Problem Set 2 – Solutions

Problem 2.1

(a) Using the abstraction of signals as objects presented in class,

$$Y = F(X - GY) \quad \Longleftrightarrow \quad (1 + FG)Y = FX$$

which implies that the system function is

$$\frac{Y}{X} = \frac{F(D)}{(1 + F(D)G(D))}.$$

(b) If

$$F(D) = \frac{N_F(D)}{D_F(D)} \quad \& \quad G(D) = \frac{N_G(D)}{D_G(D)},$$

the system function becomes the following rational expression

$$\frac{Y}{X} = \frac{N_F(D)}{D_F(D) + N_F(D)N_G(D)/D_G(D))} = \frac{N_F(D)D_G(D)}{D_F(D)D_G(D) + N_F(D)N_G(D))}.$$

Problem 2.2

Part I:

(a) The difference equation is

$$y[n] = x[n] + 10,$$

one can simplify it by defining the input to be $(x[n] + 10)$, which is a signal that can be denoted by abuse of notation by $x[n]$. The difference equation becomes

$$y[n] = x[n],$$

and the system function is

$$\frac{Y}{X} = 1.$$

(b) The impulse response of the system is $\delta[n]$, obtained by looking at the system function (expanded in Taylor series in "D"), or by checking the difference equation.

Part II:

(c) As in part I, we redefine the input and consider a system $y[n] = x[n] + ay[n-1]$. Its block diagram is that of a first order system presented in class.

(d) The signal $y[n]$ is the position of your leg and $x[n]$ is that of your partner. They are real signals, so it does not make sense to choose a complex value for a.

(e) This depends on your personality: For the aggressive "go for it" type, one would say that after a short dance, we'd like to be "closer".

For the player, "shy" type who plays "hard to get", the requirement could be that of a response that gets you further apart (and hence requires from your partner more effort).

Assuming you're of the first type,: we have a first order system, the impulse response of which is $a^n u[n]$. More importantly, if the partner moves closer, i.e., $x[n] = u[n]$, then the response would be

$$y[n] = \sum_0^n a^k = \frac{1-a^{n+1}}{1-a}.$$

As n goes to infinity, $y[n]$ goes to $1/(1-a)$, provided that $|a| < 1$. If you'd like to get closer, you would choose a negative.

(f) One needs to choose $|a| < 1$. Otherwise, a step forward from your partner will lead you to either leaving the room, or bumping heads!

Part III:

(g) To generate a wavy movement, and based on what we have seen in class, you need a second order system with complex conjugate poles (as your response is naturally real and not only in your dreams). Starting with the system function, our target is:

$$\frac{1}{(1-ae^{j\omega_o}D)(1-ae^{-j\omega_o}D)} = \frac{(1-e^{-2j\omega_o})^{-1}}{(1-ae^{j\omega_o}D)} + \frac{(1-e^{2j\omega_o})^{-1}}{(1-ae^{-j\omega_o}D)}.$$

The response is

$$\begin{aligned}
&(1-e^{-2j\omega_o})^{-1}a^n e^{j\omega_o n}u[n] + (1-e^{2j\omega_o})^{-1}a^n e^{-j\omega_o n}u[n] \\
&= a^n\left[(1-e^{-2j\omega_o})^{-1}e^{j\omega_o n} + (1-e^{2j\omega_o})^{-1}e^{-j\omega_o n}\right]u[n] \\
&= \tfrac{a^n}{|1-e^{-2j\omega_o}|^2}\left[e^{j\omega_o n} + e^{-j\omega_o n} - e^{j(\omega_o n+2\omega_o)} - e^{-j(\omega_o n+2\omega_o)}\right]u[n] \\
&= \tfrac{a^n}{(1-\cos 2\omega_o)^2+(\sin 2\omega_o)^2}\left[2\cos\omega_o n - 2\cos\omega_o(n+2)\right]u[n] \\
&= \tfrac{4}{(1-\cos 2\omega_o)^2+(\sin 2\omega_o)^2}a^n \sin\omega_o(n+1)\sin\omega_o u[n].
\end{aligned}$$

To guarantee that the amplitude gets halved every five steps, we choose

$$a^5 = 0.5 \quad \Leftrightarrow \quad a = \left(\frac{1}{5}\right)^{1/5}.$$

Problem 2.3

(b) Examining the block diagram, we can write:

$$\begin{aligned} & Y = (-3/2)DY + D[X - (1/2)DY] = -(3/2)DY - (1/2)D^2Y + DX \\ \iff & \left[1 + (3/2)D + (1/2)D^2\right] Y = DX \\ \iff & \frac{Y}{X} = \frac{D}{[1 + (3/2)D + (1/2)D^2]} \end{aligned}$$

(a) The difference equation may be readily obtained:

$$\begin{aligned} & y[n] + (3/2)y[n-1] + (1/2)y[n-2] = x[n-1] \\ \iff & y[n] = x[n-1] - (3/2)y[n-1] - (1/2)y[n-2]. \end{aligned}$$

(c) To find the poles of the system, we proceed as explained in class:

i) We replace D by z^{-1}, we obtain

$$\frac{Y}{X} = \frac{z}{[z^2 + (3/2)z + (1/2)]}$$

ii) We solve for the roots of the denominator and we obtain:

$$-1 \quad \& \quad -(1/2).$$

Problem 2.4

Using complex quantities is equivalent to having *two* real quantities for each, the real part and the imaginary part.

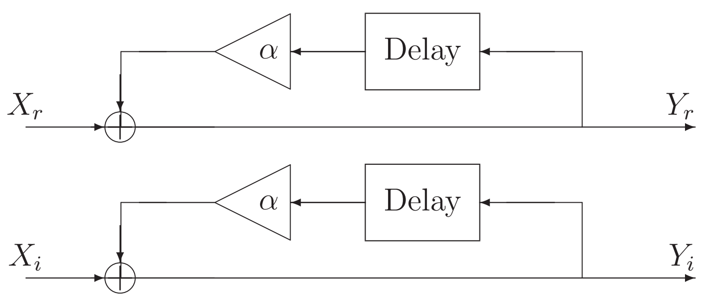
(a) The difference equation of this first order system is

$$y[n] - \alpha y[n-1] = x[n].$$

Taking the real and imaginary parts:

$$\begin{cases} y_r[n] - \alpha\, y_r[n-1] = x_r[n] \\ y_i[n] - \alpha\, y_i[n-1] = x_i[n] \end{cases}$$

In essence, we have a complex input, i.e., two inputs; and a complex output, i.e., two outputs:

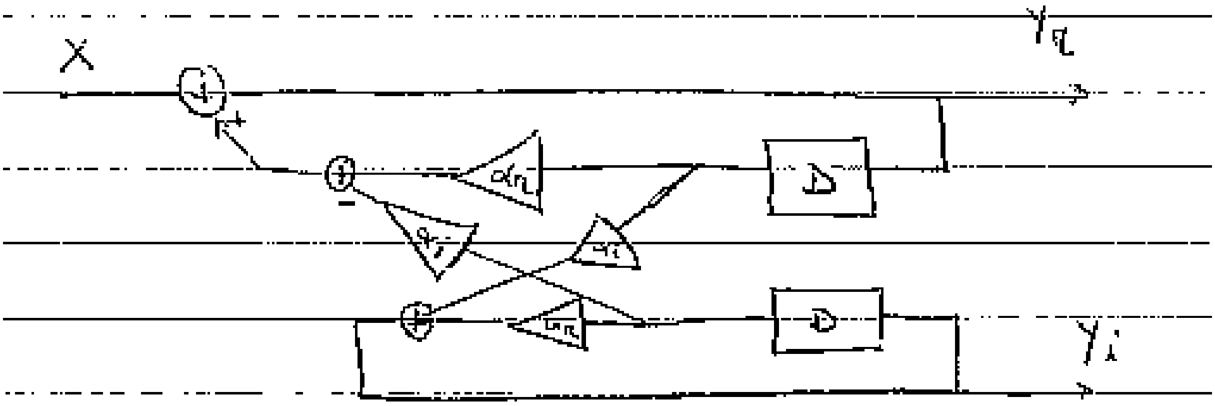


Note that understandably we needed two registers because we need to store one complex number.

(b) Despite the fact that the input is real, the output is still complex because of the multiplication by the complex number $\alpha = \alpha_r + j\alpha_i$.

$$\begin{cases} y_r[n] - \alpha_r\, y_r[n-1] + \alpha_i\, y_i[n-1] = x[n] \\ y_i[n] - \alpha_r\, y_i[n-1] - \alpha_i\, y_r[n-1] = 0 \end{cases}$$

Drawing a corresponding block diagram:



Problem 2.5

Consider a DT system with the following system function:

$$\frac{Y}{X} = \frac{2 - 10D + 12D^2}{1 - D - (1/4)D^2 + (1/4)D^3}.$$

(b) The difference equation of the system is

$$y[n] - y[n-1] - (1/4)y[n-2] + (1/4)y[n+3] = 2x[n] - 10x[n-1] + 12x[n-2].$$

(a) A block diagram of the system may be drawn as presented in class. A direct form of type I or type II may be readily obtained by using the appropriate values of the coefficients $\{a_l\}$'s and $\{b_l\}$'s

(c) To obtain a cascade of 0-th and 1-st order systems, we decompose the system function into polynomials of either degree 0 or 1. This may be done by polynomial factorization of both the numerator and denominator:

$$\frac{Y}{X} = \frac{2(1-2D)(1-3D)}{(1-D)(1-1/2D)(1+1/2D)},$$

where now one can see that X is passed by a multiplier (equal to 2) then through two 0-th order systems: $(1-2D)$ and $(1-3D)$ followed serially by the first order systems: $1/(1-D)$, $1/(1-1/2D)$ and $1/(1+1/2D)$.

The block diagrams for these systems were presented in class.

(d) To obtain a parallel decomposition, we perform a partial fraction expansion:

$$\frac{Y}{X} = \frac{A}{(1-D)} + \frac{B}{(1-1/2D)} + \frac{C}{(1+1/2D)},$$

where

$$A = \frac{2(1-2)(1-3)}{(1-1/2)(1+1/2)} = \frac{16}{3}, \quad B = -15, \quad C = \frac{35}{3}.$$

Problem 2.6

We expand the system function as a Taylor series expansion (in D) and simply "read" the impulse response:

$$\begin{aligned}(1-\alpha D)^{-k} &= 1 + k\alpha D + \frac{(-k)(-k-1)}{2}\alpha^2 D^2 + \cdots \\ &= 1 + k\alpha D + \frac{(k)(k+1)}{2}\alpha^2 D^2 + \cdots + \frac{(k)(k+1)\cdots(k+n-1)}{n!}\alpha^n D^n + \cdots\end{aligned}$$

Therefore, the impulse response is

$$h[n] = \frac{(n+k-1)!}{n!(k-1)!}\alpha^n u[n] = \begin{pmatrix} n+k-1 \\ n \end{pmatrix} \alpha^n u[n].$$