AMERICAN UNIVERSITY OF BEIRUT
Department of Electrical and Computer Engineering
EECE340- Signals and Systems –Summer 2011
Pset 2 Solutions

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Problem 1

(a)

$$s^3 + s^2 + s + k = 0$$

Routh Hurwitz =>

i. all coefficients are to be positive => K>0
ii. none of the coefficients are 0 => K = 0
iii. R-H array:

s^3	1	1
s^2	1	K
s^1	1-K	0
s^0	K	

The number of roots in RHP is equal to number of sign changes in first column => Stability => in LHP => 1-K>0 => K<1 AND K>0 => 0<K<1

Range of K is 0<K<1

(b)

$$D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$(a_1a_2-a_3a_0)/a_2$	0
s^0	a_0	

To be stable:

i. a_0, a_1, a_2, a_3 are all positive or all negative
ii. a_0, a_1, a_2, a_3 are all different than zero
iii. According to the R-H array:
No change of sign must occur in the first column =>
If all coefficients are positive then $a_1a_2-a_3a_0 >0$ => $a_1a_2>a_3a_0$
If all coefficients are negative then $a_1a_2-a_3a_0 >0$ => $a_1a_2>a_3a_0$ since a_2 is negative.

(C)

For part (a) $a_0 = 1$, $a_1=1$, $a_2=1$, $a_3=K$

a_0, a_1, a_2, a_3 are all positive and different than zero

$a_1a_2>a_3a_0$ since 1>K

⇨ Stable

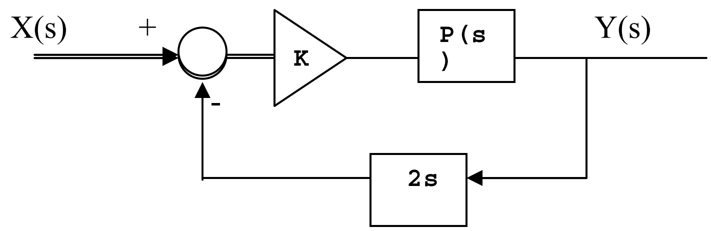
Problem 2

a) $\frac{d^2y}{dt^2} - 4y = 10\,i \quad \leftrightarrow \quad s^2\,Y(s) - s\,y(s) - y'(s) - 4\,Y(s) = -10\,I(s)$

$v(t) = 2\,\frac{dy(t)}{dt} \quad \leftrightarrow \quad V(s) = 2sY(s) - 2y(s)$

Then the system, in the s-domain looks like:

$$P(s) = \frac{-10}{s^2 - 4} = \frac{Y(s)}{I(s)}$$



Stability:

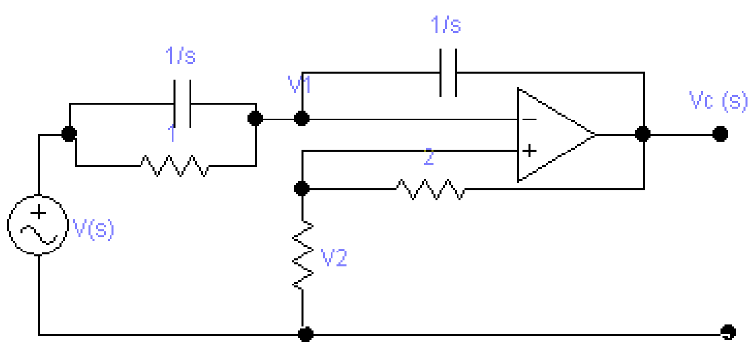
➔ zeros $(1 + 2s.K.\,P(s))$ are in L.H.P

zeros $\frac{(s^2 - 4 - 2sK \times 10)}{s^2 - 4}$ = roots $(s^2 - 20sK - 4)$ are in LHP.

By R-H, a necessary condition for stability is that all coefficients of the equation are the same sign, but this is not satisfied in $s^2 - 20sK - 4$ (at least in power of $s^2 + s^{\circ}$)

➔ System is not stable for all K.

b)



$V_1 = V_2$

$$V_1 = V_2 = \frac{R_1}{R_1 + R_2}\,Vc = \frac{1}{3}\,Vc(s) \qquad (R_1 = 1;\ R_2 = 2)$$

$$V - V_1 = \frac{R_1\,.\,(Cs)^{-1}}{R_1 + (Cs)^{-1}} \times I = \frac{R_1}{R_1\,Cs + 1} \times I = \frac{1}{s+1} \times I. \qquad (R_1 = 1;\ C = 1)$$

$$I = \frac{V - Vc}{(s+1)^{-1} + s^{-1}} = \frac{s\,(s+1)}{2s+1}(V - Vc)\ ;$$

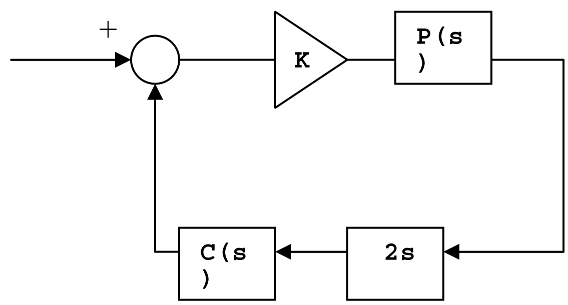
$$I = \frac{V - Vc}{s^{-1}} = \frac{-2s}{3} Vc \; ;$$

$$\frac{s\,(s+1)}{2s+1} V = \left[\frac{s\,(s+1)}{2s+1} - \frac{2s}{3} \right] Vc$$

$$\Rightarrow \frac{Vc(s)}{V(s)} = \frac{-\,3(s+1)}{s-1}$$

c)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K\,P(s)}{1 + 2.s.K.\,C(s)P(s)}$$



$$\Rightarrow$$

$$H(s) = \left[\frac{\dfrac{-10K}{s^2 - 4}}{1 + \dfrac{2.s.k\,.3(s+1)}{s-1} \times \dfrac{10}{s^2 - 4}} \right]$$

$$H(s) = \frac{-10K\,(s-1)}{s^3 + (60K-1)s^2 + (60K-4)s + 4}$$

$$C(s) = \frac{-\,3(s+1)}{s-1}$$

$$P(s) = \frac{-\;10}{s^2 - 4}$$

d) roots ($s^3 + (60K-1)s^2 + (60K-4)s + 4 = 0$) are in LHP.
For system to be stable, use R-H criterion:

s^3	1	$60K-4$
s^2	$60K-1$	4
s^1	$\dfrac{(60K-1)(60K-4) - 4}{60K-1}$	0
s°	4	

$60K-1 > 0$ and $\dfrac{3600K^2 - 300K}{60K - 1} > 0$

$\Rightarrow K > 1/60$ and $300K\,(12K-1) > 0$

$\Rightarrow K > 1/12.$

$\Rightarrow$

$$\boxed{K > 1/12}$$

Problem 3

a- $p(s) = \frac{8}{s} \cdot \frac{1}{s-3} = \frac{8}{s(s-3)}$

$p(s)$ is unstable since it has a pole $(s=3)$ in the RHS of the s-plane

b- $k(s) = k_o$

$\therefore \frac{V(s)}{U(s)} = \frac{8}{s(s-3)+8k_o} = \frac{8}{s^2-3s+8k_o}$

Taking the equation $s^2-3s+8k_o$ and finding its roots we get:

$S_{1,2} = 3 \pm \sqrt{9-32k_o}$

we find the $3+\sqrt{9-32k_o}$ is +ve for all values of k_o $\therefore$ we always have a pole in the RHS of the s-plane for any value of k_o, then the system is unstable.

c- $k(s) = k_o + k_1(s)$

$\therefore \frac{V}{U} = \frac{8}{s(s-3)+8(k_o+k_1 s)} = \frac{8}{s^2-3s+8k_o+8k_1 s}$

$= \frac{8}{s^2+(8k_1-3)s+8k_o}$

Using Routh Harwitz criterion

s^2	1	$8k_o$
s^1	$8k_1-3$	0
s^0	$8k_o$	

$8k_1 - 3 > 0 \Rightarrow k_1 > 3/8$

$k_0 > 0$

So, k_0 should be greater than zero and k_1 greater than $3/8$ for the system to be stable

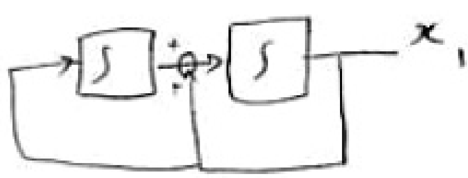
Problem 4

a-1: Write down the differential equations which describe the dynamics of his weight increase x_1 and his exercise ability x_2.

a-2: Draw a block diagram representation of the ordinary differential equation governing x_1 with a minimum number of integrators.

$$\dot{x}_1 = x_1 - x_2$$
$$\dot{x}_2 = -2x_1$$

$$\ddot{x}_1 = \dot{x}_1 - \dot{x}_2 = \dot{x}_1 + 2x_1$$
$$\Rightarrow \ddot{x}_1 - \dot{x}_1 - 2x_1 = 0$$
$$\Rightarrow x_1 = \int x_1 + \iint 2x_1$$



b) (5%) Find the weight of Homer for all time as his exercise program progresses. Assume that his weight initially was $x_1(0) = 120$ Kg and the rate $x'(0) = 0$. Is there a problem with this exercise plan?

$\ddot{x}_1 - 2x_1 - \dot{x}_1 = 0$ $\qquad x_1(0) = 120$

$\qquad x_1'(0) = 0$

$y_{hom}(\tau) = Ae^{s\tau}$

$\Rightarrow s^2 - s - 2 = 0 \quad \Rightarrow (s-2)(s+1) = 0$

$s = 2, -1$

$\Rightarrow x_1(\tau) = Ae^{2\tau} + Be^{-\tau} \quad \tau \geq 0$

$$\left.\begin{aligned} A + B &= 120 \\ 2A - B &= 0 \end{aligned}\right\} \Rightarrow 3A = 120$$

$A = 40, \; B = 80$

$\therefore \; x_1(\tau) = 40e^{2\tau} + 80e^{-\tau} \quad \tau \geq 0$

$e^{2\tau}$ is increasing with time

So Homer might die!!

c) (5%) Lisa Simpson observed how her father's weight is progressing and decided to help him. In addition to the earlier exercise dynamics (B above), she noticed that the rate of his exercise ability x_2' will increase in direct proportion to its current state x_2. Therefore, she decided to give him some reward to boost this ability (such as watching TV at the end of each exercise). In this case, the rate x_2' will be increasing at a steady constant rate R. That is,

$$x_2' = -2x_1 + k_2x_2 + k_1x_1 + Ru(t)$$

Could this scheme work? If so, find the constants R, k_1 and k_2 such that his weight stabilizes at half its original, that is, $x_1 = \frac{1}{2}(x_1(0)) = 60$ Kg.

$$\dot{x}_1 = x_1 - x_2$$

$$x_2' = -2x_1 + k_1x_1 + k_2x_2 + Ru(t)$$

$$\Rightarrow \ddot{x}_1 = \dot{x}_1 - \dot{x}_2 = \dot{x}_1 + 2x_1 - k_1x_1 - k_2x_2 - Ru(t)$$

$$= \dot{x}_1 + (2-k_1)x_1 - k_2(x_1 - \dot{x}_1) - Ru(t)$$

$$\ddot{x}_1 = (1+k_2)\dot{x}_1 + (2-k_1-k_2)x_1 - Ru(t)$$

$$\therefore \quad \ddot{x}_1 - (1+k_2)\dot{x}_1 + (-2+k_1+k_2)x_1 = -Ru(t)$$

Taking Laplace:

$$s^2X_1(s) - sx(0) - x'(0) - (1+k_2)sX_1(s) - (1+k_2)x_1(0)$$
$$+ (k_1+k_2-2)X_1(s) = -\frac{R}{s}$$

$$\therefore X_1(s) = \frac{-R}{s(s^2-(1+k_2)s+(k_1+k_2-2))} + \frac{[(1+k_2)+s]x(0)}{s^2-(1+k_2)s+(k_1+k_2-2)}$$

for stability: $1+k_2 < 0 \Rightarrow \boxed{k_2 < -1}$

$k_1+k_2-2 > 0 \Rightarrow \boxed{k_1 > 2-k_2}$

choose $\boxed{k_2 = -3} \Rightarrow k_1 > 5$ choose $\boxed{k_1 = 6}$

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$$x_1(s) = \frac{-R}{s(s^2+2s+1)} + \underbrace{x\,s\,R}_{\to 0} \qquad x_1(t)\Big|_{t\to\infty} = \lim_{s\to 0} sX_1(s) = -R$$

$$\Rightarrow \boxed{R = -60}$$

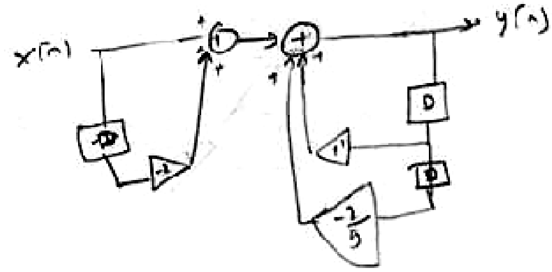
$$x_1(t)\Big|_{t\to\infty} = \frac{-R}{k_1+k_2-2}$$

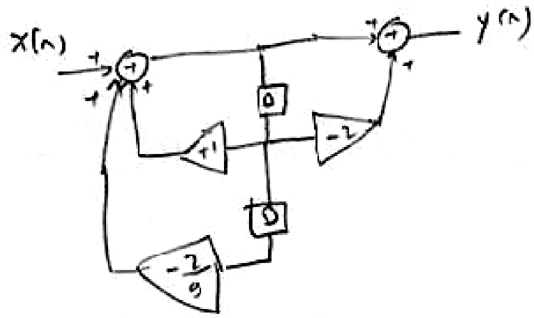
Problem 5

PROBLEM 2

$$y(n) + \frac{2}{9} y(n-2) - y(n-1) = x(n) + 2x(n-1)$$

START with non-canonical form (Direct form I)





$$Y(z)\left(1 + \frac{2}{9} z^{-2} - 1\, z^{-1}\right) = \left(1 - 2z^{-1}\right) X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - z^{-1} + \frac{2}{9} z^{-2}} = \frac{1 - 2z^{-1}}{\left(1 - \frac{2}{3} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)}$$

$$H(z) : \frac{1 - 2z^{-1}}{\left(1 - \frac{2}{3} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)}$$

if system is causal $\Rightarrow$ it is stable. (poles inside disc)

(b) $x[n] = u(n) - u(n-2) = \delta(n) + \delta(n-1)$

lets find $h[n] \longleftrightarrow H(z) = \dfrac{1-2z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)}$

$$H(z) = \frac{A}{1-\frac{1}{3}z^{-1}} + \frac{B}{1-\frac{2}{3}z^{-1}}$$

$$A: \frac{1+6}{1-2} = 5, \quad B = \frac{1-3}{1-\frac{1}{2}} = -4$$

$$H(z) = \frac{5}{1-\frac{1}{3}z^{-1}} - \frac{4}{1-\frac{2}{3}z^{-1}}$$

$$h(n) = 5\left(\tfrac{1}{3}\right)^n u(n) - 4\left(\tfrac{2}{3}\right)^n u(n)$$

$\therefore$ $y(n) = 5\left(\frac{1}{3}\right)^n u(n) - 4\left(\frac{2}{3}\right)^n u(n) + 5\left(\frac{1}{3}\right)^{n-1} u(n-1) - 4\left(\frac{2}{3}\right)^{n-1} u(n-1)$

d) $x(n) = (-2)^n \quad \forall n$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{-\infty}^{\infty} h[k]\,(-2)^{n-k}$$

$$= (-2)^n \sum_{-\infty}^{\infty} h(k)(-2)^{-k}$$

$$= (-2)^n H(-2)$$

$$= (-2)^n \left[\frac{1+1}{\left(1+\frac{1}{6}\right)\left(1+\frac{1}{3}\right)}\right] = 0$$

$$(-2)^n \left(\frac{2 \times 6 \times 3}{7 \times 4}\right) = \left(\frac{9}{7}\right)(-2)^n$$

Problem 6

tions:

(a) $y[n] - \alpha y[n-1] = 2x[n]$

$$\begin{aligned} r - \alpha &= 0 \\ y^{(h)}[n] &= c_1 \alpha^n \end{aligned}$$

(d) $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$

$$\begin{aligned} r^2 + r + \frac{1}{4} &= 0 \\ r &= -\frac{1}{2}, \ -\frac{1}{2} \\ y^{(h)}[n] &= c_1\left(-\frac{1}{2}\right)^n + c_2 n\left(-\frac{1}{2}\right)^n \end{aligned}$$

Problem7

(a) $y[n] - \frac{2}{5}y[n-1] = 2x[n]$

(i) $x[n] = 2u[n]$

$$\begin{aligned} y^{(p)}[n] &= ku[n] \\ k - \frac{2}{5}k &= 4 \\ k &= \frac{20}{3} \\ y^{(p)}[n] &= \frac{20}{3}u[n] \end{aligned}$$

(ii) $x[n] = -(\frac{1}{2})^n u[n]$

$$\begin{aligned} y^{(p)}[n] &= k\left(\frac{1}{2}\right)^n u[n] \\ k\left(\frac{1}{2}\right)^n - \frac{2}{5}\left(\frac{1}{2}\right)^{n-1} k &= -2\left(\frac{1}{2}\right)^n \\ k &= -10 \\ y^{(p)}[n] &= -10\left(\frac{1}{2}\right)^n u[n] \end{aligned}$$