# Chapter 12

**The Discrete-Time Fourier Series (DTFS)**

# Definition

Consider a Discrete-Time signal x[n] of period N or frequency $\Omega_0=(2\pi/N)$. The DTFS representation of x[n] is defined as:

$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_0 n}$$

Where

$$x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n}$$

We say that x[n] and X[k] are a DTFS pair

# Example

Let x[n] be a periodic Discrete-time signal of period N=5. Over one period x[n] is defines as:

$$x[n] = \begin{cases} 1 & n = 0 \\ -0.5 & n = 1 \\ 0 & n = 2 \\ 0 & n = 3 \\ 0.5 & n = 4 \end{cases}$$

Determine the Frequency-domain representation of x[n]

Answer:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n} = \frac{1}{5}\sum_{n=0}^{4} x[n]e^{-jk\frac{2\pi}{5}n} = \frac{1}{5}\left[1 - 0.5e^{-jk\frac{2\pi}{5}} + 0.5e^{-jk\frac{8\pi}{5}}\right]$$

# EXAMPLE

Determine the DTFS coefficients of

$$x[n] = 1 + \sin\left(\frac{n\pi}{12} + \frac{3\pi}{8}\right)$$

Solution:

$$X[k] = \begin{cases} -\dfrac{e^{-j3\pi/8}}{2j} & k = -1 \\ 1 & k = 0 \\ \dfrac{e^{-j3\pi/8}}{2j} & k = 1 \\ 0 & -11 \leq k \leq 12 \end{cases}$$

# EXAMPLE

Find the DTFS coefficients of the N-periodic impulse train

$$x[n] = \sum_{\ell=-\infty}^{\infty} \delta[n - \ell N]$$

**Solution:**

Since there is only one nonzero value in x[n] per period, it is convenient to evaluate X[k] over the interval $n$ = 0 to $n$ = N - 1 to obtain

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jkn2\pi/N} = \frac{1}{N}$$

# EXAMPLE: The Inverse DTFS

One period of the DTFS coefficients of a signal is given by

$$X[k] = \left(\frac{1}{2}\right)^k \qquad 0 \le k \le 9$$

Find the discrete- time domain signal x[n] assuming N=10.

**Answer:**

$$x[n] = \frac{1-(1/2)^{10}}{1-(1/2)^{j(\pi/5)n}}$$

# EXAMPLE: DTFS Representation of a Square wave

Find the DTFS coefficients for the N-periodic square wave given by

$$x[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{Over the rest of the period} \end{cases}$$

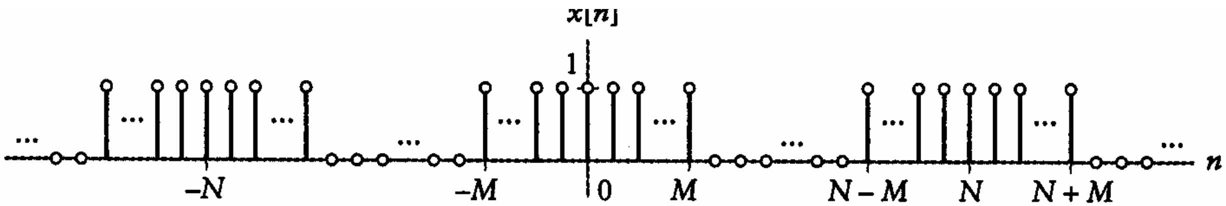


**FIGURE 3.11** Discrete-time square wave for Example 3.6.

That is, each period contains 2M+1 consecutive ones and the remaining {N-(2M+1)} values are zero. Note that this definition requires that N > 2M + 1.

# Solution

The period is N, so $\Omega_0$=2π/N. It is convenient in this case to evaluate the DTFS of x[n] over indices *n* = -M to *n* = N - M - 1. We thus have

$$X[k] = \frac{1}{N} \sum_{n=-M}^{N-M-1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=-M}^{M} e^{-jk\Omega_0 n}$$

We perform the change of variable on the index of summation by letting m=n+M to obtain

$$X[k] = \frac{1}{N} \sum_{m=0}^{2M} e^{-jk\Omega_0 (m-M)} = \frac{e^{jk\Omega_0 M}}{N} \sum_{m=0}^{2M} e^{-jk\Omega_0 m}$$

# Solution

Now, for k=0, ±N, ±2N, ±3N, …, we have

$$e^{jk\Omega_0} = e^{-jk\Omega_0} = 1$$

$$X[k] = \frac{1}{N}\sum_{m=0}^{2M} 1 = \frac{2M+1}{N} \qquad \text{for } k = 0, \pm N, \pm 2N, \ldots$$

for k=0, ±N, ±2N, ±3N, …, we may sum the geometric series to obtain

$$X[k] = \frac{e^{jk\Omega_0 M}}{N}\left(\frac{1-e^{-jk\Omega_0(2M+1)}}{1-e^{-jk\Omega_0}}\right) = \frac{1}{N}\frac{\sin\left(k\Omega_0(2M+1)/2\right)}{\sin\left(k\Omega_0/2\right)}$$

# Solution

Substituting $\Omega_0=2\pi/N$ by its value in the equation

$$X[k] = \frac{e^{jk\Omega_0 M}}{N}\left(\frac{1-e^{-jk\Omega_0(2M+1)}}{1-e^{-jk\Omega_0}}\right) = \frac{1}{N}\frac{\sin\left(k\Omega_0(2M+1)/2\right)}{\sin\left(k\Omega_0/2\right)}$$

we obtain

$$X[k] = \frac{1}{N}\frac{\sin\left(k\pi(2M+1)/N\right)}{\sin\left(k\pi/N\right)}$$

# Solution

The above equation is also valid for k=0, ±N, ±2N, ±3N, … using L"Hopital's rule.

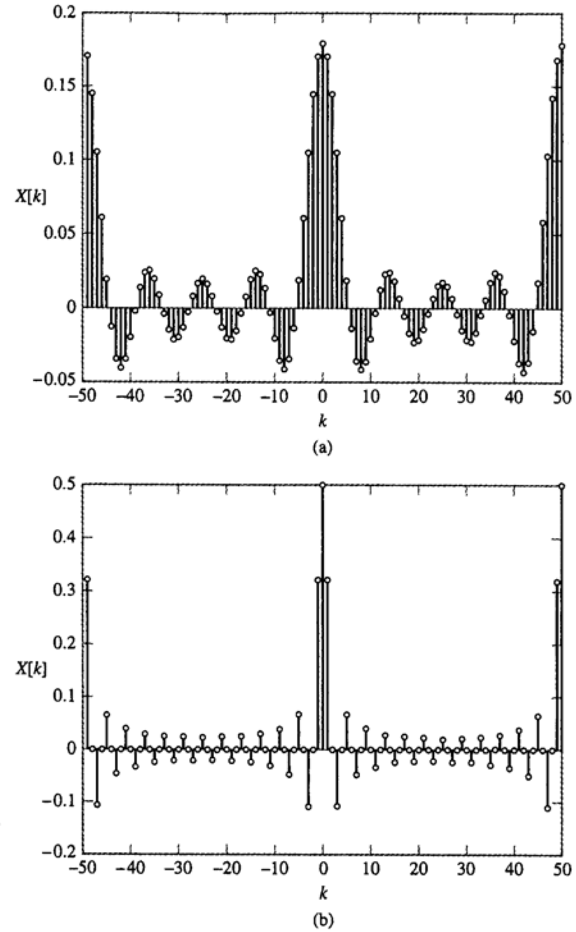


**FIGURE 3.12** The DTFS coefficients for the square wave shown in Fig. 3.11, assuming a period $N = 50$: (a) $M = 4$. (b) $M = 12$.

x[n]

# Convolution of periodic signals

- This subsection addresses the convolution of two signals that are periodic functions of time. The convolution of periodic signals does not occur naturally in the context of evaluating the input-output relationships of systems, since any system with a periodic impulse response is unstable. However, the convolution of periodic signals often occurs in the context of signal analysis and manipulation.

- The discrete-time convolution of two N-periodic sequences x[n] and z[n] is defined as:

$$y[n] = x[n] \otimes z[n] = \sum_{k=0}^{N-1} x[k] z[n-k]$$

# Convolution of periodic signals

- This is the periodic convolution of x[n] and z[n]. The signal y[n] is N periodic, so the DTFS is the appropriate representation for all three signals: x[ n], z[n], and y[n].

$$y[n] = x[n] \otimes z[n] \xleftrightarrow{\text{DTFT}, \frac{2\pi}{N}} Y[k] = X[k] \cdot Z[k]$$

- Thus, convolution of time signals transforms to multiplication of DTFS coefficients