Chapter 13

The Discrete-Time Fourier Transform (DTFT)

The Discrete-Time Fourier Transform (DTFT)

The DTFT is used to represent a discrete-time non-periodic signal as a superposition of complex sinusoids. DTFT would involve a continuum of frequencies on the interval $-\pi<\Omega\leq\pi$, where Ω has units of radians. Thus, the DTFT representation of a time-domain signal involves an integral over frequency, namely,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

where

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{n=\infty} x[n]\, e^{-j\Omega n}$$

The Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{n=\infty} x[n]\, e^{-j\Omega n}$$

is the DTFT of the signal x[n]. We say that $X(e^{j\Omega})$ and x[n] are a DTFT pair. The DTFT is used primarily to analyze the action of discrete-time systems on discrete-time signals

The Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{n=\infty} x[n]\, e^{-j\Omega n}$$

The infinite sum above converges if x[n] has finite duration and is finite valued, If x[n] is of infinite duration, then the sum converges only for certain classes of signals. If x[n] is absolutely summable, That is;

$$\sum_{n=-\infty}^{n=\infty} \left| x[n] \right| < \infty$$

The Discrete-Time Fourier Transform (DTFT)

Many physical signals encountered in engineering practice satisfy these conditions. However, several common non-periodic signals, such as the unit step u[n], do not. In some of these cases, we can define a transform pair that behaves like the DTFT by including impulses in the transform. This enables us to use the DTFT as a problem-solving tool, even though, strictly speaking, it does not converge. One example of such usage is given later in the section.

Example:

Find the DTFT of the exponential sequence

$$x[n] = a^n u[n]$$

Solution

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{n=\infty} x[n]\, e^{-j\Omega n} = \sum_{n=0}^{\infty} (ae^{-j\Omega})^n$$

$$X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

Example:

$$X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

If a is a real-valued quantity, then

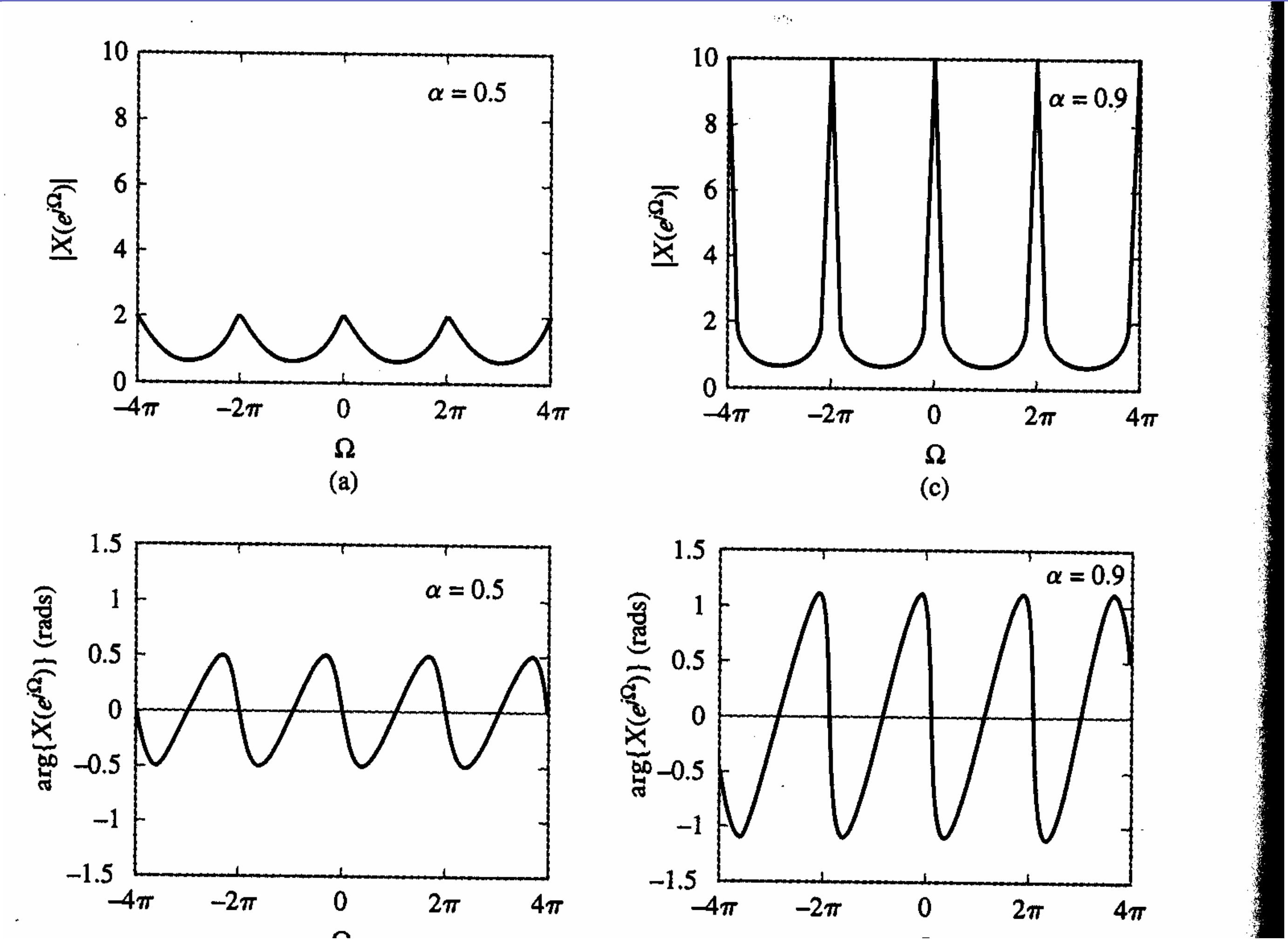
$$X(e^{j\Omega}) = \frac{1}{1 - a\cos\Omega + ja\sin\Omega}$$

Accordingly, the magnitude and phase spectra are given by:

$$\left|X(e^{j\Omega})\right| = \frac{1}{\sqrt{(1 - a\cos\Omega)^2 + a^2\sin^2\Omega}} = \frac{1}{\sqrt{1 + a^2 - 2a\cos\Omega}}$$

$$\arg\left\{X\left(e^{j\Omega}\right)\right\} = -\arctan\left(\frac{a\sin\Omega}{1 - a\cos\Omega}\right)$$

Example



Example

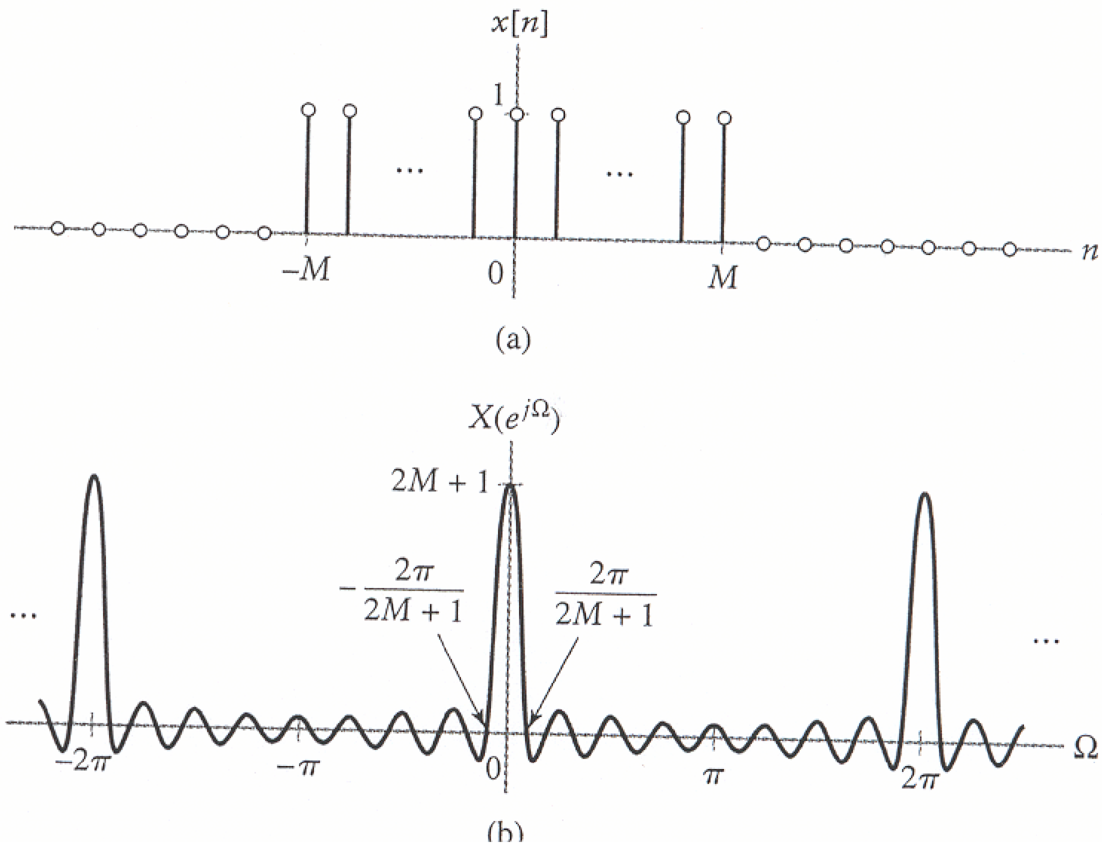
Find the DTFT of

$$x[n] = \begin{cases} 1 & |n| \leq M \\ 0 & \text{Otherwise} \end{cases}$$

solution

$$X\left(e^{j\Omega}\right) = \frac{\sin\left(\Omega(2M+1)/2\right)}{\sin\left(\Omega/2\right)}$$

Example


$x[n]$
1
...
...
$-M$
0
M
n
(a)
$X(e^{j\Omega})$
$2M+1$
$-\frac{2\pi}{2M+1}$
$\frac{2\pi}{2M+1}$
...
...
-2π
$-\pi$
0
π
2π
Ω
(b)

Example

Find the DTFT of the following time-domain signal

$$x[n] = \begin{cases} 2^n & 0 \le n \le 9 \\ 0 & \text{Otherwise} \end{cases}$$

Answer

$$X\left(e^{j\Omega}\right) = \frac{1 - 2^{10} e^{-j10\Omega}}{1 - 2e^{-j\Omega}}$$

Properties of the DTFT

1. **Linearity and Symmetry Properties**

$$z[n] = ax[n] + by[n] \xleftrightarrow{\text{DTFT}} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

The linearity property is used to find Fourier representations of signals that are constructed as sums of signals whose representations are already known

Properties of the DTFT

2. Symmetry Properties: Real and Imaginary Signals

a. **For real-valued signals**

$$X^*(e^{j\Omega}) = X(e^{-j\Omega})$$

b. **For Imaginary-valued signals**

$$X^*(e^{j\Omega}) = -X(e^{-j\Omega})$$

Properties of the DTFT

3. Convolution Property

A similar property, to that of continuous non-periodic signals, holds for discrete-time non periodic signals

$$z[n] = x[n] * y[n] \xleftrightarrow{\text{DTFT}} Z(e^{j\Omega}) = X(e^{j\Omega})Y(e^{j\Omega})$$

The proof of this result closely parallels that of the continuous-time case.

Properties of the DTFT

4. Shifting Property

$$x[n-n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(e^{j\Omega})$$

Where n_0 is an integer value. For $n_0=1$, then

$$x[n-1] \xleftrightarrow{\text{DTFT}} e^{-j\Omega} X(e^{j\Omega})$$

Properties of the DTFT

The shifting property can be used to determine the Frequency response of a discrete-time signal represented by a difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Taking the DTFT of both sides of this equation, we obtain

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{i\Omega})} = \frac{\sum_{k=0}^{M} b_k (e^{-j\Omega})^k}{\sum_{k=0}^{N} a_k (e^{-j\Omega})^k}$$

Example

Find the difference corresponding to the system with frequency response

$$H(e^{j\Omega}) = \frac{1 + 2e^{-j2\Omega}}{3 + 2e^{-j\Omega} - 3e^{-j3\Omega}}$$

Answer

3y[n]+2y[n-1]-3y[n-3]=x[n]+2x[n-2]

Properties of the DTFT

5. Multiplication by n

$$x[n] \leftrightarrow X(e^{j\Omega})$$

$$nx[n] \leftrightarrow j\frac{d\left[X(e^{j\Omega})\right]}{d\Omega}$$

Properties of the DTFT

6. Multiplication Property

The multiplication property defines the Fourier representation of a product of discrete-time signals. If x[n] and z[n] are discrete-time nonperiodic signals, then the DTFT of the product y[n] = x[n]z[n] is given by the convolution of the their DTFT's and multiplication by 1/2π.

$$y[n] = x[n] \cdot z[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} X(e^{j\Omega}) \otimes Z(e^{j\Omega})$$

Inverse of DTFT by Partial fraction expansion

Example

Find the Inverse DTFT of

$$X(e^{j\Omega}) = \frac{5 - \frac{5}{6}e^{-j\Omega}}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}}$$

$$X(e^{j\Omega}) = \frac{5 - \frac{5}{6}e^{-j\Omega}}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}} = \frac{5 - \frac{5}{6}e^{-j\Omega}}{\left(1 + \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{3}e^{-j\Omega}\right)}$$

Example

Using Partial Fraction Expansion

$$X(e^{j\Omega}) = \frac{4}{\left(1+\frac{1}{2}e^{-j\Omega}\right)} + \frac{1}{\left(1-\frac{1}{3}e^{-j\Omega}\right)}$$

Then

$$x[n] = 4(-1/2)^{n}u[n] + (1/3)^{n}u[n]$$

Example

Find the frequency and impulse responses of the discrete-time system described by the following difference equations:

y[n-2] + 5y[n-1] + 6y[n] = 8x[n - 1] + 18x[n]

Answer

$$H(e^{j\Omega}) = \frac{8e^{-j\Omega} + 18}{(e^{-j\Omega})^2 + 5e^{-j\Omega} + 6}$$

$$h[n] = 2(1/4)^n u[n] + (-1/2)^n u[n]$$

Parseval's Theorem for Energy Signals

$$\sum_{n=-\infty}^{n=\infty} \left| x[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\Omega}) \right|^2 d\Omega$$

Example

Let

$$x[n] = \frac{\sin(Wn)}{\pi n}$$

Use Parseval's theorem to evaluate the energy of x[n]

Solution

Using the DTFT, we have

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \leq W \\ 0 & W < |\Omega| \leq \pi \end{cases}$$

It follows that the energy of x[n]=W/π