AMERICAN UNIVERSITY OF BEIRUT
Department of Electrical and Computer Engineering
EECE340 Signals and Systems- Summer 2011
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Problem Set 5 Solutions

Problem 1

a)

$$x[n] = \left(\frac{1}{3}\right)^{|n|} u[-n-2]$$

$$\Rightarrow X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}$$

$$= \sum_{k=-\infty}^{-2} \left(\frac{1}{3}\right)^{-k} e^{-j\Omega k}$$

$$= \sum_{k=2}^{\infty} \left(\frac{1}{3} e^{j\Omega}\right)^{k}$$

$$= \frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1 - \frac{1}{3}e^{j\Omega}$$

$$= \frac{\frac{1}{9}e^{2j\Omega}}{1-\frac{1}{3}e^{j\Omega}}$$

b)

$$x[n] = \sin\left(\frac{\pi n}{2}\right) + \cos(n)$$

$$\Rightarrow X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}$$

$$= \frac{\pi}{j}\left(\sum_{k=-\infty}^{\infty} \delta\left(w-\frac{\pi}{2}-2\pi k\right) - \sum \delta\left(w-\frac{\pi}{2}+2\pi k\right)\right) + \pi\left(\sum \delta(w-1-2\pi k) - \sum \delta(w-1+2\pi k)\right)$$

c)

$$X(e^{j\Omega}) = 4e^{2j\Omega} - e^{j\Omega} + 5 + 3e^{-3j\Omega} - 16e^{-12j\Omega}$$

$$\Rightarrow x[n] = 4\delta[n+2] - \delta[n+1] + 5\delta[n] + 3\delta[n-3] - 16\delta[n-12]$$

$$(u\sin g --- that :: DTFT^{-1}(e^{j\Omega n^0}) = \delta[n-n^0])$$

d)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega}d\Omega$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi/2} X(e^{j\Omega})e^{j\Omega}d\Omega + \int_{-\pi/4}^{\pi/4} X(e^{j\Omega})e^{j\Omega}d\Omega + \int_{\pi/2}^{\pi} X(e^{j\Omega})e^{j\Omega}d\Omega\right)$$

$$= \frac{1}{2j\pi n}[e^{-j\frac{\pi}{2}n} - e^{-j\pi n}] + \frac{1}{2j\pi n}[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}] + \frac{1}{2j\pi n}[e^{j\pi n} - e^{j\frac{\pi}{2}n}]$$

$$= \frac{1}{2j\pi n}[e^{-j\frac{\pi}{2}n} - e^{j\frac{\pi}{2}n}] + \frac{1}{2j\pi n}[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}] + \frac{1}{2j\pi n}[e^{j\pi n} - e^{-j\pi n}]$$

$$= \frac{-1}{\pi n}\sin(\frac{\pi}{2}n) + \frac{1}{\pi n}\sin(\frac{\pi}{4}n)$$

e)

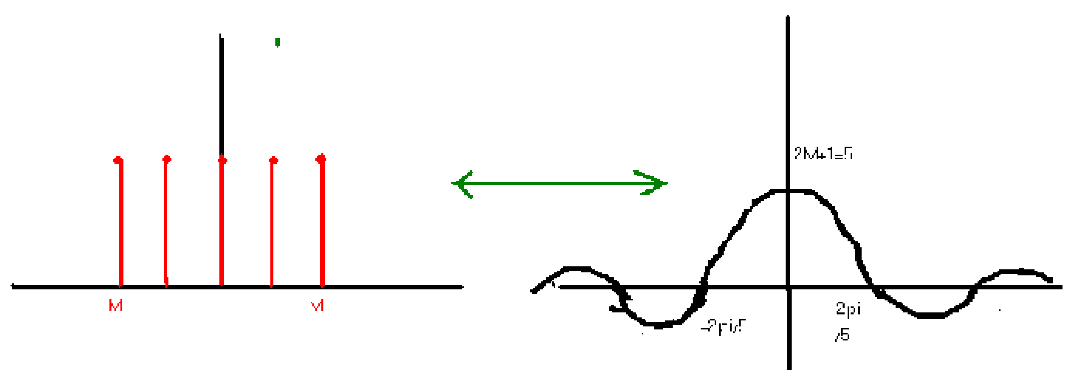
$$X(e^{j\Omega}) = \frac{1+3e^{-j\Omega}}{1+\frac{1}{4}e^{-j\Omega}}$$

$$\Rightarrow x[n] = \left(\frac{-1}{4}\right)^{n}u[n] + 3\left(\frac{-1}{4}\right)^{n-1}u[n-1]$$

<u>Problem 2</u>

$$h[n] = \frac{\sin(\frac{\pi n}{3})}{\pi n}$$

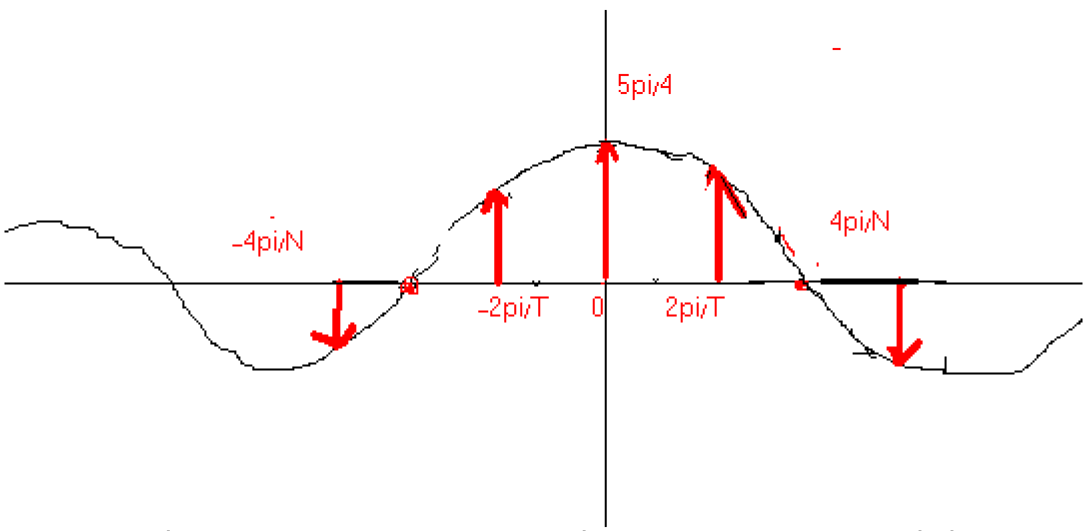
xt[n] is the non periodic version of x[n]…



$$then X_T(e^{j\Omega}) = \frac{\sin\frac{(2M+1)\Omega}{2}}{\sin\frac{\Omega}{2}} = \frac{\sin(\frac{5\Omega}{2})}{\sin\frac{\Omega}{2}}$$

$$=> X(e^{j\Omega}) = \frac{2\pi}{N}\sum_{n=-\infty}^{\infty} X_T(e^{j\Omega})\Big| \delta\Omega - \frac{2\pi}{N}n) ---------- N=8--\Omega = \frac{2\pi}{N}$$

$$= \frac{2\pi}{8}\sum_{n=-\infty}^{\infty} \frac{\sin(\frac{5}{2}\frac{2\pi}{8})}{\sin\frac{\Omega}{2}\frac{2\pi}{8}}\Bigg| \delta(\Omega - \frac{2\pi}{8}n)$$



After passing through the low pass filter H,the only surviving terms are Ω between
-pi/3 and pi/3.i.e n=0, n=1, n=-1

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

$$= \frac{5\pi}{4}\delta(w) + \frac{\pi}{4}\frac{\sin\frac{5\pi}{8}}{\sin\frac{\pi}{8}}[\delta(w+\frac{\pi}{4}) + \delta(w-\frac{\pi}{4})]$$

over.one.period

$$=> y[n] = \frac{5}{8} + \frac{1}{4}\frac{\sin\frac{5\pi}{8}}{\sin\frac{\pi}{8}}\cos(\frac{\pi}{4}n)$$

Problem 3

- $x[n] = \sin\left(\frac{\pi}{8}n\right) - 2\cos\left(\frac{\pi}{4}n\right)$

$$\Rightarrow \tilde{X}(e^{j\Omega}) = \frac{\pi}{j}\left[\sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{\pi}{8} - 2\pi k\right) - \delta\left(\Omega + \frac{\pi}{8} - 2\pi k\right)\right]$$

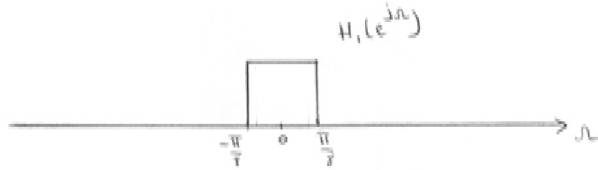
$$- 2\pi\left[\sum_{-\infty}^{\infty} \delta\left(\Omega - \frac{\pi}{4} - 2\pi k\right) + \delta\left(\Omega + \frac{\pi}{4} - 2k\pi\right)\right]$$

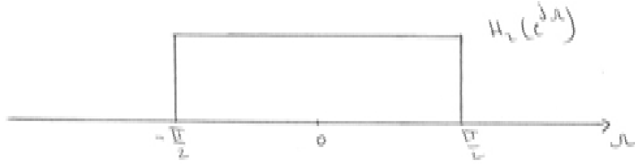
- $h[n] = \dfrac{\sin\left(\frac{\pi}{8}n\right)\sin\left(\frac{\pi}{2}n\right)}{\pi n}$

$$\Rightarrow H(e^{j\Omega}) = \left[F\left\{\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\right\} * F\left\{\frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}\right\}\right]\left(\frac{1}{2\pi}\right)$$

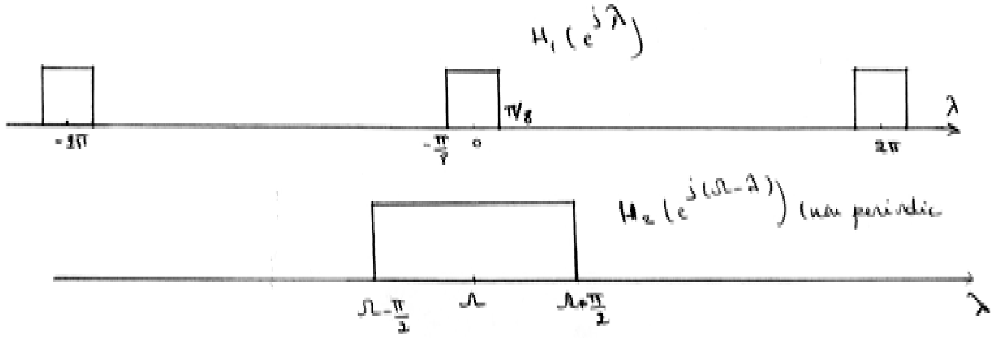
$$= \left[H_1(e^{j\Omega}) \circledast H_2(e^{j\Omega})\right]\left(\frac{1}{2\pi}\right)$$

where:





To do circular convolution, choose one to be non periodic and perform regular convolution over one period



① start at $\Omega = -\pi \Rightarrow \Omega + \frac{\pi}{2} = -\frac{\pi}{2} \Rightarrow$ no intersection $\Rightarrow H_1 * H_2 = 0$

② for $\Omega + \frac{\pi}{2} > -\frac{\pi}{8}$ & $\Omega + \frac{\pi}{2} < \frac{\pi}{8}$

$$\Rightarrow -\frac{5}{8}\pi < \Omega < -\frac{3\pi}{8} \quad : H_1 * H_2 = \int_{-\frac{\pi}{8}}^{\Omega+\frac{\pi}{2}} d\lambda = \Omega + \frac{5}{8}\pi$$

③ for $\Omega + \frac{\pi}{2} > \frac{\pi}{8}$ & $\Omega - \frac{\pi}{2} < -\frac{\pi}{8}$

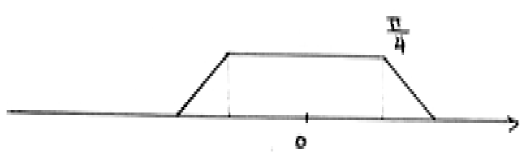
$$\Rightarrow -\frac{3\pi}{8} < \Omega < \frac{3\pi}{8} \quad : H_1 * H_2 = \frac{\pi}{4}$$

④ for $\Omega - \frac{\pi}{2} > -\frac{\pi}{8}$ and $\Omega - \frac{\pi}{2} < \frac{\pi}{8}$

$$\Rightarrow \frac{3\pi}{8} < \Omega < \frac{5\pi}{8} \quad : H_1 * H_2 = \int_{\Omega-\frac{\pi}{2}}^{\frac{\pi}{8}} d\lambda = \frac{5\pi}{8} - \Omega$$

⑤ for $\Omega > \frac{5\pi}{8} \Rightarrow$ no intersection $\Rightarrow H_1 * H_2 = 0$

thus, we have : $H_1 \circledast H_2$ looks as follows:



Accordingly, we get $\tilde{H}(e^{j\Omega}) = \frac{1}{2\pi} H_1 \circledast H_2$ as follows

Problem 4

(i) We have

$$X(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}}.$$

Therefore,

$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1}{1-\frac{3}{4}e^{-j\omega}}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{-2}{1-\frac{1}{2}e^{-j\omega}} + \frac{3}{1-\frac{3}{4}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = 3\left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n].$$

(ii) We have

$$X(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}.$$

Therefore,

$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{3}{(1-\frac{1}{4}e^{-j\omega})^2} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] - 3(n+1)\left(\frac{1}{4}\right)^n u[n].$$

(iii) $x(n) = (-1)^n u(n) = e^{j\pi n} u(n)$

if $y(n) = u(n) \longrightarrow Y(e^{j\Omega}) = \frac{1}{1-e^{-j\Omega}} + \pi \sum_k \delta(\Omega - 2\pi k)$

$x(n) = e^{j\pi n} y(n) \Rightarrow X(e^{j\Omega}) = Y(e^{j(\Omega-\pi)})$

$\therefore X(e^{j\Omega}) = \frac{1}{1-e^{-j(\Omega-\pi)}} + \pi \sum_k \delta(\Omega - \pi - 2\pi k))$

$= \frac{1}{1+e^{-j\Omega}} + \pi \sum_k \delta(\Omega - \pi(2k+1))$

$h(n) = (\frac{1}{2})^n u(n) \Rightarrow H(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$

output $z(n) = h(n) * x(n) \Rightarrow Z(e^{j\Omega}) = \left(\frac{1}{1-\frac{1}{2}e^{-j\Omega}}\right)\left(\frac{1}{1+e^{-j\Omega}} + \pi \sum \delta(\Omega - (2k+1)\pi)\right)$

$\therefore Z(e^{j\Omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\Omega})(1+e^{-j\Omega})} + \pi \sum_k \frac{1}{1+\frac{1}{2}e^{-j(2k+1)\pi}} \delta(\Omega - (2k+1)\pi)$

$= \frac{A}{1+e^{-j\Omega}} + \frac{B}{1-\frac{1}{2}e^{-j\Omega}} + \pi \sum_k \frac{2}{3} \delta(\Omega - (2k+1)\pi)$

where $A = \frac{2}{3}$ & $B = \frac{1}{3}$

$\therefore Z(e^{j\Omega}) = \frac{\frac{1}{3}}{1-\frac{1}{2}e^{-j\Omega}} + \frac{2}{3}\left[\frac{1}{1+e^{-j\Omega}} + \pi \sum_k \delta(\Omega - (2k+1)\pi)\right]$

$\Rightarrow z(n) = \frac{1}{3}(\frac{1}{2})^n u(n) + \frac{2}{3}(-1)^n u(n)$

Problem 5

(a) Since the two systems are cascaded, the frequency response of the overall system is

$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\ &= \frac{2 - e^{-j\omega}}{1 + \frac{1}{8}e^{-j3\omega}} \end{aligned}$$

Therefore, the Fourier transforms of the input and output of the overall system are related by

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2 - e^{-j\omega}}{1 + \frac{1}{8}e^{-j3\omega}}.$$

Cross-multiplying and taking the inverse Fourier transform, we get

$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1].$$

b) We may rewrite the overall frequency response as

$$H(e^{j\omega}) = \frac{4/3}{1 + \frac{1}{2}e^{-j\omega}} + \frac{(1 + j\sqrt{3})/3}{1 - \frac{1}{2}e^{j120}e^{-j\omega}} + \frac{(1 - j\sqrt{3})/3}{1 - \frac{1}{2}e^{-j120}e^{-j\omega}}.$$

Taking the inverse Fourier transform we get

$$h[n] = \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1 + j\sqrt{3}}{3}\left(\frac{1}{2}e^{j120}\right)^n u[n] + \frac{1 - j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j120}\right)^n u[n].$$

Problem 6

The code is given below.
For x(t)=cos(2pi 55 t)

Theoretically, X(w) should have a peak at 55 Hz. Since we are effectively plotting the sampled version of x(t) (called xs(t)), we should observe Xs(w) which is periodic with period 2*pi*Fs. Plotting things in Hz and not rad/sec, we should then have a peak at 55 Hz, and another at 200-55=145 Hz.

Now the signal is sampled and also we have only a finite length segment of it (for 0.5 sec). Therefore Xs(w) is really the original signal convolved with a sinc function in frequency (since in time it is windowed by a rectangular pulse).

Practically, taking a 100 pt DFT will give a sampled version of the signal X(w) at intervals of Fs/N =200/100 = 2Hz, starting from 0 Hz. Accordingly, the DFT will not "hit" the peak at 55 Hz , but rather give samples around it.

On the other hand, taking a 200 pt DFT will give a sampled version of the signal X(w) at intervals of Fs/N =200/200 = 1Hz, starting from 0 Hz. That is we are having a better sampled version of Xs(w) and should thus look more like the theoretical result. This time, the DFT will "hit" the peak at 55 Hz , as is shown in the figure.

For x(t)=cos(2pi 110 t),

Theoretically, X(w) should have a peak at 110 Hz. Importantly, the sampled signal now is aliased since Fs=200 Hz, which is less the Nyquist rate. Therefore, Xs(w) will have another peak at 200-110 Hz=90 Hz.

Again, we have a finite window of the data, so Xs(w) should be convolved with a sinc in frequency (since it is effectively multiplied with a rectangular function in time) .

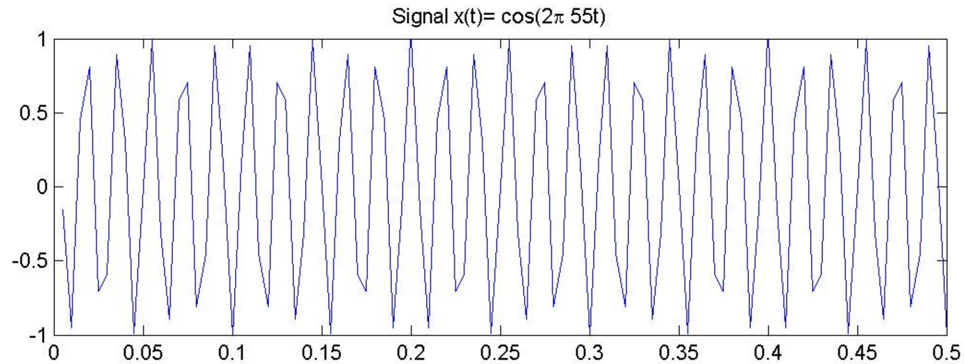
Practically, taking a 100 pt DFT will give a sampled version of the signal X(w) at intervals of Fs/N =200/100 = 2Hz, starting from 0 Hz. Accordingly, the DFT now "hits" the peaks at 90 and 110 Hz , but one might be fooled into thinking that the frequency is actually 90 Hz.

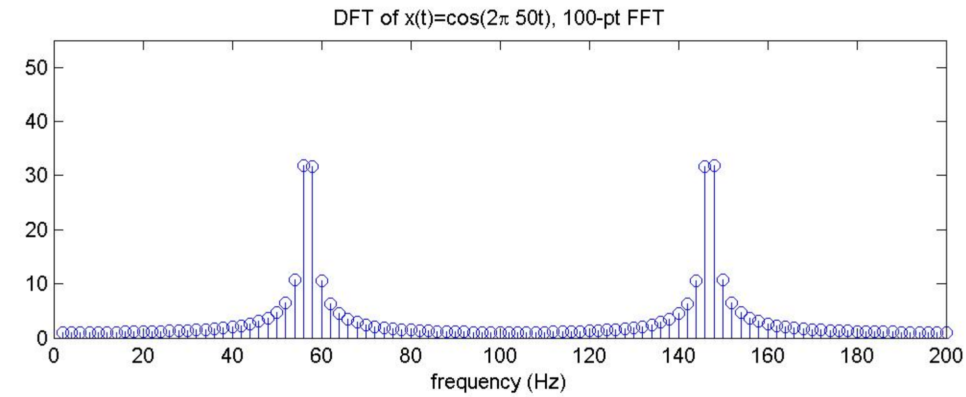
Taking the 200 pt DFT will give closer samples in frequency and the shape of X(w) is more apparent.

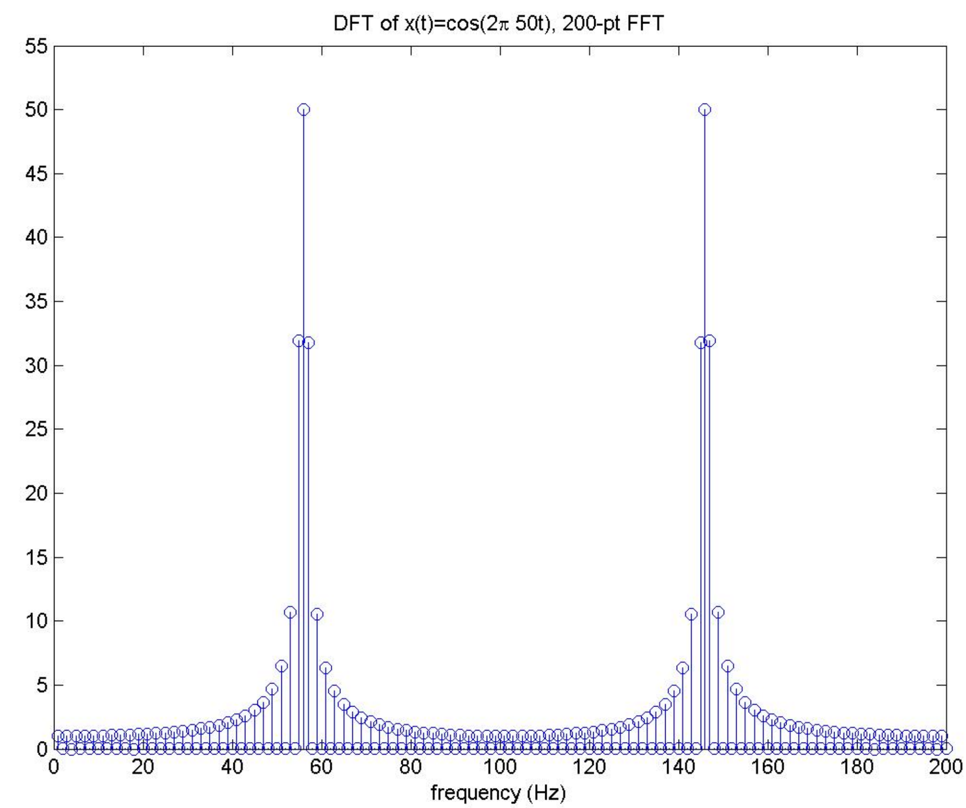
Finally, I also added a 2048 pt DFT which will give a much finer representation of the signal Xs(w) close to what one expects theoretically.

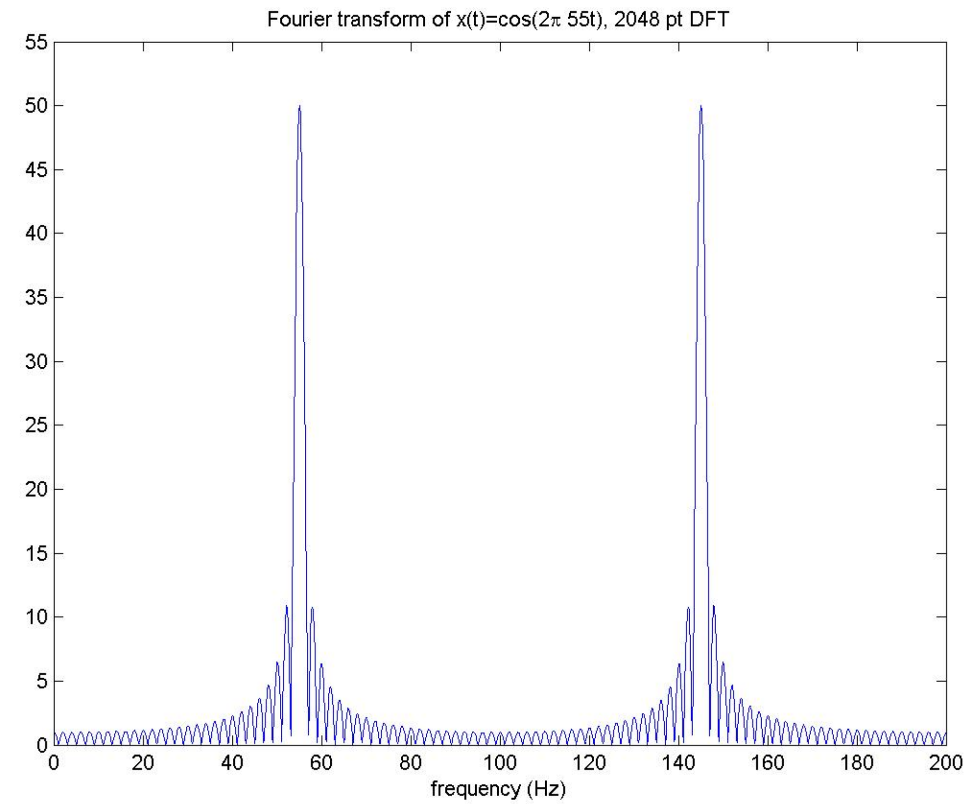
```
Fs=200; %Hz

%CREATE THE TIME VECTOR

t=1/Fs:1/Fs:0.5;

%CREATE THE SIGNAL X(t)
x=cos(2*pi*55*t);
%.*(1+cos(2*pi*30*t));
figure(1);
subplot(211);
plot(t,x);title('Signal x(t)= cos(2\pi 55t)')

N=length(t);
%FIND THE DFT X[k]
fx1=abs(fft(x));

%CREATE THE CORRESPONDING FREQUENCY VECTOR: K-> 2*PI*fS/N
fs1=(1/N:1/N:1)*Fs;

%PLOT AS  A DISCRETE SAMPLED FUNCTION
subplot(212); stem(fs1,fx1);

%add some labeling and make it look nice
title('DFT of x(t)=cos(2\pi 50t), 100-pt FFT');
xlabel('frequency (Hz)');axis([0 200 0 55]);


%NOW INCREASE THE SIZE OF THE DFT TO 200 POINTS AND REPEAT

N1=2*N;

figure(2);

fx2=abs(fft(x,N1)); fs2=(1/N1:1/N1:1)*Fs;
stem(fs2,fx2);
%add some labeling and make it look nice
title('DFT of x(t)=cos(2\pi 50t), 200-pt FFT');
xlabel('frequency (Hz)'); axis([0 200 0 55]);


figure(3);
N2=2048;
fx2=abs(fft(x,N2)); fs2=(1/N2:1/N2:1)*Fs;
plot(fs2,fx2);
title('Fourier transform of x(t)=cos(2\pi 55t), 2048 pt DFT');
xlabel('frequency (Hz)'); axis([0 200 0 55]);

%NOW LET FO=110 HZ AND REPEAT
x=cos(2*pi*110*t);


figure(4);
subplot(211); plot(t,x);
```

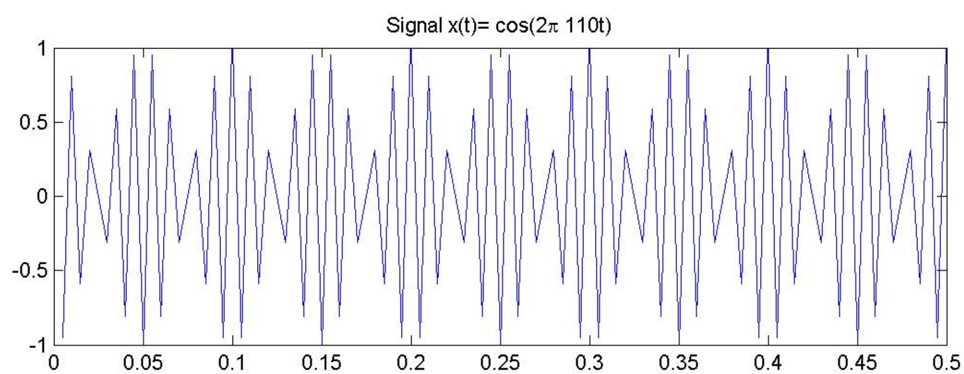
```
N=length(t);
%FIND THE DFT X[k]
fx1=abs(fft(x));title('Signal x(t)= cos(2\pi 110t)')

%CREATE THE CORRESPONDING FREQUENCY VECTOR: K-> 2*PI*fS/N
fs1=(1/N:1/N:1)*Fs;

%PLOT AS  A DISCRETE SAMPLED FUNCTION
subplot(212); stem(fs1,fx1);
%add some labeling and make it look nice
title('DFT of x(t)=cos(2\pi 110t), 100-pt FFT');
xlabel('frequency (Hz)');axis([0 200 0 55]);


%NOW INCREASE THE SIZE OF THE DFT TO 200 POINTS AND REPEAT

N1=2*N;

figure(5);
fx2=abs(fft(x,N1)); fs2=(1/N1:1/N1:1)*Fs;
stem(fs2,fx2);
%add some labeling and make it look nice
title('DFT of x(t)=cos(2\pi 100t), 200-pt FFT');
xlabel('frequency (Hz)'); axis([0 200 0 55]);


N1=2048

figure(6);
fx2=abs(fft(x,N1)); fs2=(1/N1:1/N1:1)*Fs;
plot(fs2,fx2); title('Fourier transform of x(t)=cos(2\pi 100t), 2048 pt
DFT');
xlabel('frequency (Hz)'); axis([0 200 0 55]);
```

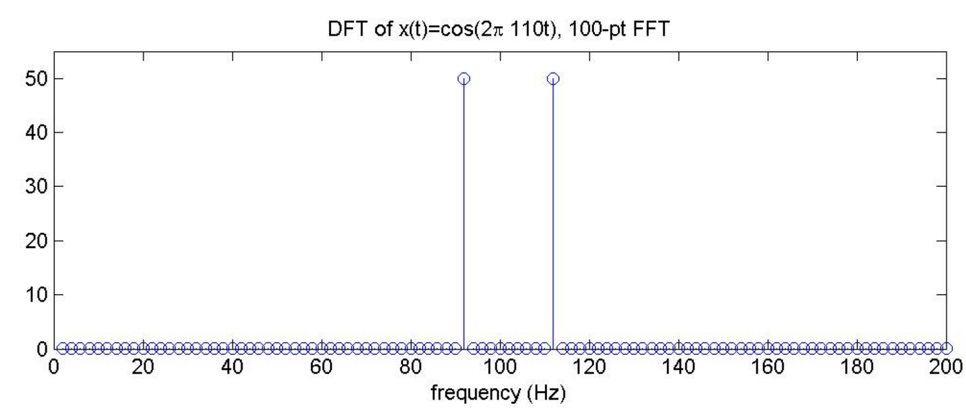

Signal x(t)= cos(2π 55t)
1
0.5
0
-0.5
-1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5


DFT of x(t)=cos(2π 50t), 100-pt FFT
50
40
30
20
10
0
0
20
40
60
80
100
120
140
160
180
200
frequency (Hz)

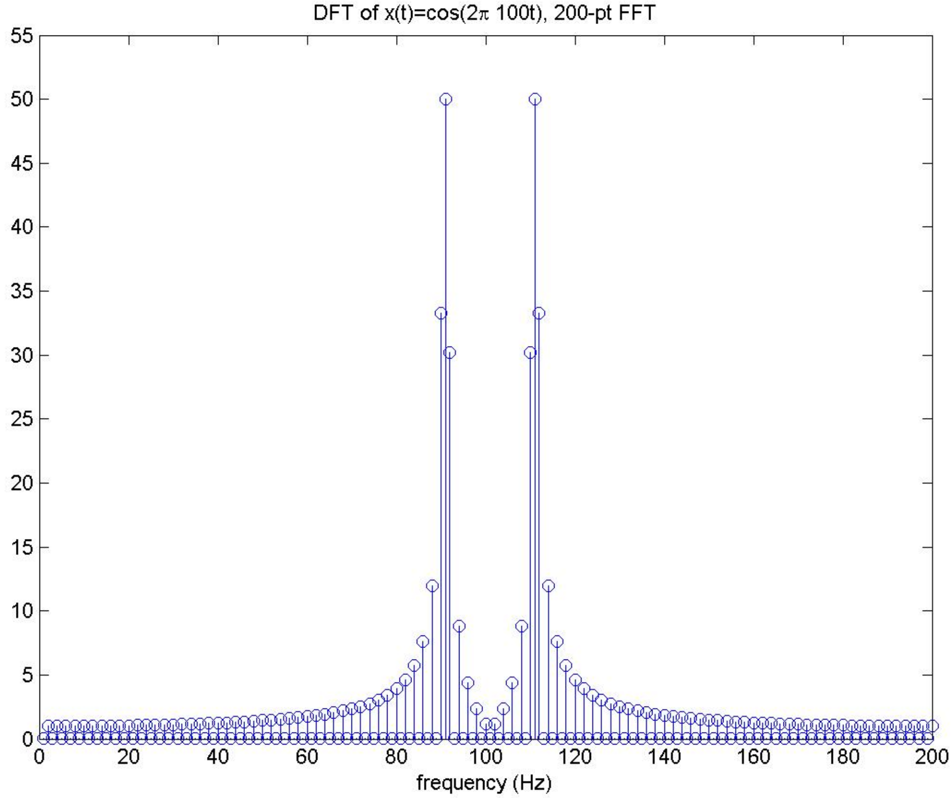

DFT of x(t)=cos(2π 50t), 200-pt FFT
55
50
45
40
35
30
25
20
15
10
5
0
0
20
40
60
80
100
120
140
160
180
200
frequency (Hz)


Fourier transform of x(t)=cos(2π 55t), 2048 pt DFT
frequency (Hz)

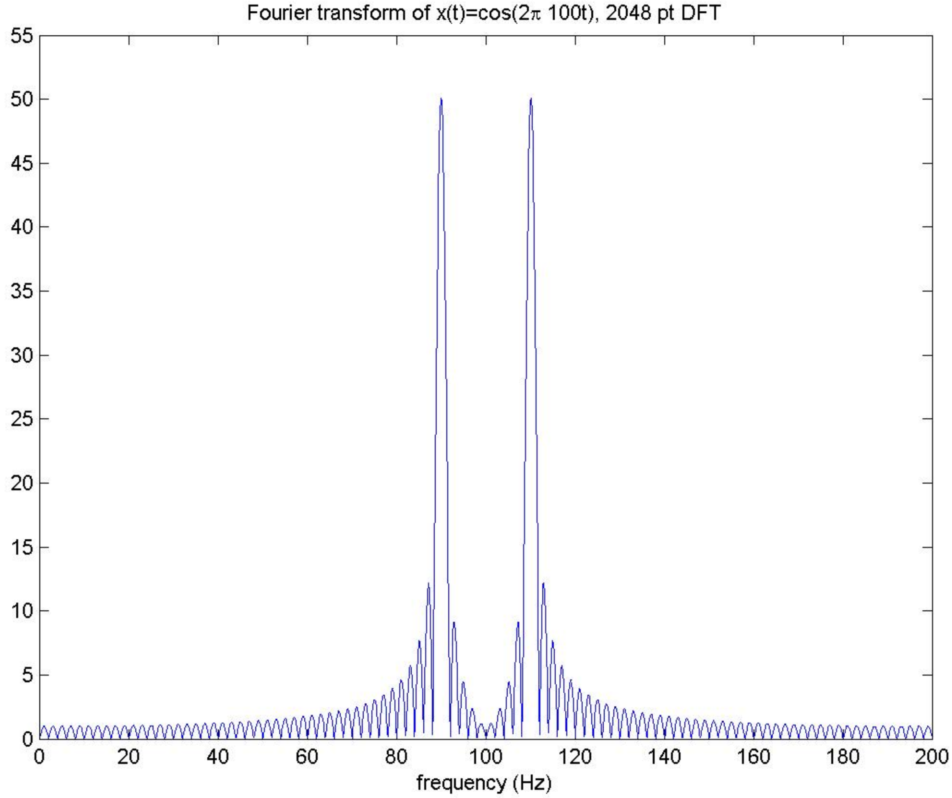

Signal x(t)= cos(2π 110t)


DFT of x(t)=cos(2π 110t), 100-pt FFT
frequency (Hz)

DFT of x(t)=cos(2π 100t), 200-pt FFT



Fourier transform of x(t)=cos(2π 100t), 2048 pt DFT



Problem 7

(a) $\tilde{Y}_1(k) = [2\ 1\ 0\ 1\ 2]$; $\tilde{Y}_2(k) = [0\ 1\ 2\ 0\ 0]$

$z(n) = y_1(n) \circledast_8 y_2(n)$

since we have 8-pt DFT & $y_1(n)$, $y_2(n)$ are real

then DFT is symmetric @ 4 pts. $\Rightarrow \tilde{Y}_1(k) = [2\ 1\ 0\ 1\ 2\ 1\ 0\ 1]$

$\tilde{Y}_2(k) = [0\ 1\ 2\ 0\ 0\ 0\ 2\ 1]$

$\tilde{Z}(k) = \left(\tilde{Y}_1(k)\, \tilde{Y}_2(k)\right) = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 1]$

$z(n) = \frac{1}{8} \sum \tilde{Z}(k) e^{-j\frac{2\pi}{8}nk} = \frac{1}{8}\left[e^{-j\frac{2\pi}{8}n} + e^{-j\frac{2\pi}{8}7n}\right]$

$= \frac{1}{8}\left(e^{-j\frac{2\pi}{8}n} + e^{+j\frac{2\pi}{8}n}\right]$

$z(n) = \frac{1}{4}\cos\frac{\pi}{4}n \qquad n = 0 \ldots 7$