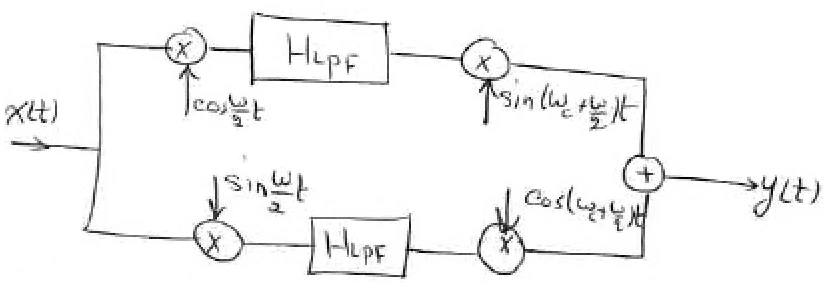
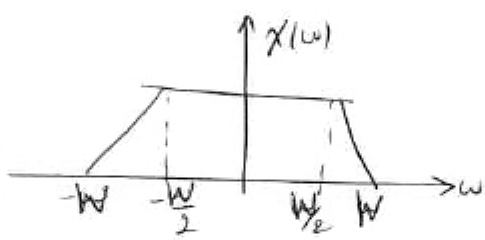
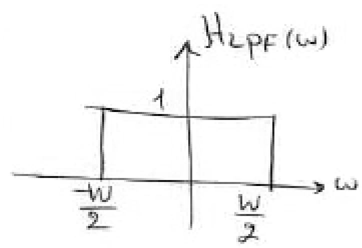
AMERICAN UNIVERSITY OF BEIRUT
Department of Electrical and Computer Engineering
EECE340- Signals and Systems –Summer 2011
Pset 4 Solutions

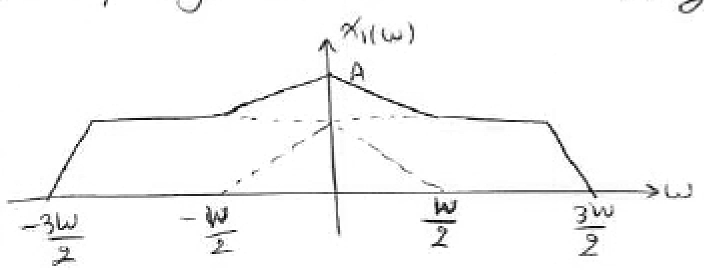
Problem 1

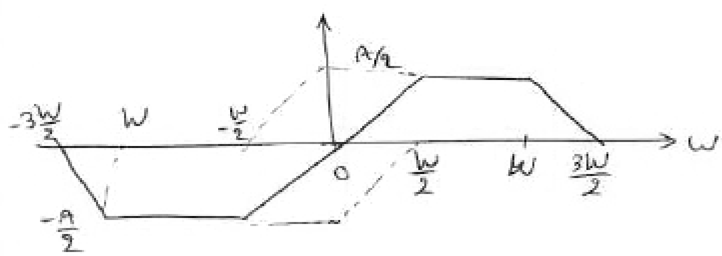




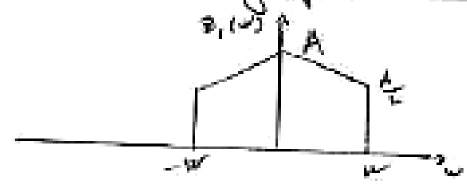


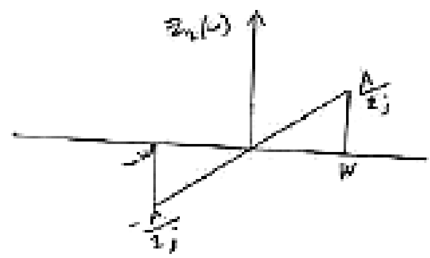
In frequency we can trace all the signals as follows:



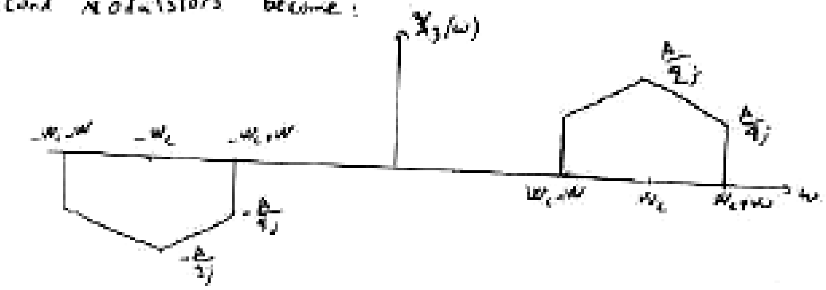


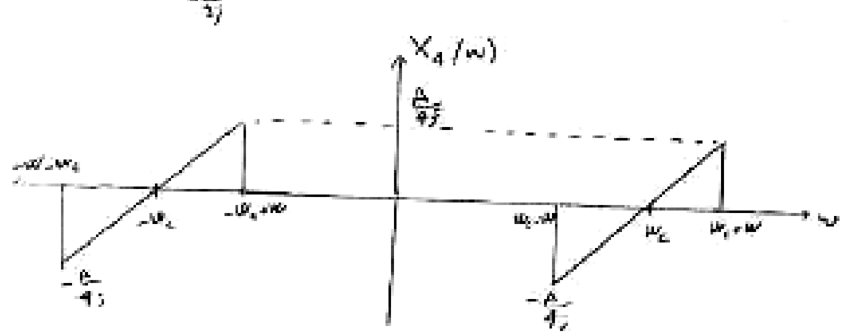
The output of the LP filters, denoted by $z_1(t)$ & $z_2(t)$ have the following spectra:



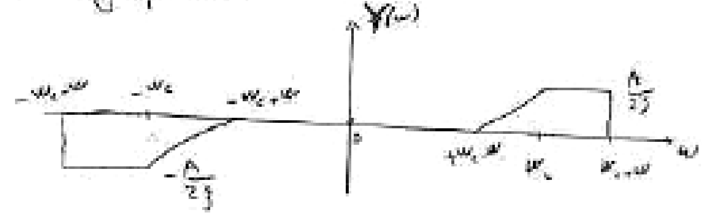


Now, taking $\omega_c >> W$ (for modulation), the output of the second modulators become:

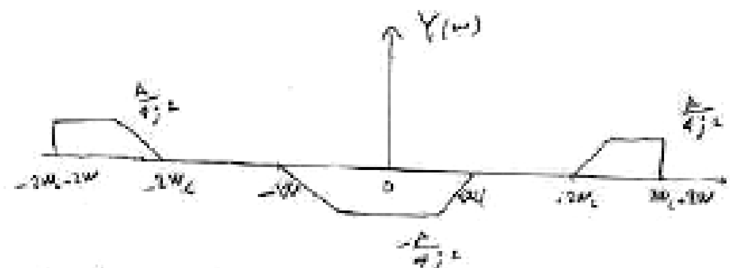




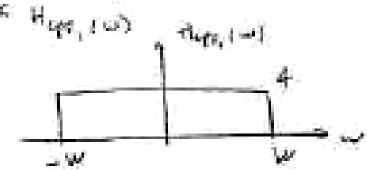
The Final output $y(t) = x_s(t) + x_v(t)$ Thus has the following spectrum.



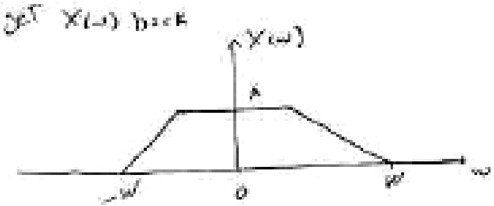
b) To retrieve $x(t)$ from $y(t)$, multiply by $\sin(\omega_c + W)t$



Then low pass filter with $H_{LPF_1}(\omega)$

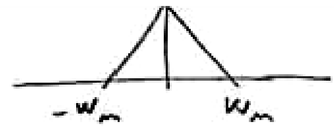


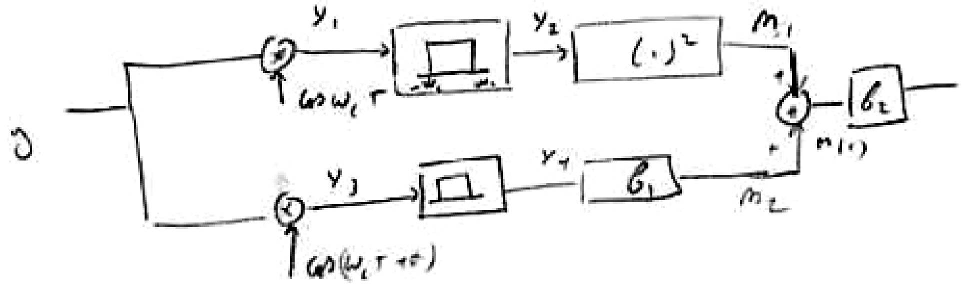
To get $X(\omega)$ back



Problem 2

$y(t) = [A + x(t)]\cos(w_c t + \theta)$





$$y_1(t) = y(t)\cos w_c t = (A + x(t))\cos w_c t \cos(w_c t + \theta)$$

$$= \frac{1}{2}(A + x(t))\cos(2w_c t + \theta) + \frac{1}{2}(A + x(t))\cos\theta$$

$y_2(t)$ will have only second term since first is centered at $2w_c$

$$y_2(t) = \frac{1}{2}(A + x(t))\cos\theta$$

$$\left[m_1(t) = \frac{1}{4}(A + x(t))^2\cos^2\theta\right]$$

$$y_3(t) = [A + x(t)]\cos(w_c t + \alpha)\cos(w_c t + \theta)$$

$$= \frac{1}{2}[A + x(t)]\cos(2w_c t + \alpha + \theta) + \frac{1}{2}[A + x(t)]\cos(\alpha - \theta)$$

Again, the first term is low-pass filtered in $y_4(t)$

$$y_4(t) = \frac{1}{2}[A + x(t)]\cos(\alpha - \theta)$$

choose g_1 such that $m_2(t) = \frac{1}{4}(A + x(t))^2\sin^2\theta$ $\Big\}$ $g_1(x) = x^2$, $\alpha = \pm 90^\circ$

Then $m(t) = \frac{1}{4}(A + x(t))^2(\cos^2\theta + \sin^2\theta)$

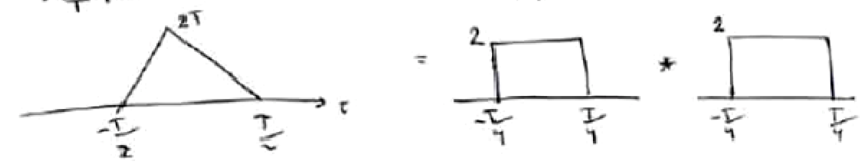
choose $g_2(x) = x^{\frac{1}{2}} \Rightarrow z(t) = \sqrt{m(t)} = \frac{1}{2}(A + x(t))$

9

Problem 3

(a) The non-periodic version of $x(t)$ is

$X_T(t)$ $\qquad\qquad$ $x_{T_1}(t) * x_{T_1}(t)$



Then:

$$x(t) = x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\Rightarrow X(\omega) = X_T(\omega) \cdot \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T} n\right)$$

$$X_T(\omega) = X_{T_1}(\omega) \cdot X_{T_1}(\omega) = \left(T \cdot \frac{\sin \frac{T}{4}\omega}{\frac{\omega T}{4}}\right)^2 = \left(\frac{4 \sin \frac{\omega T}{4}}{\omega}\right)^2$$

Thus:

$$X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} X_T\left(\frac{2\pi}{T} n\right) \delta\left(\omega - \frac{2\pi}{T} n\right)$$

$$= \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \left(4 \frac{\sin \frac{2\pi\, nT}{T \cdot 4}}{\frac{2\pi}{T} n}\right)^2 \delta\left(\omega - \frac{2\pi}{T} n\right)$$

$$X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} 4T^2 \frac{\sin^2 \frac{\pi}{2} n}{\pi^2 n^2} \delta\left(\omega - \frac{2\pi}{T} n\right)$$

$$\boxed{X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} 4T \frac{\sin^2 \frac{\pi n}{2}}{\pi^2 n^2} \delta\left(\omega - \frac{2\pi}{T} n\right)}$$

(b) $$C_{xn} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{0} 2T\left(1+\frac{2}{T}t\right)e^{-j\frac{2\pi}{T}nt}\,dt + \frac{1}{T}\int_{0}^{\frac{T}{2}} 2T\left(1-\frac{2}{T}t\right)e^{-j\frac{2\pi}{T}nt}\,dt$$

$$= 2\int_{-\frac{T}{2}}^{0} e^{-j\frac{2\pi}{T}nt}\,dt + \frac{4}{T}\int_{-\frac{T}{2}}^{0} t\,e^{-j\frac{2\pi}{T}nt}\,dt + 2\int_{0}^{\frac{T}{2}} e^{-j\frac{2\pi}{T}nt}\,dt - \frac{4}{T}\int_{0}^{\frac{T}{2}} t\,e^{-j\frac{2\pi}{T}nt}\,dt$$

using integration by parts, we have

$$\int t e^{\alpha t}\,dt = \frac{1}{\alpha}\left(t e^{\alpha t} - \frac{e^{\alpha t}}{\alpha}\right)$$

$$\Rightarrow C_{xn} = 2\left(\frac{-T}{j2\pi n}\right)\left[e^{-j\frac{2\pi}{T}nt}\right]_{-\frac{T}{2}}^{0} + \frac{4}{T}\left(\frac{-T}{j2\pi n}\right)\left[te^{-j\frac{2\pi}{T}nt} + \frac{T}{j2\pi n}e^{-j\frac{2\pi}{T}nt}\right]_{-\frac{T}{2}}^{0}$$

$$+ 2\left(\frac{-T}{j2\pi n}\right)\left[e^{-j\frac{2\pi}{T}nt}\right]_{0}^{\frac{T}{2}} - \frac{4}{T}\left(\frac{-T}{j2\pi n}\right)\left[te^{-j\frac{2\pi}{T}nt} + \frac{T}{j2\pi n}e^{-j\frac{2\pi}{T}nt}\right]_{0}^{\frac{T}{2}}$$

$$= \frac{-2T}{j2\pi n}\left[1 - e^{j\pi n}\right] - \frac{4}{j2\pi n}\left[\frac{T}{2}e^{j\pi n} + \frac{T}{j2\pi n}\left(1-e^{-j\pi n}\right)\right]$$

$$- \frac{2T}{j2\pi n}\left[e^{-j\pi n} - 1\right] + \frac{4}{j2\pi n}\left[\frac{T}{2}e^{-j\pi n} + \frac{T}{j2\pi n}\left(e^{-j\pi n}-1\right)\right]$$

$$= \frac{2T}{j2\pi n}\left[e^{j\pi n} - e^{-j\pi n}\right] - \frac{4T}{j2\pi n}\left[e^{j\pi n} - e^{-j\pi n}\right] - \frac{8T}{(j2\pi n)^2}$$

$$+ \frac{4T}{(j2\pi n)^2}\left[e^{j\pi n} + e^{-j\pi n}\right]$$

$$\Rightarrow C_{xn} = \cancel{\frac{2T}{\pi n}\sin \pi n} - \cancel{\frac{2T}{\pi n}\sin \pi n} + \frac{8T}{4\pi^2 n^2} - \frac{4T}{2\pi^2 n^2}\cos \pi n$$

$$\Rightarrow \boxed{C_{xn} = \frac{2T}{\pi^2 n^2}\left(1-\cos \pi n\right)}$$

using $1-\cos 2\alpha = 2\sin^2\alpha$, Then:

$$\boxed{C_{xn} = \frac{4T}{\pi^2 n^2}\sin^2 \frac{\pi}{2} n}$$

(c) $C_{xn} = \frac{1}{T} X_T\left(\frac{2\pi}{T} n\right)$ is verified as can be seen in parts (a) + (b):

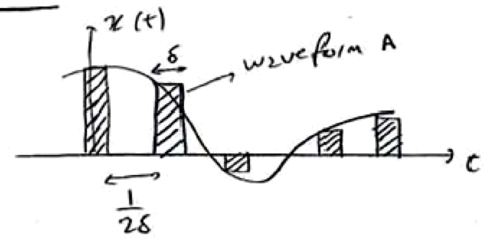
part (a) : $\frac{1}{T} X_T\left(\frac{2\pi}{T} n\right) = \frac{1}{T} \frac{4T^2 \sin^2 \frac{\pi}{2} n}{\pi^2 n^2} = 4T\frac{\sin^2 \frac{\pi}{2} n}{\pi^2 n^2}$

part (b) : $C_{xn} = 4T \frac{\sin^2 \frac{\pi n}{2}}{\pi^2 n^2}$

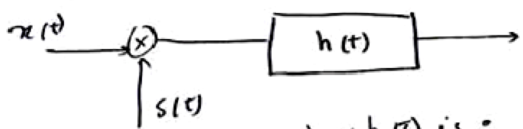
Thus $X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_{xn}\,\delta\left(\omega - \frac{2\pi}{T} n\right)$ as expected.

Problem 4

(a)

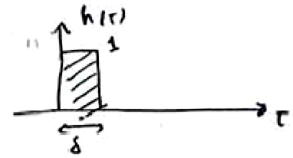


pulse waveform A can be viewed as follows

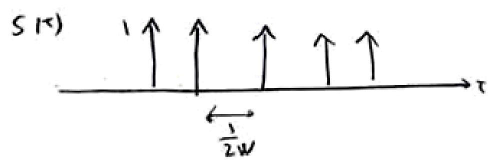


$y(t) = [s(t)\, x(t)] * h(t)$

where $h(t)$ is :



and $s(t)$ is $\sum_{k=-\infty}^{\infty} \delta(t - \frac{k}{2W}) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$ $\quad T_s = \frac{1}{2W}$



In Frequency, we have

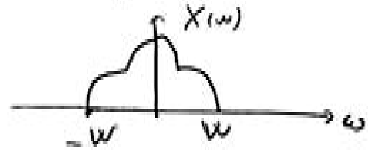
$$Y(\omega) = \frac{1}{2\pi}[S(\omega) * X(\omega)] \cdot H(\omega)$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi}{T_s}) * X(\omega)\right] \cdot \overbrace{\left[e^{-\frac{j\delta}{2}} \frac{\sin\frac{\delta\omega}{2}}{\frac{\omega\delta}{2}} \cdot \delta\right]}^{H(\omega)}$$

$$Y(\omega) = \left[\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - \frac{2\pi k}{T_s})\right] \cdot \delta e^{-\frac{j\delta}{2}} \operatorname{sinc}(\frac{\delta\omega}{2})$$

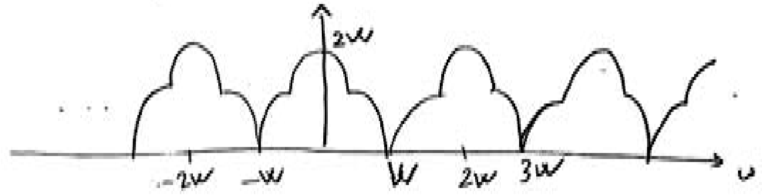
$$= 2W \sum_{k=-\infty}^{\infty} X(\omega - 4\pi k W)\ \delta e^{-\frac{j\delta}{2}} \operatorname{sinc}(\frac{\delta\omega}{2})$$

GRAPHICALLY: let $X(\omega)$ be band limited as follows:

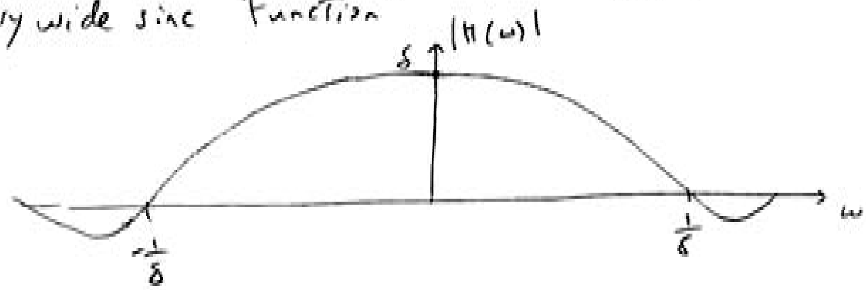


Then:

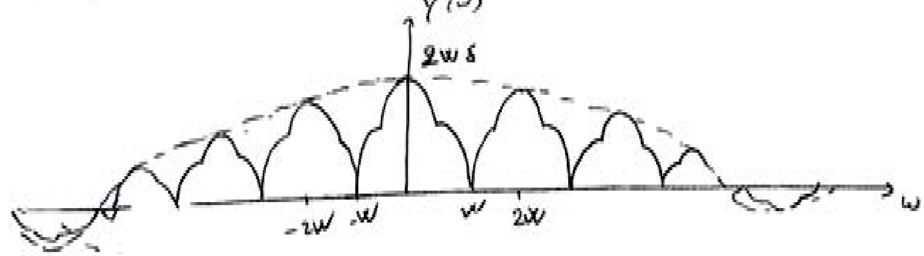
$x(t) \cdot s(t) \rightarrow \frac{1}{2\pi} X(\omega) * S(\omega)$



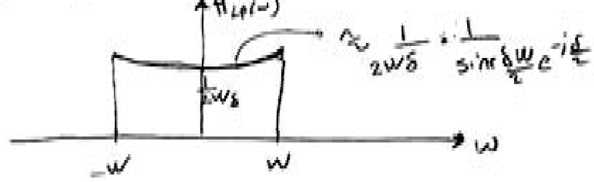
Now, since $h(t)$ is a short pulse $\delta << \frac{1}{2W}$, Then $H(\omega)$ is a very wide sinc Function



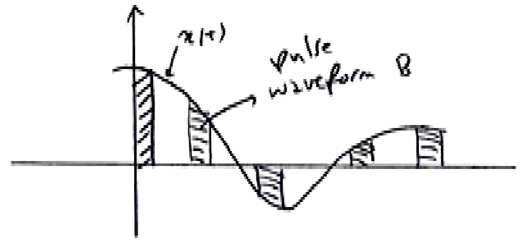
Therefore $Y(\omega)$ has $H(\omega)$ as an envelope



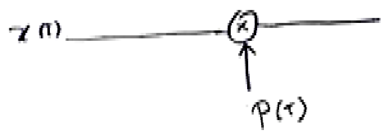
To Recover $x(t)$, We need To use a low pass Filter with a cutoff at W & whose magnitude has a shape That reverses The effect of $H(\omega)$ between $-W$ & W. That is



(b)

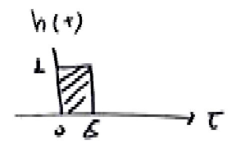


Pulse waveform B can be viewed as simply multiplying $x(t)$ by a periodic square wave:

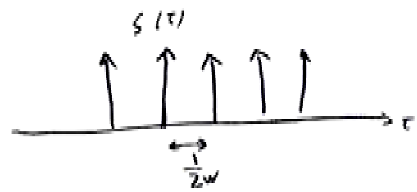


$y(t) = x(t)\,p(t)$

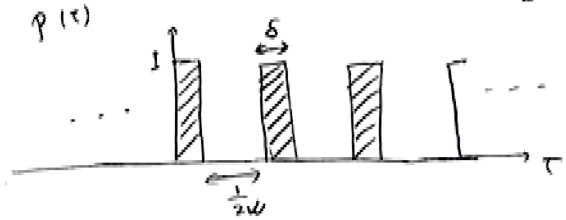
$p(t) = s(t) * h(t)$ where



and



That is $p(t)$



In the frequency domain we have:

$$Y(\omega) = \frac{1}{2\pi}\left[P(\omega) * X(\omega)\right] = \frac{1}{2\pi}\left[S(\omega)\,H(\omega)\right] * X(\omega)$$

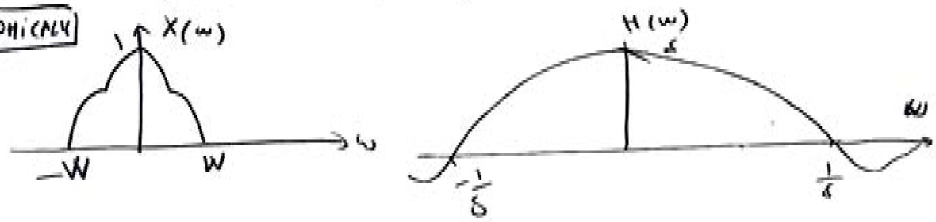
$$Y(\omega) = \frac{1}{2\pi}\left[\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T_s}\right)\cdot H(\omega)\right] * X(\omega)$$

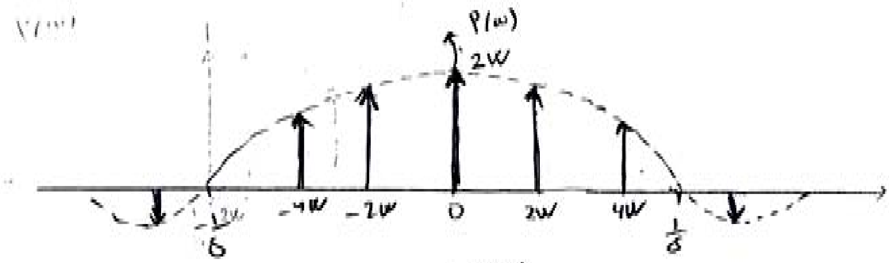
$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} H\left(\frac{2\pi}{T_s}k\right)\delta\left(\omega - \frac{2\pi}{T_s}k\right) * X(\omega)$$

$$= 2W \sum_{k=-\infty}^{\infty} H\left(\frac{2\pi}{T_s}k\right) X\left(\omega - \frac{2\pi}{T_s}k\right)$$

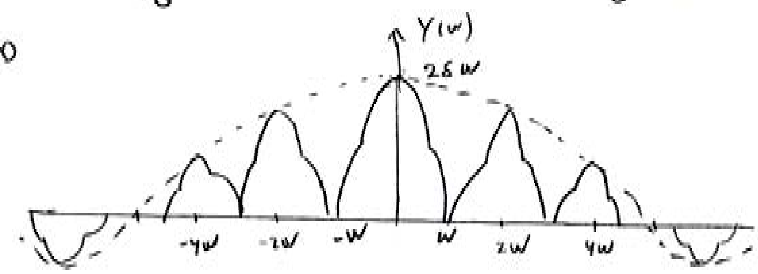
$$= 2W \sum_{k=-\infty}^{\infty} \delta e^{-jk} \operatorname{sinc}\left(\frac{\pi\delta}{T_s}k\right) X\left(\omega - \frac{2\pi}{T_s}k\right) \quad ⊛$$

$\boxed{\text{GRAPHICALLY}}$





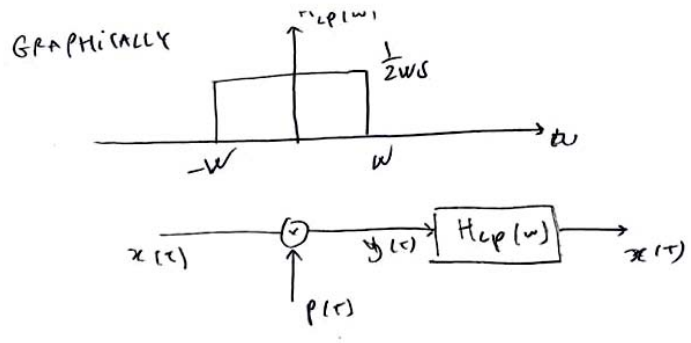
AND



To Recover $x(t)$ we can see, we can see that in equation ⊛ above, we have at the origin (and in figure) $2W \cdot H(0)\, X(\omega) = 2W\delta\, X(\omega)$

$\Rightarrow$ use a Low pass Filter with bandwidth W & magnitude $\frac{1}{2W\delta}$

4

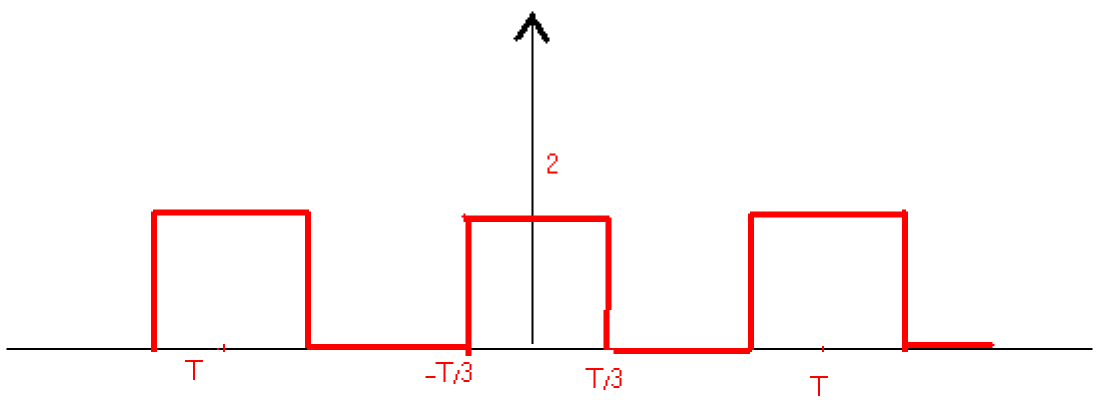


Problem 5

$$\delta^{'}(t) = \delta(t) + 1$$

a) Let : $=> \delta^{'}(w) = \delta(w) + 2\pi\delta(w)$ and let's plot $\delta^{'}(w)$, then to go back

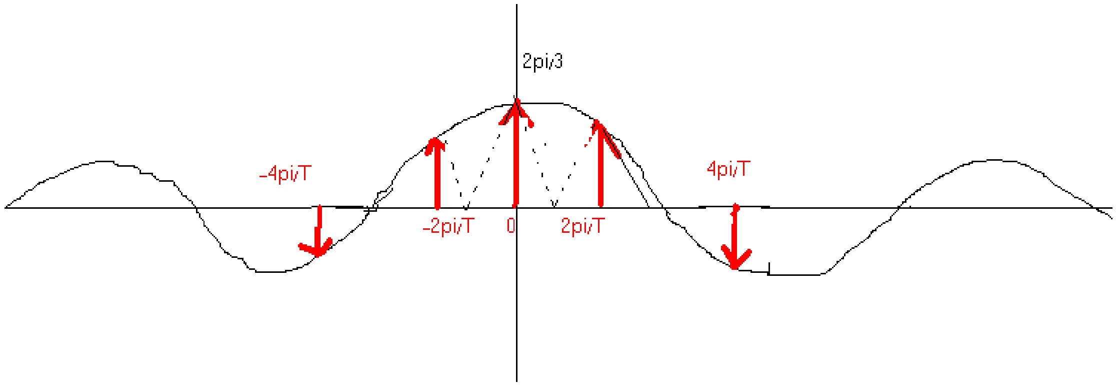
$\delta(w)$, we simply modify the amplitude at w=0, and we make it $\delta^{'}(0) - 2\pi$



$$\delta'_T(t) = 2\,\mathrm{Re}\,ct(\frac{t}{\frac{2T}{3}})$$

$$=> \delta'_T(w) = 2.\frac{2T}{3}\sin c(\frac{wT}{3}) = \frac{4T}{3}\sin c(\frac{wT}{3})$$

$$=> \delta'(w) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta'_T(\frac{2\pi k}{T})\delta(w - \frac{2\pi}{T}k)$$



$$At w = 0, \delta'(0) = \frac{2\pi}{T}.\frac{4T}{3} = \frac{8\pi}{3}$$

$$=> \delta(0) = \frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$$

<u>No aliasing occurs if :</u>

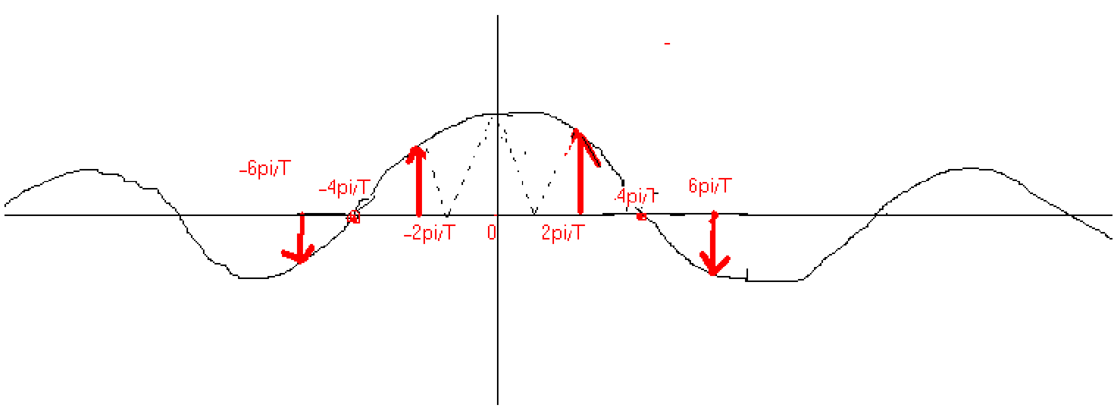
$$\frac{2\pi}{T} - w_m > w_m \;=> T \leq \frac{\pi}{w_m}$$

b) we do the same as above, but now replacing Δ by T/4.

$$\delta'_T(t) = 2\,\mathrm{Re}\,ct\left(\frac{t}{\frac{2T}{4}}\right)$$

$$=> \delta'_T(w) = 2.\frac{2T}{4}\sin c\left(\frac{wT}{4}\right) = T\sin c\left(\frac{wT}{4}\right)$$

$$=> \delta'(w) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta'_T\left(\frac{2\pi k}{T}\right)\delta\left(w - \frac{2\pi}{T}k\right)$$

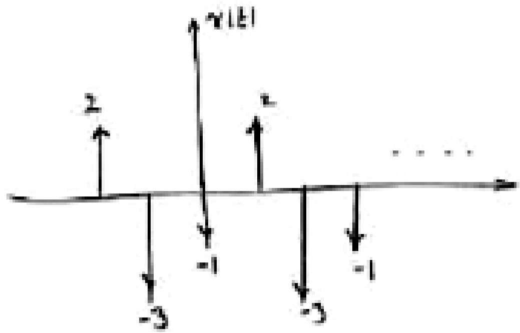


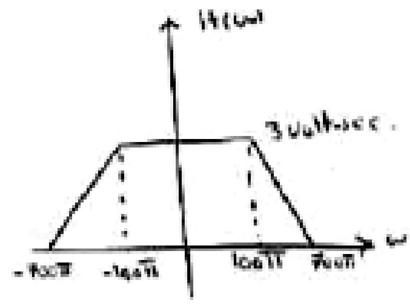
$$At w = 0, \delta'(0) = \frac{2\pi}{T}\sin c(\frac{2\pi kT}{T.4}).T = 2\pi$$
$$=> \delta(0) = 2\pi - 2\pi = 0$$

No aliasing occurs if:

$$\frac{2\pi}{T} - w_m \geq -\frac{2\pi}{T} + w_m$$
$$=> \frac{4\pi}{T} \geq 2w_m$$
$$=> T \leq \frac{2\pi}{w_m}$$

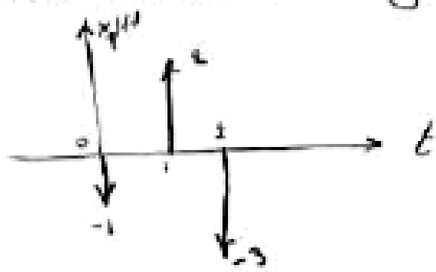
Problem 6





x(t) is a periodic signal.

let $x_T(t)$ be the following figure:



$=> x_T(t) = -\delta(t) + 2\delta(t-1) - 3\delta(t-2)$.

$x(t) = x_T(t) * \sum \delta(t-nT)$.

where $T = 3\times 10^{-3}$ sec.

$$\Rightarrow X(\omega) = \cancel{X_T(\omega)\,\delta(\omega)} = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} X_T\left(\frac{2\pi n}{T}\right) \delta\left(\omega - \frac{2\pi n}{T}\right).$$

but $X_T(\omega) = -1 + 2e^{-j\omega} - 3e^{-j2\omega}$

$$= -1 + 2e^{-\frac{jn\pi 2000}{3}} - 3e^{-\frac{jn\pi 4000}{3}}.$$

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

$$= \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} X_T\left(\frac{2\pi n}{T}\right) H\left(\frac{2n\pi}{T}\right) \delta\left(\omega - \frac{2\pi n}{T}\right).$$

$$= \frac{2000\pi}{3} \sum_{n=-\infty}^{\infty} X_T\left(\frac{2000\pi n}{3}\right) H\left(\frac{2000\pi n}{3}\right) \delta\left(\omega - \frac{2000\pi n}{3}\right)$$

$$H_1(\omega) = \begin{cases} 0 & \omega < -700\pi \\ \frac{\omega}{200\pi} + \frac{7}{2} & -700\pi < \omega < -100\pi \\ 3 & -100\pi < \omega < 100\pi \\ \frac{-\omega}{200\pi} + \frac{7}{2} & 100\pi < \omega < 700\pi \\ 0 & \omega > 700\pi \end{cases}$$

$n=0 \Rightarrow H_1(0) = 3$

$n=1 \Rightarrow H_1\left(\frac{2000\pi}{3}\right) = \frac{1}{6}$.

$n=2 \Rightarrow H_1\left(\frac{4000\pi}{3}\right) = 0$

$n=-1 \Rightarrow H_1\left(-\frac{2000\pi}{3}\right) = 1/6$

$n=-2 \Rightarrow H_1\left(-\frac{4000\pi}{3}\right) = 0.$

$$\Rightarrow H_1\left(\frac{2000n\pi}{3}\right) = \begin{cases} 0 & n \le -2 \\ 1/6 & n = -1 \\ 3 & n = 0 \\ 1/6 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

2

$$n=0 \Rightarrow X_1(\omega) = -2$$
$$n=1 \Rightarrow X_1(\omega) = -1 + 2e^{-j\frac{2000\pi}{3}} - 3e^{-j\frac{4000\pi}{3}}$$
$$n=-1 \Rightarrow X_1(\omega) = -1 + 2e^{-j\frac{2000\pi}{3}} - 3e^{-j\frac{4000\pi}{3}}.$$

$$Y(\omega) = \frac{2000\pi}{3}\Big[-6\delta(\omega) + \frac{1}{6}X_1\Big(\frac{2000\pi}{3}\Big)\delta\Big(\omega - \frac{2000\pi}{3}\Big) + \frac{1}{6}X_1\Big(-\frac{2000\pi}{3}\Big)\delta\Big(\omega + \frac{2000\pi}{3}\Big)\Big]$$

$$y(t) = -\frac{6000}{3} + \frac{1000\pi}{9}\Big[-1 + 2e^{-j\frac{2000\pi}{3}} - 3e^{-j\frac{4000\pi}{3}}\Big]e^{j\frac{2000\pi t}{3}}$$
$$+ \frac{1000\pi}{9}\Big[-1 + 2e^{j\frac{2000\pi}{3}} - 3e^{j\frac{4000\pi}{3}}\Big]e^{-j\frac{2000\pi}{3}t}$$

$$= -2000 + \frac{2000\pi}{9}\Big[-\cos\frac{2000\pi t}{3} + 2\cos\Big[\frac{2000\pi}{3}(t-1)\Big] - 3\cos\Big[\frac{2000\pi t}{3}(t-2)\Big]\Big].$$