EECE 340 Signals & Systems

Midterm Solutions

Problem M.1:

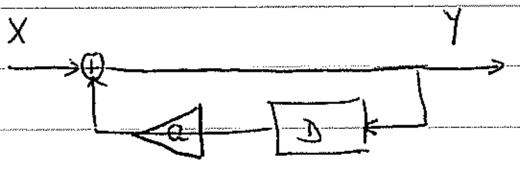
I a) The difference equation is $y[n] = x[n] + 10$
one can simplify it by defining the input to be $(x[n]+10)$! a signal that can be denoted by $x[n]$ (By abuse of notation)

$\Rightarrow y[n] = x[n]$

The system function is $\frac{Y}{X} = 1$!

b) The impulse response is $\delta[n]$, obtained by looking @ the sys function (expanded in Taylor series in "D")

II a) As in part I, we (re)define the input and consider a sys function $y[n] = x[n] + a\,y[n-1]$
The block diagram is:



b) $y[n]$ represents the position of the leg and $x[n]$ is that of your partner. They are real signals, so it DOES NOT make sense to choose a complex value for "a"

c) This is dependent on one's personality: For the agressive "go for it" type, one would say that after a short dance, we'd like to be "closer"! For the player, "shy" type playing hard-to-get require a "response" that gets us "further" apart requiring from the partner more effort

We will assume the first case here: we have a 1^{st} order system, the impulse response of which is $a^n u[n]$. More importantly, if the partner moves closer, i.e. $x[n] = u[n]$, then the response is

$$y[n] = \sum_{k=0}^{\infty} a^{n-k} u[n-k] = a^n \sum_{0}^{n} a^{-k} = \sum_{0}^{n} a^k$$

$$= \frac{1-a^{n+1}}{1-a}$$

As $n \to \infty$ $y[n] \to \frac{1}{1-a}$

(provided that $|a|<1$)

and if we'd like to be closer we choose a negative

f) For the response to be "stable" i.e. your behavior being "sane" we need to choose $|a|<1$
for otherwise a step forward from your partner will lead to you leaving the room, or bumping heads with him.

III - g] To generate the wavy movement, and based on what we have seen in class you need a "2nd order system" with complex conjugate poles (as your response is naturally real and not only in your head)

Starting with the system function, our target is

$$\frac{1}{(1-\alpha e^{j\omega_0} z^{-1})(1-\alpha e^{-j\omega_0} z^{-1})} = \frac{(1-e^{-2j\omega_0})^{-1}}{1-\alpha e^{j\omega_0} z^{-1}} + \frac{(1-e^{2j\omega_0})^{-1}}{1-\alpha e^{-j\omega_0} z^{-1}}$$

$\Rightarrow$ The response is

$$\left[(1-e^{-2j\omega_0})^{-1}\alpha^n e^{j\omega_0 n} + (1-e^{j2\omega_0})^{-1}\alpha^n e^{-j\omega_0 n}\right]u[n]$$

$$= \alpha^n \left[\frac{e^{j\omega_0 n}}{1-e^{-j2\omega_0}} + \frac{e^{-j\omega_0 n}}{1-e^{j2\omega_0}}\right] u[n]$$

$$= \frac{1}{|1-e^{-j2\omega_0}|^2}\alpha^n\left[e^{j\omega_0 n} + e^{-j\omega_0 n} - e^{j(\omega_0 n+2\omega_0)} - e^{-j(\omega_0 n+2\omega_0)}\right]u[n]$$

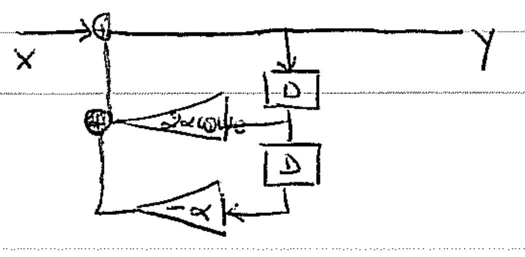
$$= \frac{1}{(1-\cos 2\omega_0)^2+(\sin 2\omega_0)^2}\alpha^n\left[2\cos\omega_0 n - 2\cos\omega_0(n+2)\right]u[n]$$

$$= \frac{4}{(1-\cos 2\omega_0)^2+(\sin 2\omega_0)^2}\alpha^n \sin\omega_0(n+1)\sin\omega_0 \, u[n]$$

Finally, to guarantee that the amplitude gets halved every five steps, we choose

$$\alpha^5 = \frac{1}{2} \Leftrightarrow \alpha = \left(\frac{1}{2}\right)^{1/5}$$

h] The sys function is $\frac{1}{1-2\alpha\cos\omega_0 z^{-1}+\alpha z^{-2}} \Leftrightarrow \frac{1}{1-2\alpha\cos\omega_0 D+\alpha D^2}$



Problem 11.2

a) $y[n] = 2^n \left(\underbrace{n\,u[n] + x[n-1]}_{\Updownarrow \mathcal{Z}} \right)$

$$\frac{z^{-1}}{(1-z^{-1})^2} + z^{-1}X(z) \qquad ROC \supset \{r_0<|z|<r_1\} \cap \{|z|>$$

$$Y(z) = \frac{2z^{-1}}{(1-2z^{-1})^2} + 2z^{-1}X\left(\frac{z}{2}\right) \qquad ROC \supset \{2r_0<|z|<2r_1\} \cap \{|z|>2\}$$

b) $h[n] = \begin{cases} \left(\frac{1}{4}\right)^{n/2} u[n], & n \text{ even} \\ 0 & \text{o.w.} \end{cases}$

Finding the $\mathcal{Z}$-transform: $H(z) = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k z^{-2k}$

$$= \frac{1}{1-\frac{1}{4}z^{-2}} \qquad \left(|z| > \frac{1}{2}\right)$$

The fundamental modes are the roots of $H(z)$ which are $z^{-2} = 4$

$$z = \pm\frac{1}{2}$$

c) We know that $H(z) = K.\dfrac{(z-e^{j\pi/3})(z-e^{-j\pi/3})}{z}$

$$= K\frac{1}{z}\left(z^2 - 2\cos\frac{\pi}{3}\, z + 1\right)$$

$$= K\left(z - 2\cos\frac{\pi}{3} + z^{-1}\right)$$

$$\Rightarrow h[n] = \begin{cases} K & n=1 \\ -2K\cos\frac{\pi}{3} & n=0 \\ K & n=-1 \\ 0 & \text{o.w.} \end{cases} = \begin{cases} K & n=-1 \\ -K & n=0 \\ K & n=1 \\ 0 & \text{o.w.} \end{cases}$$

When the input is $x[n]=1$, The output is constant equal to K

$$\Rightarrow \quad h[n] = \begin{cases} 3 & n=-1 \\ -3 & n=0 \\ 3 & n=1 \\ 0 & \text{o.w.} \end{cases}$$

d) $$X(z) = \frac{z^{-2}}{1-3z^{-1}+2z^{-2}} \qquad \{|z|>2\} \cup \{\infty\}$$

$$= \frac{1}{z^2-3z+2}$$

Since $\{\infty\} \in$ ROC, the signal is causal and we need to determine it for $n \geq 0$ only

Next, we find the poles: $\quad p_0 = 1 \quad p_1 = 2$

$$\Rightarrow \quad X(z) = \frac{1}{(z-1)(z-2)}$$

Using contour integration: $x[n] = \sum_{\substack{p_i:\ \text{poles of } X(z)z^{n-1}=f(z) \\ \text{"inside contour"}}} \text{Res}(f(z), p_i)$

we have two poles @ 1 and 2 and for $n=0$ @ pole @ zero all simple

$*\ n \geq 1, \quad x[n] = -1 + 2^{n-1}$

$*\ n=0, \quad x[n] = -1 + \frac{1}{2} + \frac{1}{2} = 0$

$$\Rightarrow \quad x[n] = -u[n-1] + 2^{n-1}u[n-1]$$

$$= (2^{n-1}-1)\,u[n-2]$$

e) $x[n] = u[n]$ & $h[n] = \left(\frac{1}{7}\right)^n u[n]$

$$\tilde{X}(z) = \frac{1}{1-z^{-1}} \qquad \tilde{H}(z) = \frac{1}{1-\left(\frac{1}{7}\right)z^{-1}}$$

$$\tilde{Y}(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{(1-\frac{1}{7}z^{-1})}$$

$$y[0] = \lim_{z\to\infty} \tilde{Y}(z) = 1$$

$$y[\infty] = \lim_{z\to 1} (z-1)\tilde{Y}(z)$$

$$= \lim_{z\to 1} \frac{z}{1-\frac{1}{7}z^{-1}} = \frac{7}{6}$$

Problem M.3

a) We are told $f[n]$ is causal and takes non-negative values. Checking the convolution

$$(f*f)[n] = \sum_{k=0}^{n} f[k]\,f[n-k] \qquad n \geq 0$$

Evaluating for different values of n

$(f*f)[0] = f[0]^2$

$(f*f)[1] = f[0]f[1] + f[1]f[0]$

But $f*f = u$

$\Rightarrow f[0]^2 = 1 \Rightarrow \boxed{f[0]=1}$

b) $(f*f)[1] = 1 \Rightarrow 2f[1] = 1 \Rightarrow \boxed{f[1] = \frac{1}{2}}$

c) We are looking for a function f that is causal. Let $F(z)$ be its $\mathcal{Z}$ transform

The identity is $F(z)^2 = \dfrac{1}{1-z^{-1}}$ on $|z|>1$

$$\Rightarrow F(z) = \pm\sqrt{\frac{1}{1-z^{-1}}} = \pm(1-z^{-1})^{-1/2}$$

To identify $f[n]$, we use a Taylor series expansion of the RHS:

$$F(z) = \pm\left[1 + \tfrac{1}{2}z^{-1} + \tfrac{1}{2!}\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)z^{-2} + \tfrac{1}{3!}\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)z^{-3} + \dots\right]$$

Since we're told that $f[n]$ is non negative, we deduce

$$f[n] = \begin{cases} \frac{1}{n!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2n-1}{2}\right) & n \geq 1 \\ 1 & n = 0 \end{cases}$$

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Problem M.4

a) * Let $z[n] = a\,x[n]$ and denote by $v[n]$ & $y[n]$
The output of the sys when driven
by $z[n]$ and $x[n]$ respectively.

$y[n] = \left(\frac{-1}{2}\right)^n \left[x[n]+2\right]$

$v[n] = \left(\frac{-1}{2}\right)^n \left(z[n]+2\right) = \left(\frac{-1}{2}\right)^n \left[a\,x[n]+2\right] \neq a\,y[n]$
for $a \neq 1$

$\Rightarrow$ The sys (A) is NOT linear.

* Applying $z[n]$ to sys B,

$$v[n] = \sum_{k=1}^{n} \left(z^2[k+1] - z[k]\right) = \sum_{k=1}^{n} a^2 x^2[k+1] - a\,x[k]$$

$\neq a\,y[n]$ whenever $a \neq 1$

$\Rightarrow$ sys B is NOT linear either

b) Now let $z[n] = x[n-N]$

* In sys A $v[n] = \left(\frac{-1}{2}\right)^n \left(z[n]+2\right)$

$= \left(\frac{-1}{2}\right)^n \left(x[n-N]+2\right) \neq y[n-N]$
for all N

SYS A is NOT TI

* In sys B $v[n] = \sum_{k=1}^{n} z[k+1]^2 - z[k]$

$$= \sum_{k=1}^{n} x[k-N+1]^2 - x[k-N]$$

$$= \sum_{k'=1-N}^{n-N} x[k'+1]^2 - x[k']$$

SYS B is NOT TI $\neq y[n-N]$

c) * Sys A $y[n]$ depends on $x[k]$, $k \leq n$

$\Rightarrow$ IT IS Causal

* Sys B $y[n] = x^2[n+1] - x[n] + \sum_{k=1}^{n-1} x[k+1]^2 - x[k]$

for $n \geq 2$

$\Rightarrow$ it does depend on "Future" inputs

$\Rightarrow$ sys B is NOT causal

d) Naturally we mean by stability BIBO stability

* Sys A is NOT stable

for $x[n] = 1 \quad \forall n$ BI

$y[n] = 3 \left(\frac{-1}{2}\right)^n$ Not bounded for $n<0$!

* Sys B is NOT stable

for $x[n] = 2 \quad \forall n$ BI

$y[n] = \sum_{k=1}^{n} 2 = 2n \quad n \geq 1$ NOT BO.