USE FOR ANSWER

EECE 340

Signals & Systems

Midterm Solutions

Mezze

a) Since the systems are stable and all are rational fcts of z, it is equivalent to saying that ROC includes the jω-axis.
Determining causality requires the ROC to extend outward and include "∞"

i) The fction is $\dfrac{z^3 - 4/3\, z^2 + 1/2\, z}{(z - 1/2)(z - 1/3)}$

The poles are @ 1/2 and 1/3
=> the impulse response is right-sided
plugging in ∞ yields ∞ => it's Not causal

Alternatively, the degree of the numerator is higher than the degree of the denominator => not causal

ii) Finding the poles:

$\Delta = 1/4 + 3/4 = 1$

roots: $\dfrac{-1/2 \pm 1}{2} = \begin{cases} 1/4 \\ 3/4 \end{cases}$

Both inside the U.C.
=> The impulse response is right-sided
degree of numerator ≤ that of denominator
=> sys is causal

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iii) Multiplying by z^3 $\dfrac{z^4 + z^3}{z^4 + 4/3\, z^3 - 1/2\, z - 2/3}$

Looking @ denominator, it's equal to

$$z^3(z + 4/3) - 1/2\,(z + 4/3)$$

$$= (z^3 - 1/2)(z + 4/3)$$

⇒ There exists a pole @ $-4/3$

⇒ The region of conv does not extend outwar[ds]

⇒ The impulse response is not even right-sid[ed]

⇒ It's not causal

b) $z[n] = x[n+3] * y[-n+1]$

$x[n] \longleftrightarrow X(z) = \dfrac{1}{1 - 1/2\, z^{-1}}$ ROC $|z| > 1/2$

$x[n+3] \longleftrightarrow z^3 X(z) = \dfrac{z^3}{1 - 1/2\, z^{-1}}$ ROC $|z| > 1/2$

$y[n] \longleftrightarrow Y(z) = \dfrac{1}{1 - 1/3\, z^{-1}}$ ROC $|z| > 1/3$

$y[n+1] \longleftrightarrow z^{+1} Y(z) = \dfrac{z^{+1}}{1 - 1/3\, z^{-1}}$ ROC $|z| > 1/3$

$y[-n+1] \longleftrightarrow z^{-1} Y(z^{-1}) = \dfrac{z^{-1}}{1 - 1/3\, z} = \dfrac{z^{-2}}{z^{-1} - 1/3}$ ROC $|z|$ [<]

$$\Rightarrow \boxed{Z(z) = \frac{z^3}{1 - 1/2\, z^{-1}} \cdot \frac{z^{-2}}{z^{-1} - 1/3} = \frac{-3z}{(1 - 1/2\, z^{-1})(1 - 3z^{-1})}}$$

ROC equal to the intersection

$$1/3 < |z| < 3$$

c) Expanding the exponentials using Taylor series

$$X(z) = \left[1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots\right] + \left[1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{n!}\dots\right]$$

$$= \dots + \frac{1}{n!} z^{+n} + \dots + \frac{1}{2!} z^{+2} + z^{+1} + 2 + z^{-1} + \frac{1}{2!} z^{-2} + \dots + \frac{1}{n!}\dots$$

By identification

$$\boxed{x[n] = \begin{cases} 2 & n = 0 \\ \frac{1}{n!} & n \geq 1 \\ \frac{1}{(-n)!} & n \leq -1 \end{cases}}$$

d) We are free here to use whichever method we find suitable

i) By our treatment in class and problem sets:

$$\frac{1/4\, z^{-1}}{(1 - 1/4\, z^{-1})^2} \longleftrightarrow n\left(\frac{1}{4}\right)^n u[n]$$

$$12 \cdot \frac{1/4\, z^{-1}}{(1 - 1/4\, z^{-1})^2} \longleftrightarrow 12\, n \left(\frac{1}{4}\right)^n u[n]$$

$$z^{-2} \cdot \frac{3 z^{-1}}{(1 - 1/4\, z^{-1})^2} \longleftrightarrow \boxed{12\,[n-2]\left(\frac{1}{4}\right)^{n-2} u[n}$$

ii) $X(z) = \dfrac{z^7 - 2}{1 - z^{-7}} = \dfrac{z^7(z^7 - 2)}{z^7 - 1} = \dfrac{-z^7(z^7 - 2)}{1 - z^7}$

$$= -z^7(z^7 - 2)\left[1 + z^7 + z^{14} + z^{7\times3} + \dots\right]$$

$$= -\left[z^{7\times2} + z^{7\times3} + z^{7\times4} \dots\right]$$

$$+ 2\left[z^7 + z^{7\times2} + z^{7\times3} + z^{7\times4} + \dots\right]$$

$\Rightarrow$ In time $\boxed{x[n] = \begin{cases} 2 & n = 7 \\ 1 & n = 7k \quad k \in \mathbb{N}^* \\ 0 & \text{o.w.} \end{cases}}$

e) First note that @ ∞ the value is finite

$\Rightarrow$ ROC $\ni \{\infty\}$ $\Rightarrow$ the sequence is causal

$\Rightarrow$ need to determine it only for $n \geq 0$

We have two poles: $z^2 - 1/9 = 0$

$\Rightarrow z = \pm 1/3$

We take a contour in the ROC anti-clockwise

$\Rightarrow$ we have 2 simple poles inside the circle and they are also those of $z^{n-1} X(z) = z^{n-1} \frac{z^2 - \dots}{z^2 - \dots} \hat{=} f(z)$

$$\Rightarrow x[n] = \begin{cases} 0 & n < 0 \\ \text{Res}(f, 1/3) + \text{Res}(f, -1/3) & \end{cases}$$

Now $\text{Res}(f, 1/3) = \lim_{z \to 1/3} z^{n-1} \frac{z^2 - 1/4 z}{z + 1/3}$

$= \left(\frac{1}{3}\right)^{n-1} \frac{1/9 - 1/12}{2/3} = \left(\frac{1}{3}\right)^{n-1} \frac{1}{2} \left[\frac{1}{3} \dots\right.$

$= \left(\frac{1}{3}\right)^{n-1} \frac{1}{24}$

$\text{Res}(f, -1/3) = \lim_{z \to -1/3} z^{n-1} \frac{z^2 - 1/4 z}{z - 1/3}$

$= -\left(\frac{-1}{3}\right)^{n-1} \frac{3}{2} \left[\frac{1}{9} + \frac{1}{12}\right]$

$= -\left(-1/3\right)^{n-1} \frac{1}{2} \left[\frac{1}{3} + \frac{1}{4}\right]$

$= -\left(-1/3\right)^{n-1} \frac{7}{24}$

$$\Rightarrow x[n] = \begin{cases} 0 & n < 0 \\ \left(\frac{1}{3}\right)^{n-1} \frac{1}{24} - \left(\frac{-1}{3}\right)^{n-1} \frac{7}{24} & n \geq 0 \end{cases}$$

$$= \begin{cases} 0 & n < 0 \\ -(1/3)^{n} \, 3/4 & n \geq 0 \text{ \& odd} \\ (1/3)^{n} & n \geq 0 \text{ \& even} \end{cases}$$

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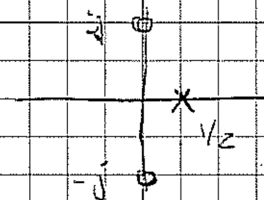
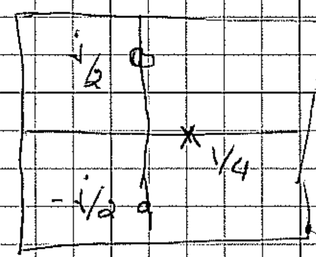
f] The pole-zero diagram of X is



i] Since $y[n] = (1/2)^n x[n]$, $Y(z) = X(2z)$

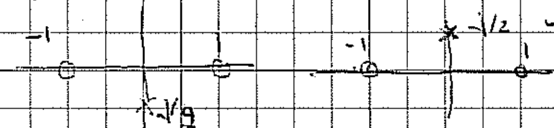
and pole-zero diag is



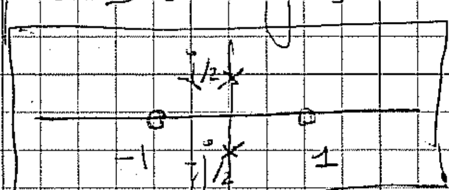
ii] $z[n] = \cos\frac{\pi n}{2}\, x[n] = \frac{1}{2}\left[e^{j\frac{\pi}{2}n} x[n] + e^{-j\pi/2\, n} x[n] \right]$

$$= \frac{1}{2}\left[(e^{j\pi/2})^n x[n] + (e^{-j\pi/2})^n x[n] \right]$$

$$Z(z) = \frac{1}{2}\left[X(e^{-j\pi/2} z) + X(e^{j\pi/2} z) \right]$$



$\Rightarrow$ its pole-zero diag is



g] Since $X(z)$ is rational and absolutely summable

$\Rightarrow$ the UC $\supset$ ROC

Additionally the poles of $X(z)$ are inside UC $\Rightarrow$

ROC is outward $\Rightarrow$ seq is <u>necessarily right-sided</u>

However, it <u>may not be causal</u>

<u>Ex</u>: $\frac{z}{1 - \frac{1}{2} z^{-1}}$ $\quad |z| > 1/2 \quad x[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1]$

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h) First note that if $s[n] = x[n] * y[n]$

then $x[n-k] * y[n] = s[n-k]$

and $x[n-k] * y[n-l] = s[n-k-l]$

Also $a^n u[n] * b^n u[n]$

$$= \sum_{k=-\infty}^{\infty} a^{n-k} u[n-k]\, b^k u[k]$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{n} a^n \left(\frac{b}{a}\right)^k = a^n \frac{\left(b/a\right)^{n+1} - 1}{b/a - 1} & n \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}\, u[n]$$

Additionally $a^n u[n] * b^n u[-n-1]$

$$= \sum_{k=-\infty}^{\infty} a^{n-k} u[n-k]\, b^k u[-k-1]$$

$$= \begin{cases} \displaystyle\sum_{k=-\infty}^{n} a^n \left(\frac{a}{b}\right)^{-k} = a^n \sum_{-n}^{\infty} \left(\frac{a}{b}\right)^k & n \leq -1 \\ \displaystyle\sum_{-\infty}^{-1} a^n \left(\frac{a}{b}\right)^{-k} = a^n \sum_{1}^{\infty} \left(\frac{a}{b}\right)^k & n > -1 \end{cases}$$

$$= \begin{cases} b^{n+1} \dfrac{1}{b-a} & n \leq -1 \quad (\text{if } a<b) \\ a^{n+1} \dfrac{1}{b-a} & n > -1 \quad (\text{if } a<b) \end{cases}$$

Therefore, $y[n] = 4^3\left[3^n u[-n-1] * \left(\frac{1}{4}\right)^{n+3} u[n+3]\right] + 4^3\left[\left(\frac{1}{3}\right)^n u[n] * \left(\frac{1}{4}\right)^{n+\ldots}\right.$

$$= 4^3 \begin{cases} \dfrac{3^{n+3}}{3 - 1/4} & n \leq -4 \\ \dfrac{(1/4)^{n+3}}{3-1/4} & n > -4 \end{cases} + 4^3 \frac{(1/3)^{n+4} - (1/4)^{n+4}}{1/3 - 1/4}$$

$$\boxed{y[n] = \begin{cases} 4^4\left[\dfrac{1}{11} 3^{n+3}\right] & n \leq -4 \\ 4^4\left[\dfrac{1}{11}\left(\dfrac{1}{4}\right)^{n+3} + \left(\dfrac{1}{3}\right)^{n+3} - 3\left(\dfrac{1}{4}\right)^{n+4}\right] & n > \end{cases}}$$

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Triangle of Leuve

I] a) The input of this sys is $y[n]$ & output $x[n]$

Since $\delta[n] = u[n] - u[n-1]$

The impulse response is $(1-0.9^{n})\,u[n] - (1-0.9^{n-1})\,u[n-1]$

$$= \delta[n] + 0.9^{n-1}(1-0.9)\,u[n-1]$$

$$x[n] = 0.1 \times 10\,\delta[n] + 0.1 \times 0.9^{n-1}\,u[n-1]$$

$$= 0.1\left[10\,\delta[n] + \frac{1}{0.9}\,0.9^{n}\,u[n-1]\right]$$

$$= \frac{0.1}{0.9}\left[9\,\delta[n] + 0.9^{n}\,u[n-1]\right]$$

$$\boxed{x[n] = \frac{1}{9}\left[8\,\delta[n] + 0.9^{n}\,u[n]\right]}$$

b) The sys fct may be obtained by taking the Z-transform

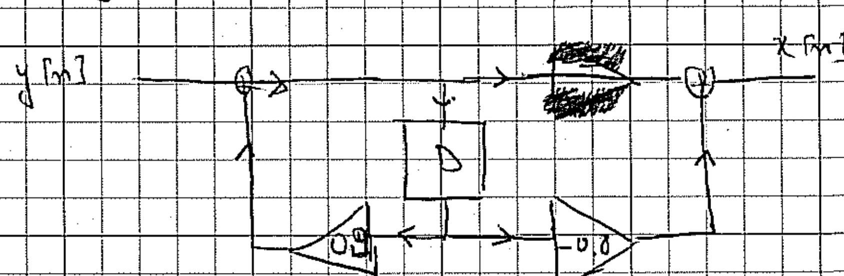
$$H(z) = \frac{1}{9}\left[8 + \frac{1}{1-0.9z^{-1}}\right] \qquad |z| > 0.9$$

c) Since $\frac{X(z)}{Y(z)} = H(z) = \frac{1}{9}\,\frac{9 - 8\times 0.9z^{-1}}{1-0.9z^{-1}} = \frac{1}{9}\,\frac{9 - 7.2z^{-1}}{1-0.9z^{-1}}$

$$\Rightarrow 9\left[X(z) - 0.9X(z)z^{-1}\right] = 9Y(z) - 7.2Y(z)z^{-1}$$

$$\Rightarrow \boxed{\left[x[n] - 0.9\,x[n-1]\right] = y[n] - 0.8\,y[n-1]}$$

d) Drawing a direct form II



II] f] The diff eq is const. coef $\Rightarrow$ $\boxed{\text{sys is LTI}}$

One can alternatively check the def.

e] The impulse response of the sys is (starting @ rest)

$y[0] = 0$

$y[1] = -b$

$y[n] = -b$

$\Rightarrow h[n] = -b\, u[n-1]$

$\Rightarrow$ does not "die" out $\Rightarrow$ $\boxed{\text{Not stable}}$!

III] g] For a given $y[n]$ $\boxed{\text{the sys is NOT linear}}$ (necessarily)

for example scaling $x[n]$ by 2 does not scale the output by 2

If $y[n]$ were a constant, the sys would be affine

The same would hold for TI! indeed a time shift in x does imply a time shift in v

$\Rightarrow$ $\boxed{\text{NOT TI}}$

If $y[n]$ were constant, then it would be

h] If y were also an input, the system would still be $\boxed{\text{NOT linear}}$, but "pretty close" kind of affine

If we consider the shifts in pair $(x[n], y[n])$

Then $\boxed{\text{YES it is TI}}$ and the pf is immediate

IV i) For $y[n] = u[n]$

$$v[n] = 1 - (1 - 0.9^n)u[n] + 1 - u[n]$$

$$= \begin{cases} 2 & n<0 \\ 0.9^n & n \geq 0 \end{cases}$$

j) The impulse response in II was $-b\,u[n-1]$

Therefore $v[n] = 1 - \delta[n] + 1 + b\,u[n-1]$

$$= \begin{cases} 2 & n<0 \\ 1 & n=0 \\ 2+b & n>0 \end{cases}$$

k) Yes, by the result of the class/PS the unit-step response is

$$v[n] = \sum_{-\infty}^{\infty} h[k]\,u[n-k] = \sum_{-\infty}^{n} h[k]$$

$$= \infty \,!$$