Review Problems

Problem 1

Given x[n] = h[n] = u[n - 1] – u[n – 5], find y[n] = x[n]*h[n] using z-transforms

$x[n] = u[n-1] - u[n-5], \quad h[n] = u[n-1] - u[n-5]$

$$\begin{aligned} x(z) &= z + z^2 + z^3 + z^4 \\ h(z) &= z + z^2 + z^3 + z^4 \\ y(z) &= x(z)h(z) \\ &= z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8 \\ y[n] &= \delta[n-2] + 2\delta[n-3] + 3\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7] + \delta[n-8] \end{aligned}$$

for $n-1 \leq 0$ $\quad n \leq 1$

$y[n] = 0$

for $n-1 \leq 4$ $\quad n \leq 5$

$$y[n] = \sum_{k=1}^{n-1} 1 = n-1$$

for $n-4 \leq 4$ $\quad n \leq 8$

$$y[n] = \sum_{k=n-4}^{4} 1 = 9-n$$

for $n-4 \geq 5$ $\quad n \geq 9$

$y[n] = 0$

$$y[n] = \delta[n-2] + 2\delta[n-3] + 3\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7] + \delta[n-8]$$

Problem 2

A recursive filter is given by:

$$H(z) = \frac{s^6 + 2s^2 + 1}{5z^5 + 4z^4 + 3z^3 + 2z^2 + z + 1}$$

Check the stability using Jury’s method

a. D(1) = 16 >0

b. $(-1)^5 D(-1) = 2 > 0$

c.

5	**4**	**3**	**2**	**1**	**1**
1	1	2	3	4	5
24	19	13	7	1	
1	7	13	19	24	
575	449	299	149		
149	299	449	575		
308424		105024			

→ The system is stable.

Problem 3

Determine the inverse Z-transform of:

$$X(z) = \frac{z(z+1)}{(z-1)(z^2 - z + \frac{1}{4})} \quad \text{with ROC } ½<|z|<1$$

$$X(z) = \frac{8}{1-z^{-1}} - \frac{6}{(1-0.5z^{-1})^2} - \frac{2}{1-0.5z^{-1}}$$

$$x[n] = -6(n-1)\left(\frac{1}{2}\right)^{n-1} u[n-1] - 2\left(\frac{1}{2}\right)^{n} u[n] - 8(1)^n u[-n-1]$$

Problem 4

An LTI system has the transfer function H(z),

$$H(z) = 1 - 3z^{-2} - 4z^{-4}$$

The input to this system is given by,

$$x[n] = 20 - 20\delta[n] + 20\cos(0.5\pi n + \pi/4) \qquad -\infty < n < \infty$$

Determine the output y[n] of the system for all n.

Solution: In the lecture notes, we derived the following transform pair:

$$\delta[n - n_0] \quad \Leftrightarrow \quad z^{-n_0}$$

Therefore,

$$h[n] = \delta[n] - 3\delta[n-2] - 4\delta[n-4].$$

This is the impulse response of an FIR filter, whose difference equation is given by,

$$y[n] = x[n] - 3x[n-2] - 4x[n-4].$$

Thus, the output $y[n]$ for the input in equation (11-2) is given by,

$$\begin{aligned} y[n] = {} & 20 - 20\delta[n] + 20\cos(0.5\pi n + \pi/4) - \\ & 3[20 - 20\delta[n-2] + 20\cos(0.5\pi(n-2) + \pi/4)] - \\ & 4[20 - 20\delta[n-4] + 20\cos(0.5\pi(n-4) + \pi/4)] \end{aligned}$$

$$\begin{aligned} y[n] = {} & -120 - 20\delta[n] + 60\delta[n-2] + 80\delta[n-4] + 20\cos(0.5\pi n + \pi/4) + \\ & 60\cos(0.5\pi n + \pi/4) - 80\cos(0.5\pi n + \pi/4) \end{aligned}$$

$$y[n] = -120 - 20\delta[n] + 60\delta[n-2] + 80\delta[n-4].$$

A more general procedure, that applies to both FIR and IIR filters is to derive the solution for $y[n]$ through the frequency response of the system:

$$H(e^{j\theta}) = H(z)\big|_{z = e^{j\theta}} = 1 - 3e^{-j2\theta} - 4e^{-j4\theta}$$

First, the input $x[n]$ can be broken up into three parts:

$$x[n] = x_1[n] + x_2[n] + x_3[n]$$

where,

$$x_1[n] = 20, \; x_2[n] = -20\delta[n] \text{ and } x_3[n] = 20\cos(0.5\pi n + \pi/4).$$

We can write the outputs corresponding to $x_1[n]$, $x_2[n]$ and $x_3[n]$:

$$H(e^{j\theta})|_{\theta=0} = 1-3-4 = -6$$

$$y_1[n] = 20|H(e^{j\theta})|_{\theta=0}\cos(0n+\angle H(e^{j\theta})|_{\theta=0}) = 20|-6|\cos(\pi) = -120$$

$$y_2[n] = -20h[n] = -20\delta[n]+60\delta[n-2]+80\delta[n-4]$$

$$H(e^{j\theta})|_{\theta=0.5\pi} = 1-3e^{-j\pi}-4e^{-j2\pi} = 1+3-4 = 0$$

$$y_3[n] = 20|H(e^{j\theta})|_{\theta=0.5\pi}\cos(0.5\pi n+\angle H(e^{j\theta})|_{\theta=0.5\pi}) = 0$$

Therefore, the total output is given by,

$$y[n] = y_1[n]+y_2[n]+y_3[n]$$

$$y[n] = -120-20\delta[n]+60\delta[n-2]+80\delta[n-4].$$

Problem 5

The signal $x(t)=e^{-8t}u(t)$ is sampled every 0.125 seconds.

a. Represent the sampled as a sequence x[n].

$$x[n]=e^{-n}u[n]$$

b. Find the z-transform X(z) of x[n].

$$X(z)=\frac{1}{1-e^{-1}z^{-1}}$$ with ROC |z|>(1/e)

Problem 6

Consider a sequence x[n] whose z-transform is given by:

$$X(z)=\frac{1/3}{1-0.5z^{-1}}+\frac{1/4}{1-2z^{-1}}$$

(note that the ROC is not specified)

a. Can x[n] be causal?

Yes, if we chose the ROC to be |z|>2

b. Can x[n] be stable?

Yes, if we chose the ROC to be 0.5|z|<2

c. Can x[n] be causal and stable?

No; 1 pole outside the unit circle.

Problem 7

Given the z-transform of x[n] to be X(z) with ROC |z|<4. Use the z-transform properties to determine the z-transform of the following signals:

a. y[n]= x[n] * x[-n]

Y(z)= X(z) X(1/z) with ROC 1/4 <|z|<4

b. y[n] = n x[n]

$Y(z) = -z\frac{dX(z)}{dz}$ with ROC |z|<4

c. y[n]=x[n-1] * x[n-3];

$Y(z)= z^{-1}X(z)z^{-3}X(z) = z^{-4}X^{2}(z)$, with ROC |z|<4