\ 0 & \text{otherwise.} \end{cases}
$$

(c) $V(z) = \dfrac{1}{z^2+z+1}$ has two poles at

$$p_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}j = e^{-j2\pi/3} \quad p_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j = e^{j2\pi/3}$$

$$V(z) = \frac{1}{(z-p_1)(z-p_2)} = \frac{z^{-2}}{(1-p_1z^{-1})(1-p_2z^{-1})} = z^{-2}\left[\frac{1-p_2/p_1}{1-p_1z^{-1}} + \frac{1-p_1/p_2}{1-p_2z^{-1}}\right]$$

Both poles have the same magnitude and lie on the UC.

i) $\{|z|>1\}\cup\{\infty\}$, and $v[n]$ is a delayed –by two– of an easily determined signal:

$$\left\{\left(1-e^{-j4\pi/3}\right)e^{j2\pi n/3} + \left(1-e^{j4\pi/3}\right)e^{-j2\pi n/3}\right\}u[n],$$

and therefore,

$$v[n] = \left\{2\cos\left[\frac{2\pi}{3}(n-2)\right] - 2\cos\left[\frac{2\pi}{3}(n-1)\right]\right\}u[n-2].$$

ii) $\{|z|<1\}$, and $v[n]$ is the delayed –by two– signal

$$-\left\{\left(1-e^{-j4\pi/3}\right)e^{j2\pi n/3} + \left(1-e^{j4\pi/3}\right)e^{-j2\pi n/3}\right\}u[-n-1],$$

and therefore,

$$v[n] = \left\{2\cos\left[\frac{2\pi}{3}(n-1)\right] - 2\cos\left[\frac{2\pi}{3}(n-2)\right]\right\}u[-n+1].$$

Problem 3.4

(a) The signal $x[n]$ is causal and hence the ROC will be easily specified: it is from largest pole (in magnitude) outwards.

$$x[n] = 5\cos(3n)u[n] = 5\frac{e^{j3n}+e^{-3jn}}{2}u[n] = \frac{5}{2}\left[e^{j3n}u[n] + e^{-3jn}u[n]\right].$$

Therefore,

$$X(z) = \frac{5}{2}\left[\frac{1}{1-e^{j3}z^{-1}} + \frac{1}{1-e^{-j3}z^{-1}}\right], \qquad ROC: |z|>1.$$

(b) since $y[n] = 5\cos(3n)$, the Z-transform does not exist. Indeed, this can be seen from 2 perspective

(i) By definition, $Y(z) = \sum_{n=-\infty}^{\infty} 5\cos(3n)z^{-n}$ if *both* limits exist.
Looking at $n\to\infty$, we note that we need $|z|>1$. When $n\to-\infty$, we need $|z|<1$.

(ii) Alternatively,

$$y[n] = \frac{5}{2}\left[e^{j3n}u[n] + e^{j3n}u[-n-1] + e^{-j3n}u[n] + e^{-j3n}u[-n-1]\right].$$

If we observe each term ”alone” we see that the requirements on $|z|$ don't intersect.

Problem 3.5

Since

$$x[n] = \left(\frac{1}{2}\right)^n u[n],$$

then

$$X(z) = \frac{1}{1-(1/2)z^{-1}}, \quad ROC = \{|z| > (1/2)\} \cup \{\infty\}.$$

That of $x[n+3]$ is hence

$$z^3 X(z) = \frac{z^3}{1-(1/2)z^{-1}}, \quad ROC = \{|z| > (1/2)\} \cup \{\infty\}.$$

Also

$$y[n] = \left(\frac{1}{3}\right)^n u[n],$$

and

$$Y(z) = \frac{1}{1-(1/3)z^{-1}}, \quad ROC = \{|z| > (1/3)\} \cup \{\infty\}.$$

That of $y[n+1]$ is hence

$$zY(z) = \frac{z}{1-(1/3)z^{-1}}, \quad ROC = \{|z| > (1/3)\} \cup \{\infty\},$$

and that of $y[-n+1]$ is

$$z^{-1}Y(z^{-1}) = \frac{z^{-1}}{1-(1/3)z} = \frac{z^{-2}}{z^{-1}-(1/3)}, \quad ROC = \{|z| < 3\}.$$

In conclusion,

$$Z(z) = \frac{z^3}{1-(1/2)z^{-1}} \frac{z^{-2}}{z^{-1}-(1/3)} = \frac{-3z}{(1-(1/2)z^{-1})(1-3z^{-1})},$$

with a region of convergence that is the intersection, namely $\{(1/2) < |z| < 3\}$.

Problem 3.6

We will determine the inverse Z-transform assuming the ROC extends outwards. For the other choices, a similar treatment may be done.

i) Using our treatment in class and the problem sets,

$$\frac{(1/4)z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)^2} \leftrightarrow n\left(\frac{1}{4}\right)^n u[n]$$
$$12\frac{(1/4)z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)^2} \leftrightarrow 12n\left(\frac{1}{4}\right)^n u[n]$$
$$z^{-2}\frac{3z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)^2} \leftrightarrow 12(n-2)\left(\frac{1}{4}\right)^{n-2} u[n-2]$$

ii) Conducting a Taylor series' expansion:

$$\begin{aligned} X(z) &= \frac{z^7-2}{1-z^{-7}} = -z^7\frac{z^7-2}{1-z^7} \\ &= -z^7(z^7-2)[1+z^7+z^{14}+z^{7\cdot 3}+\cdots] \\ &= -[z^{7\cdot 2}+z^{7\cdot 3}+z^{7\cdot 4}+\cdots]+2[z^7+z^{7\cdot 2}+z^{7\cdot 3}+z^{7\cdot 4}+\cdots]. \end{aligned}$$

In the time domain

$$x[n] = \begin{cases} 2 & n=7 \\ 1 & n=7k, k\in\mathbb{N}^* \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.7

First note that at infinity the value is finite and therefore, the ROC contains infinity. Therefore the sequence is causal and we need to determine it only for $n \geq 0$.

We have two simple poles at $\pm(1/3)$, and we take a trigonometric contour in the ROC. The poles $\pm(1/3)$ are also poles for

$$z^{n-1}X(z) = f(z) = z^{n-1}\frac{z^2-\frac{1}{4}}{z^2-\frac{1}{9}}$$

and therefore whenever $n \geq 0$,

$$x[n] = Res(f, 1/3) + Res(f, -1/3).$$

Computing the residues:

$$\begin{aligned} Res(f,1/3) &= \lim_{z\to 1/3} z^{n-1}\frac{z^2-\frac{1}{4}}{z+\frac{1}{3}} \\ &= \left(\frac{1}{3}\right)^{n-1}\frac{1}{24} \\ Res(f,-1/3) &= \lim_{z\to -1/3} z^{n-1}\frac{z^2-\frac{1}{4}}{z-\frac{1}{3}} \\ &= -\left(\frac{-1}{3}\right)^{n-1}\frac{7}{24} \end{aligned}$$

$$x[n] = \begin{cases} 0 & n<0 \\ \left(\frac{1}{3}\right)^{n-1}\frac{1}{24} - \left(\frac{-1}{3}\right)^{n-1}\frac{7}{24} & n\geq 0 \end{cases}$$

Problem 3.8

(a) Using convolutions, the response to a unit step $u[n]$ input is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$
$$= \sum_{k=0}^{\infty} h[k]u[n-k]$$
$$= \sum_{k=0}^{n} h[k]$$

where the second equation is due to the fact that $h[n]$ is causal.

(b) If $H(z)$ was the *Z-transform* of $h[n]$ then the output's *Z-transform* is

$$Y(z) = H(z)X(x) = H(z)\frac{1}{1-z^{-1}}.$$

Note here that all systems and signals are causal and we could have used the unilateral *Z-transform*.

(c) The unit-step response was found in the time domain in (a). As n goes to "∞" ,

$$y[n] \to \sum_{k=0}^{\infty} h[k],$$

and in magnitude

$$\left|\sum_{0}^{\infty} h[k]\right| \leq \sum_{0}^{\infty} |h[k]|$$

and it is finite because of stability. Therefore, $y[n]$ does tend to a constant, but not necessarily zero!

Problem 3.9

(a) Since $X(z)$ is rational and there are no poles nor zeros at "∞", $X(z)$ is of the form

$$X(z) = c\frac{z^2}{(z-2)(z-3)},$$

where c is a non-zero scalar. For the rest of the problem, we choose

$$X(z) = \frac{z^2}{(z-2)(z-3)} = \frac{1}{(1-2z^{-1})(1-3z^{-1})}.$$

(b) Performing partial fraction expansion:

$$X(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1-3z^{-1}}$$

and one can find $A = -2$ and $B = 3$.

With the ROC : $\{2 < |z| < 3\}$, the inverse transform is

$$x[n] = -2 \cdot 2^n u[n] + 3 \cdot (-3^n u[-n-1]) = -3^{n+1} u[-n-1] - 2^{n+1} u[n].$$

(c) If we where told the system is stable, then the ROC is the unique one that contains the UC, i.e., $\{|z| < 2\}$. In this case,

$$x[n] = (-2)(-2^n u[-n-1]) + 3(-3^n u[-n-1]) = (2^{n+1} - 3^{n+1}) u[-n-1].$$

(d) If the system is causal, the ROC is uniquely specified: $\{|z| > 3\}$. Then,

$$x[n] = (-2)2^n u[n] + (3)3^n u[n] = (3^{n+1} - 2^{n+1}) u[n].$$

Problem 3.10

The system function is

$$\frac{Y(z)}{X(z)} = \frac{K}{(z^2+z-2)\left[1+\frac{K}{z^2+z-2}\right]} = \frac{K}{z^2+z+(K-2)} \tag{1}$$

(a) The system is stable iff the UC is in the ROC. It's also known that the system is causal. Therefore, we require the poles to be inside the UC. Here, the poles are

$$\begin{aligned} &\triangle = 1 - 4(K-2) = 9 - 4K \\ &p_0 = \frac{-1+\sqrt{9-4K}}{2} \qquad p_1 = \frac{-1-\sqrt{9-4K}}{2}. \end{aligned}$$

Our requirements are

$$\begin{aligned} &\left|-1 \pm \sqrt{9-4K}\right| < 2 \\ \Rightarrow &\begin{cases} 1 + (9-4K) \pm 2\sqrt{9-4K} < 4 & \text{if } 9-4K \geq 0 \quad \text{i.e., } K \in \left(2, \frac{9}{4}\right] \\ 1 + (9-4K) < 4 & \text{if } 9-4K \leq 0 \quad \text{i.e., } K \in \left[\frac{9}{4}, \infty\right) \end{cases} \end{aligned}$$

In conclusion , we need to choose $K > 2$.

(b) The poles are real iff

$$9 - 4K \geq 0 \Leftrightarrow K \leq \frac{9}{4}.$$