Chapter 9

Sampling Theorem

Sampling

We live in a continuous-time world: most of the signals we encounter are CT signals, e.g. $x(t)$. How do we convert them into DT signals $x[n]$?

— Sampling, taking snap shots of $x(t)$ every T seconds.

T – sampling period

$x[n] \equiv x(nT)$, $n = ..., -1, 0, 1, 2, ...$ — regularly spaced samples

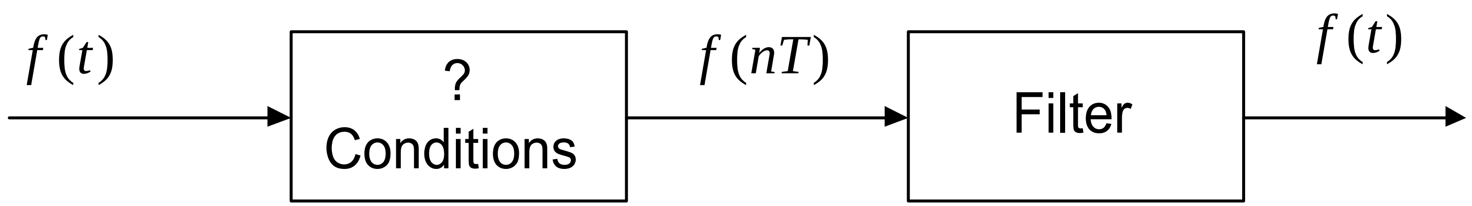
Applications and Examples

— Digital Processing of Signals
— Strobe
— Images in Newspapers
— Sampling Oscilloscope
— …

How do we perform sampling?

The Sampling Theorem

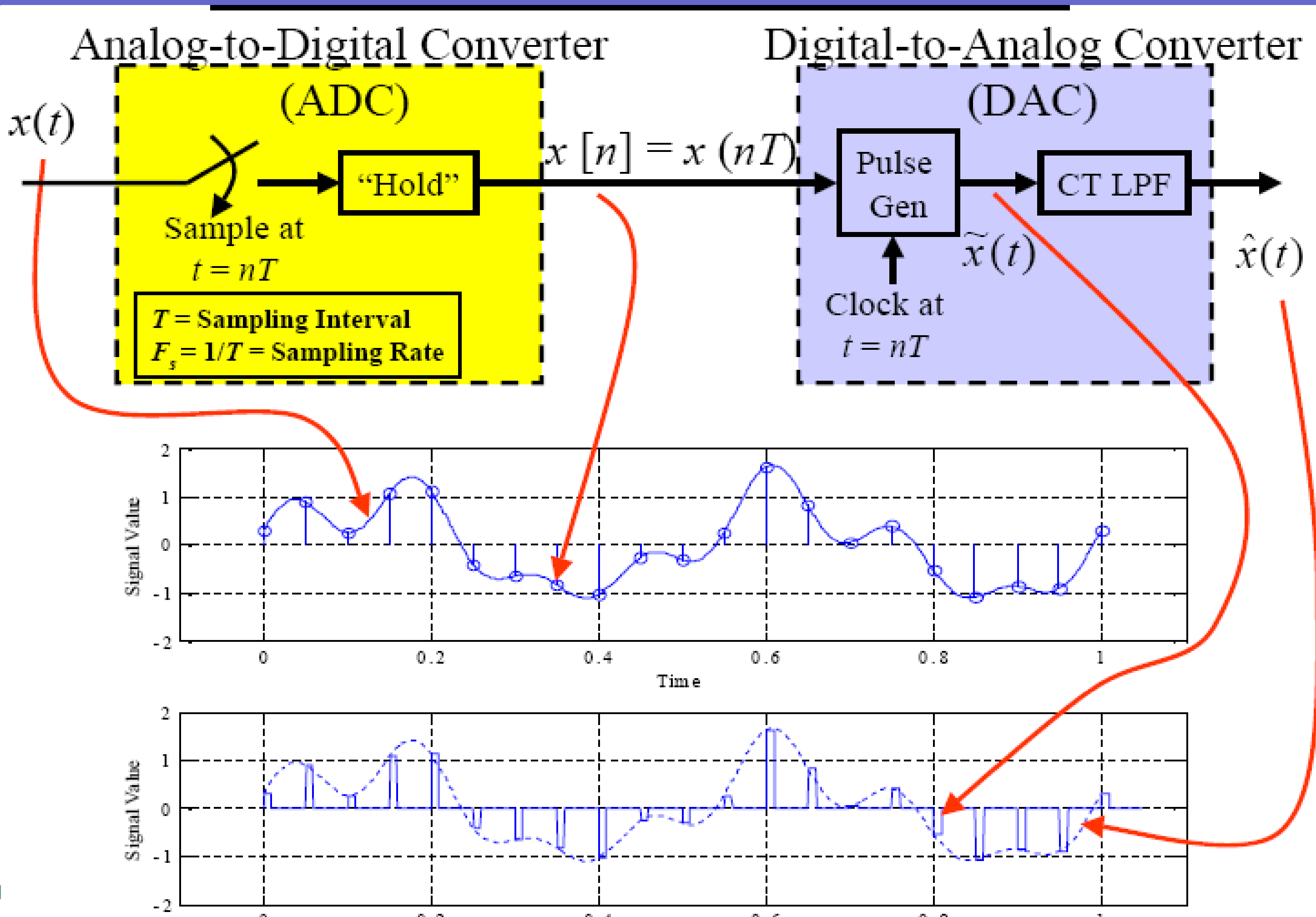
Signals bearing information are available either in continuous form f(t) or in discrete form (digital form). Therefore, we would like to determine the necessary conditions which allow us to change an analog signal to a discrete signal without loss of information, and how we recover the original signal by filtering, this link is known as the "The sampling theorem".



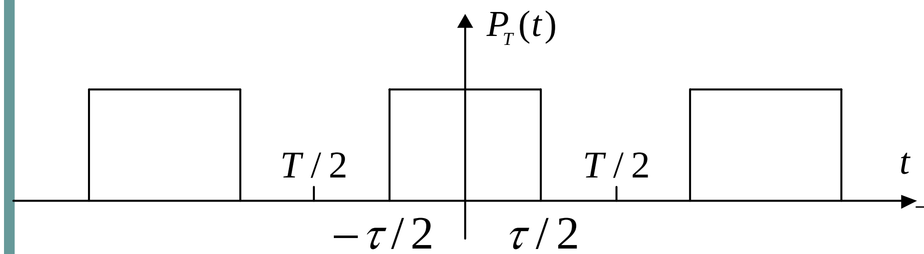
Definition

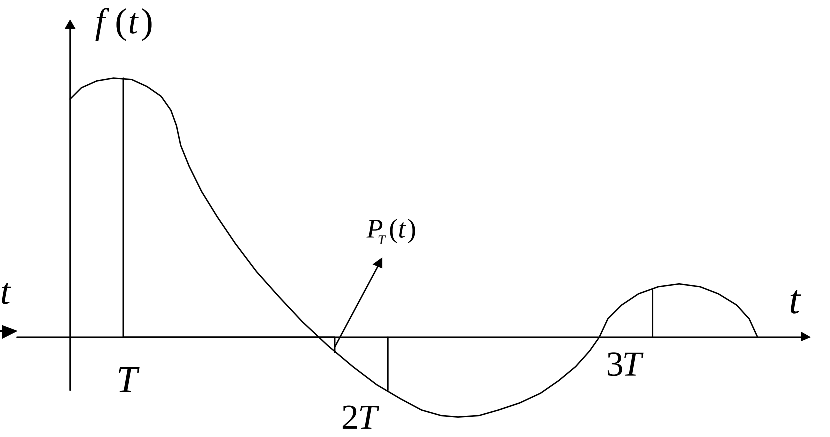
A real-valued band-limited (low-pass signals, the bandwidth is limited) having no frequency components above a frequency of W rad/s [W=2πB; B is in Hz] is determined uniquely by its values at uniform intervals that are spaced no greater than (1/2B) seconds apart.

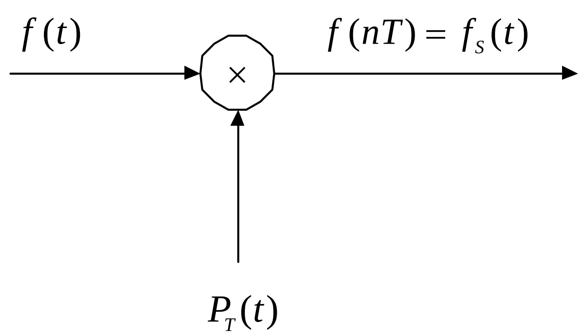
Physical Steps



Mathematical Steps







Mathematical Details

- $f_s(t)=f(t).p_T(t)$
- $P_T(t)$ is a periodic function of period T. It can be expressed in a Fourier series form.

$$P_T(t) = \sum_{n=-\infty}^{+\infty} P_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$f_S(t) = f(t) \sum_{n=-\infty}^{+\infty} P_n e^{jn\omega_0 t}$$

The Fourier Transform of $f_s(t)$

$$P_T(t) = \sum_{n=-\infty}^{+\infty} P_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$f_S(t) = f(t) \sum_{n=-\infty}^{+\infty} P_n e^{jn\omega_0 t}$$

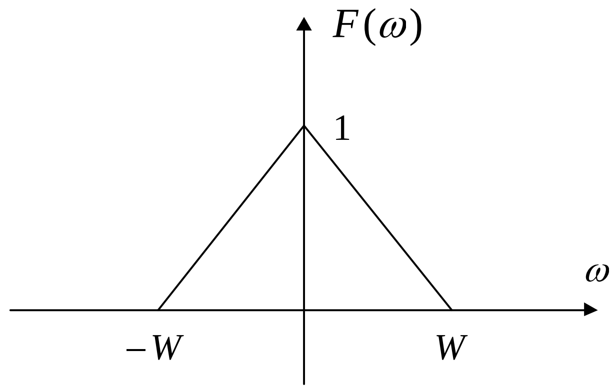
$$F_S(\omega) = \int_{-\infty}^{\infty} f_S(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t) \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t} e^{-j\omega t} dt$$

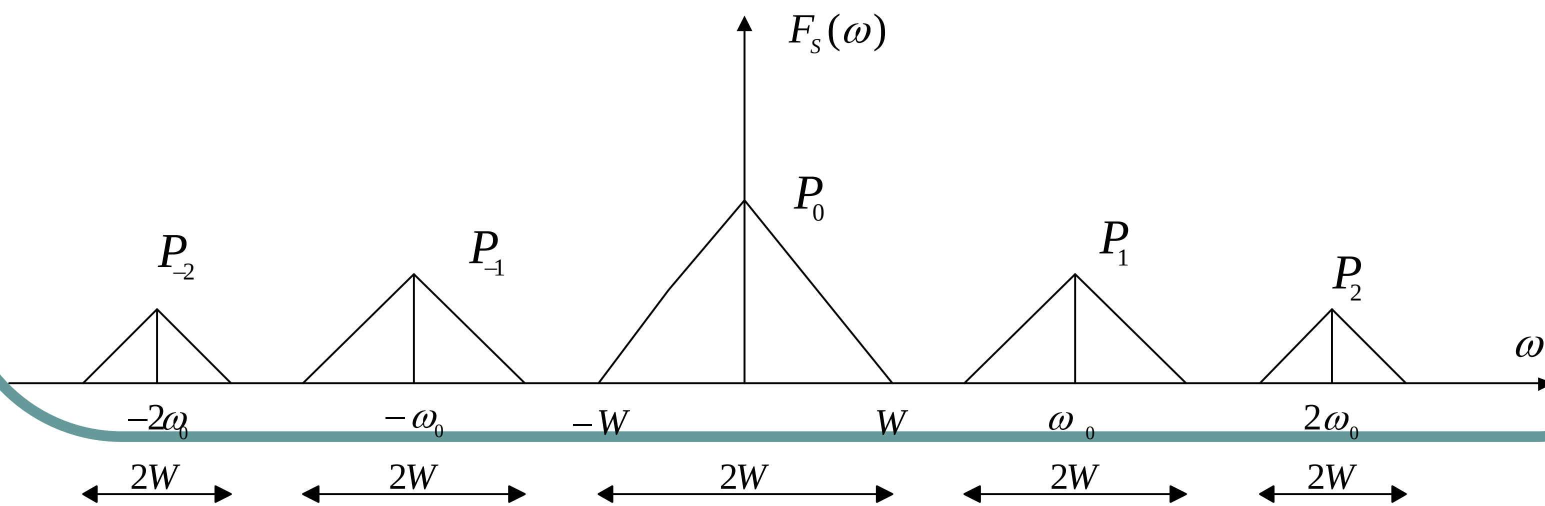
$$F_S(\omega) = \sum_{-\infty}^{\infty} P_n \int_{-\infty}^{\infty} f(t) e^{-j(\omega - n\omega_0)t}$$

$$F_S(\omega) = \sum_{n=-\infty}^{\infty} P_n F(\omega - n\omega_0)$$

Case 1: T<1/2B

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{2\pi}{\omega_0} = T;\; T < \frac{1}{2B} \Rightarrow \frac{2\pi}{\omega_0} < \frac{1}{2B} \Rightarrow \frac{1}{\omega_0} < \frac{1}{2(2\pi B)} \Rightarrow \frac{1}{\omega_0} < \frac{1}{2W} \Rightarrow \omega_0 > 2W$$



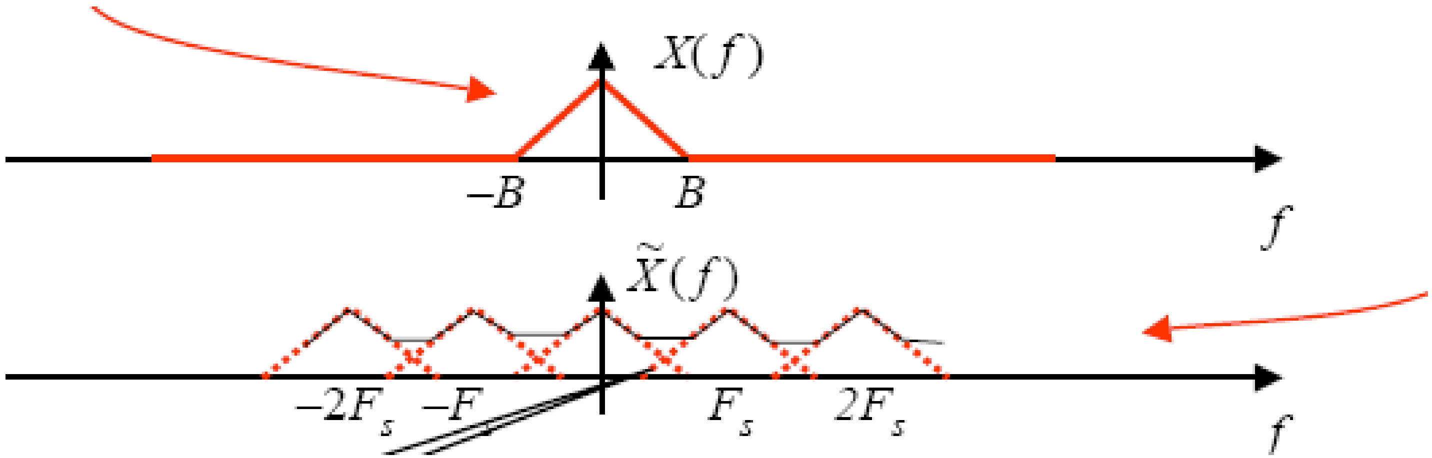


Case 1: T<1/2B

- Accordingly, the 2-signals would not interfere with each other. The original signal f(t) can be recovered from its sampled version by the use of an LPF of bandwidth W rad/s.
- Notation: **(1/2B) is called the *Nyquist interval***, **and 1/T is called the *Nyquist rate***

Under sampling
Case 2: T>1/2B

- In this case, $\omega_0 < 2W$



Accordingly, f(t) can not be recovered from its sampled version

General notes

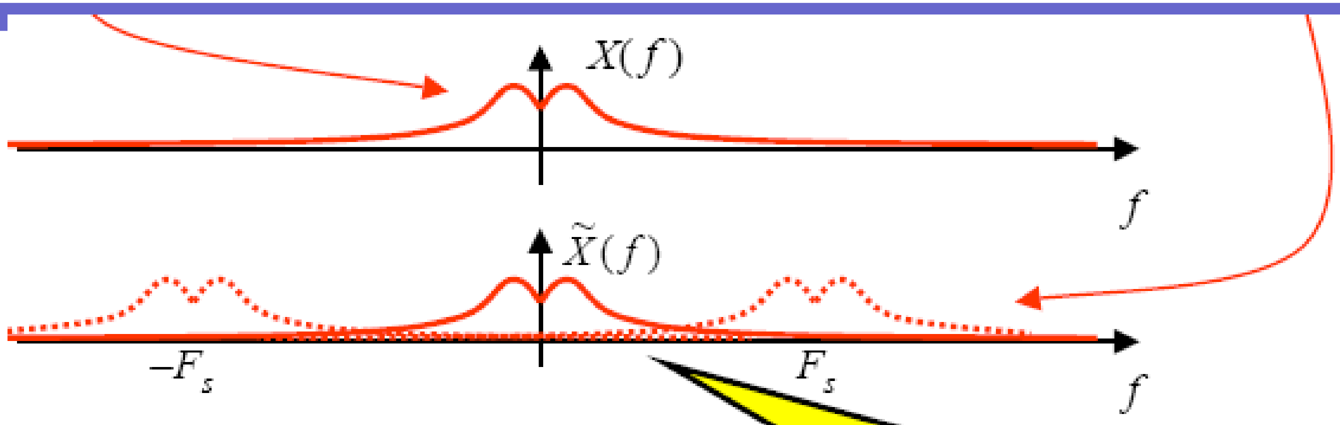
The minimum sampling rate of $F_s = 2B$ samples/sec is called the Nyquist Rate.

Sampling at the Nyquist rate is called Critical Sampling.

Sampling faster than the Nyquist rate is called Over Sampling

Note: Critical sampling is only possible if an IDEAL lowpass filter is used…. so in practice we generally need to choose a sampling rate somewhat above the Nyquist rate (e.g., $2.2B$); the choice depends on the application.

Aliasing: What if the signal f(t) is not band-limited



For Non-BL Signal Aliasing always happens regardless of F_s value

All practical signal are Non-BL!!!!

… so we choose F_s to minimize the aliasing to an level acceptable for the specific application