Chapter 8

Application of the Fourier Transform

Frequency Modulation

Characteristics of FM

- **Why need Frequency modulation?**
  - Better noise reduction
  - Improved system fidelity
- **Disadvantages**
  - Low bandwidth efficiency
  - Complex implementations
- **Applications**
  - FM radio broadcast
  - TV sound signal
  - Two-way mobile radio
  - Cellular radio
  - Microwave and satellite communications

Definition

In this modulation, the frequency of the carrier is varied linearly with the base-band signal m(t).

$$\omega_i(t) = \omega_c + k_f m(t)$$

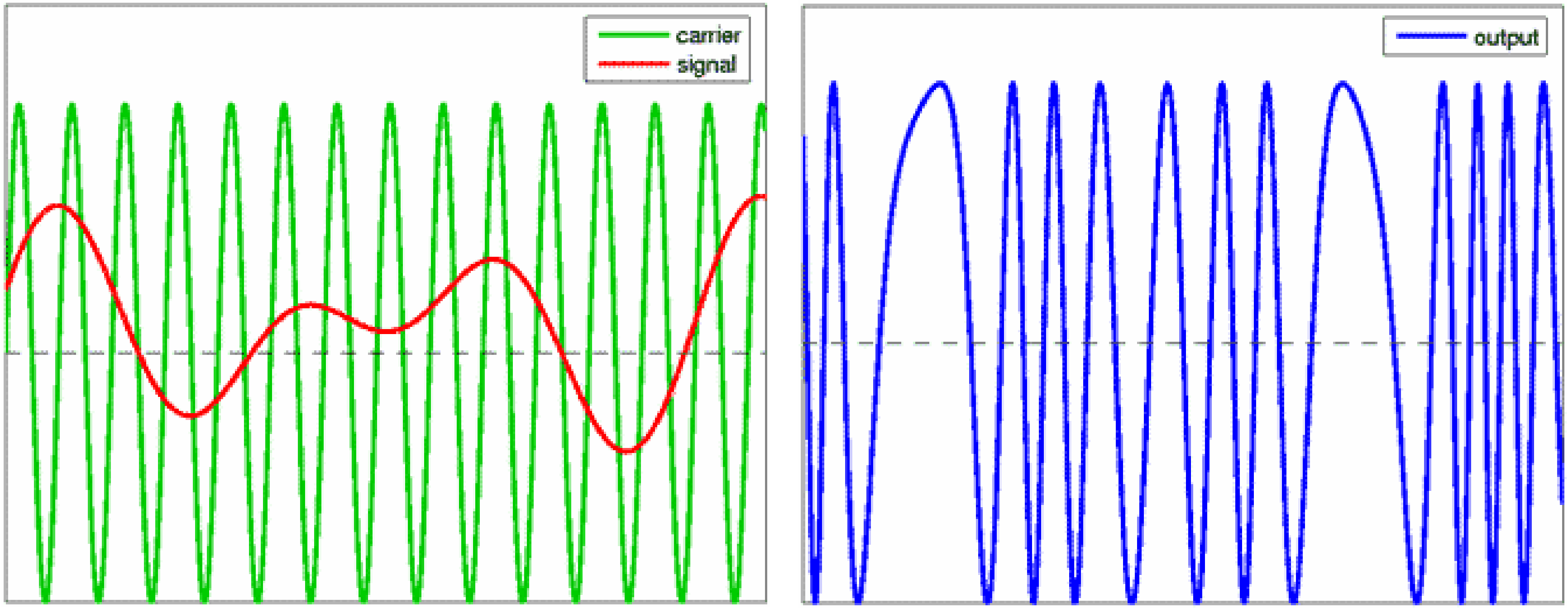
Where k_f is a constant and called frequency sensitivity.

$$\theta_i(t) = \int_0^t f_i(\tau)d\tau = \omega_c t + k_f \int_0^t m(\tau)d\tau$$

The frequency-modulated wave is therefore described in the time-domain by:

$$s(t) = A_c \cos\left[\omega_c t + k_f \int_0^t m(\tau)d\tau\right]$$

Figure of FM modulation



Single-tone modulation

In this case, the modulating signal is given by :

$$m(t) = a_m os(\omega_m t)$$

The corresponding FM wave is given by:

$$s(t) = A_c \cos\left[\omega_c t + k_f a_m \int_0^t \cos(\omega_m \tau) d\tau\right]$$

Let $\Delta\omega = k_f a_m$. $\Delta\omega$ is called the frequency deviation which represents the maximum departure of the instantaneous frequency of the FM from the carrier frequency

Single-tone modulation

With $\Delta\omega = k_f a_m$, the FM waveform becomes:

$$s(t) = A_c \cos\left[\omega_c t + \frac{\Delta\omega}{\omega_m} \sin \omega_m t\right]$$

The ratio of $\Delta\omega$ to the modulation frequency is called the modulation index of the FM wave and it is denoted by β

$$s(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$

The above is the time-representation of an FM wave for single-tone modulation

Types of FM Modulation

- Depending on the value of the modulation index β, we may distinguish two cases of frequency modulation:

1. Narrow-band FM **(NBFM)** for small β, β is small compared to 1 radian. $\beta<0.3$ is taken to be sufficient, but as high as 0.5 are used.
2. Wide-band FM **(WBFM)** when β is large compared to 1 radian

Narrow-band Frequency modulation (NBFM)

In general, the FM wave is given by:

$$s(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$

$$s(t) = A_c \cos \omega_c t \cos[\beta \sin \omega_m t] - A_c \sin \omega_c t \sin[\beta \sin \omega_m t].$$

For small β,

$$\cos[\beta \sin \omega_m t] \cong 1$$

$$\sin[\beta \sin \omega_m t] \cong \beta \sin \omega_m t$$

That is;

$$s(t) = A_c \cos \omega_c t - A_c \beta \sin \omega_c t \sin \omega_m t$$

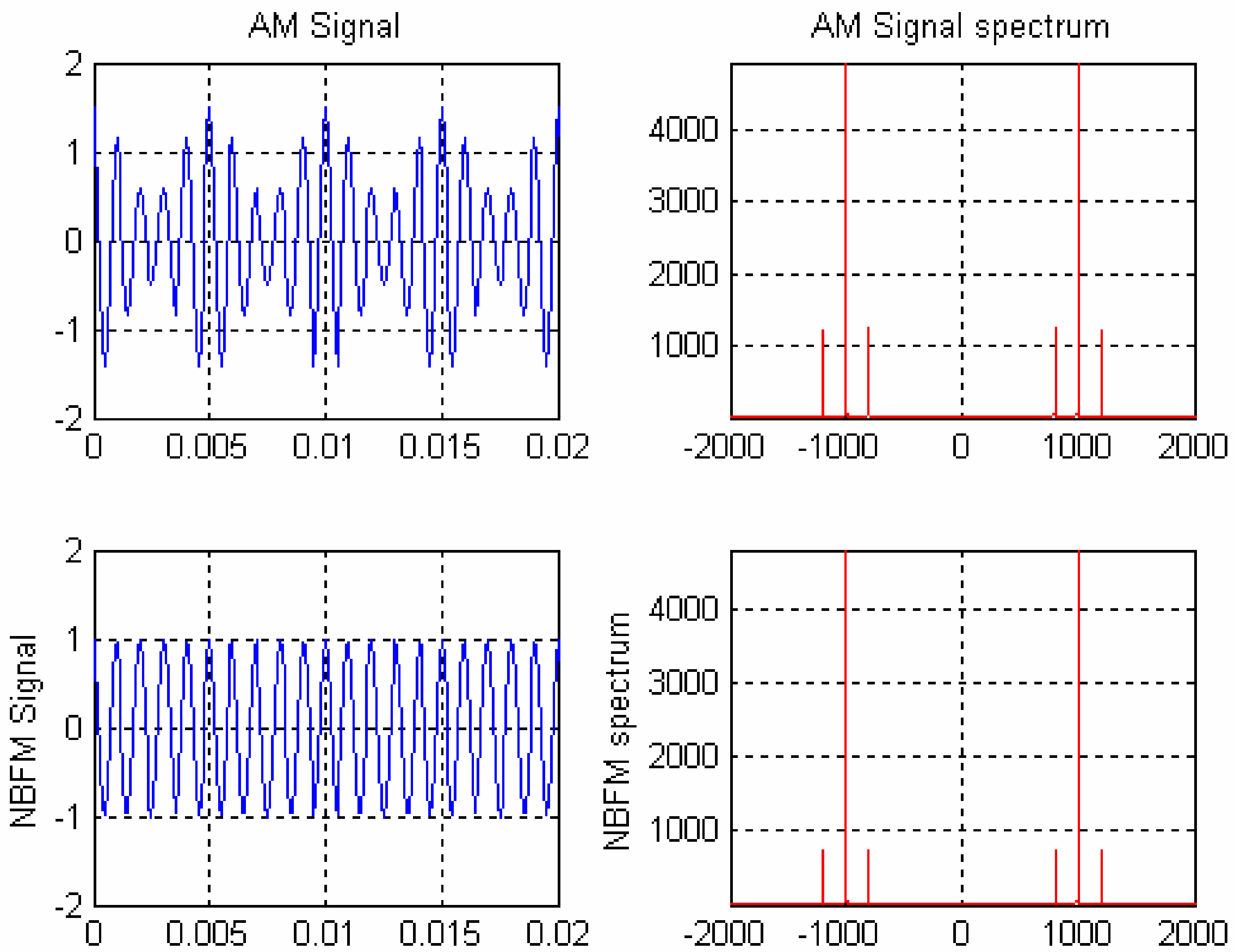
NBFM and AM

Note hat the expression representing NBFM wave is similar to that of AM for single-tone modulation

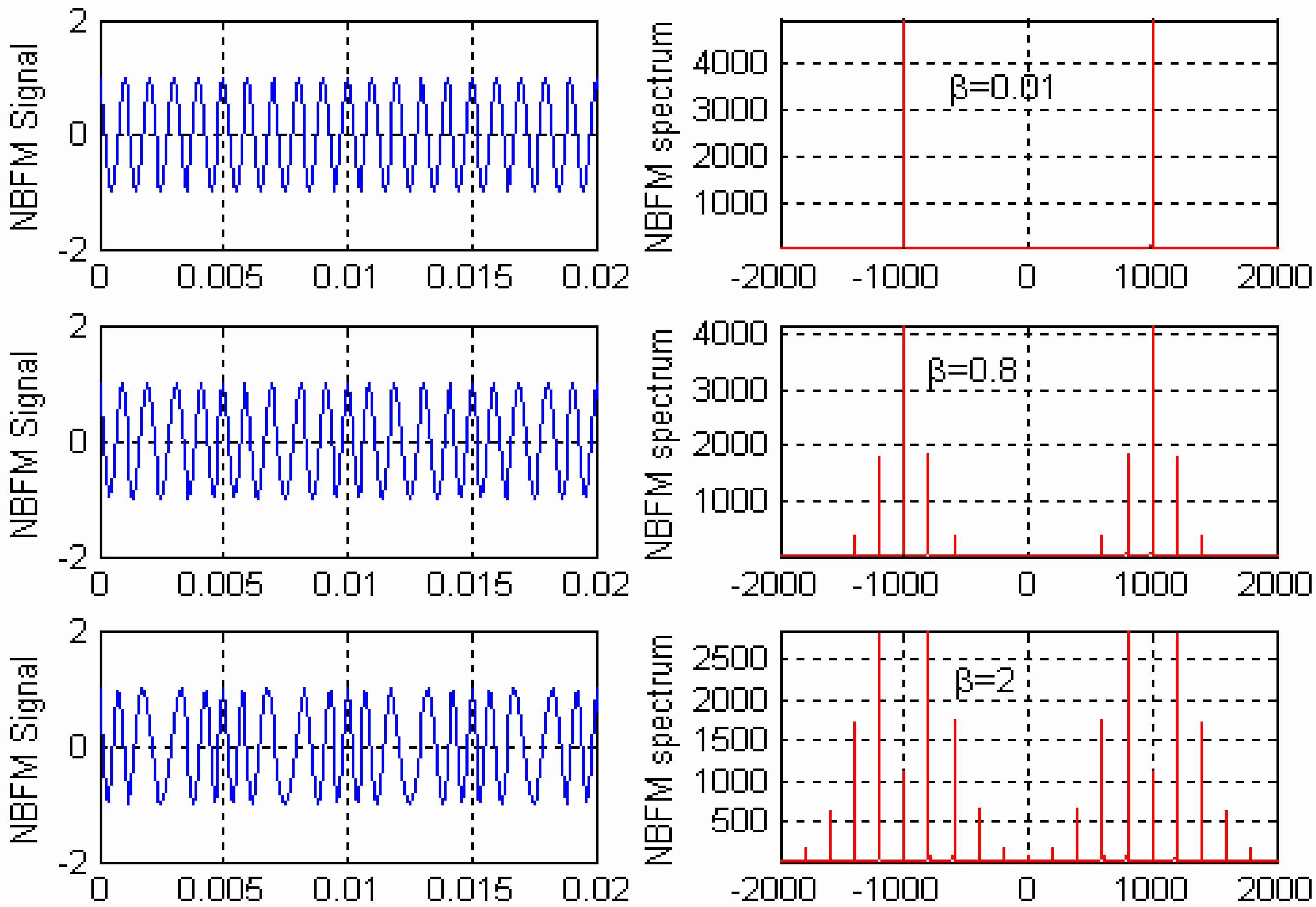
$$s_{FM}(t) = A_c \cos \omega_c t - \frac{A_c \beta}{2} \cos(\omega_c - \omega_m)t + \frac{A_c \beta}{2} \cos(\omega_c + \omega_m)t$$

$$s_{AM}(t) = A_c \cos \omega_c t + \frac{A_c \mu}{2} \cos(\omega_c - \omega_m)t + \frac{A_c \beta}{2} \cos(\omega_c + \omega_m)t$$

- Figure



AM Signal
AM Signal spectrum
NBFM Signal
NBFM spectrum
2
1
0
-1
-2
0
0.005
0.01
0.015
0.02
4000
3000
2000
1000
-2000
-1000
0
1000
2000


NBFM Signal
NBFM spectrum
β=0.01
β=0.8
β=2
0
0.005
0.01
0.015
0.02
-2000
-1000
0
1000
2000
4000
3000
2000
1000
2500
1500
500
2
-2

Spectrum of the Frequency modulation

$$s(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$

s(t) can be written in the following form

$$s(t) = \mathrm{Re}\left[A_c e^{j[\omega_c t + \beta \sin \omega_m t]}\right]$$

The Hilbert transform of s(t); $\hat{s}(t)$, is given by:

$$\hat{s}(t) = A_c \sin[\omega_c t + \beta \sin \omega_m t]$$

The pre-envelope $s_+(t)$ of s(t) is given by:

$$s_+(t) = s(t) + j\hat{s}(t) = A_c e^{j(\omega_c t + \beta \sin \omega_m t)}$$

Spectrum of the Frequency modulation

The complex envelope of s(t), $\tilde{s}(t)$ is given by:

$$\tilde{s}(t) = s_{+}(t)e^{-j\omega_c t} = A_c^{\,j\beta \sin \omega_m t}$$

$\tilde{s}(t)$ is a periodic function with period $T_0 = 1/f_m$. We may therefore express $\tilde{s}(t)$ in the form of complex Fourier series as follows:

$$\tilde{s}(t) = \sum_{n=-\infty}^{+\infty} C_n e^{j2\pi n f_m t}$$

where

$$C_n = f_m \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} \tilde{s}(t) e^{-j2\pi n f_m t} dt$$

$$C_n = A_c f_m \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} e^{j(\beta \sin 2\pi f_m t - 2\pi n f_m t)} dt$$

Spectrum of the Frequency modulation

$$C_n = A_c f_m \int_{\frac{1}{2f_m}}^{\frac{1}{2f_m}} e^{j(\beta \sin 2\pi f_m t - 2\pi n f_m t)} dt$$

Let $x = 2\pi f_m t$; $dx = 2\pi f_m dt$, then

$$C_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$$

The integral $\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx]dx$ is recognized as the nth order Bessel function of the first kind and argument β. This function is denoted as $J_n(\beta)$.

Spectrum of the Frequency modulation

$$C_n = A_c J_n(\beta)$$

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) e^{j\omega_m t}$$

The pre-envelope of s(t) is:

$$s_+(t) = \tilde{s}(t) e^{j\omega_c t}$$

$$s_+(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) e^{j(\omega_c t + n\omega_m t)}$$

and

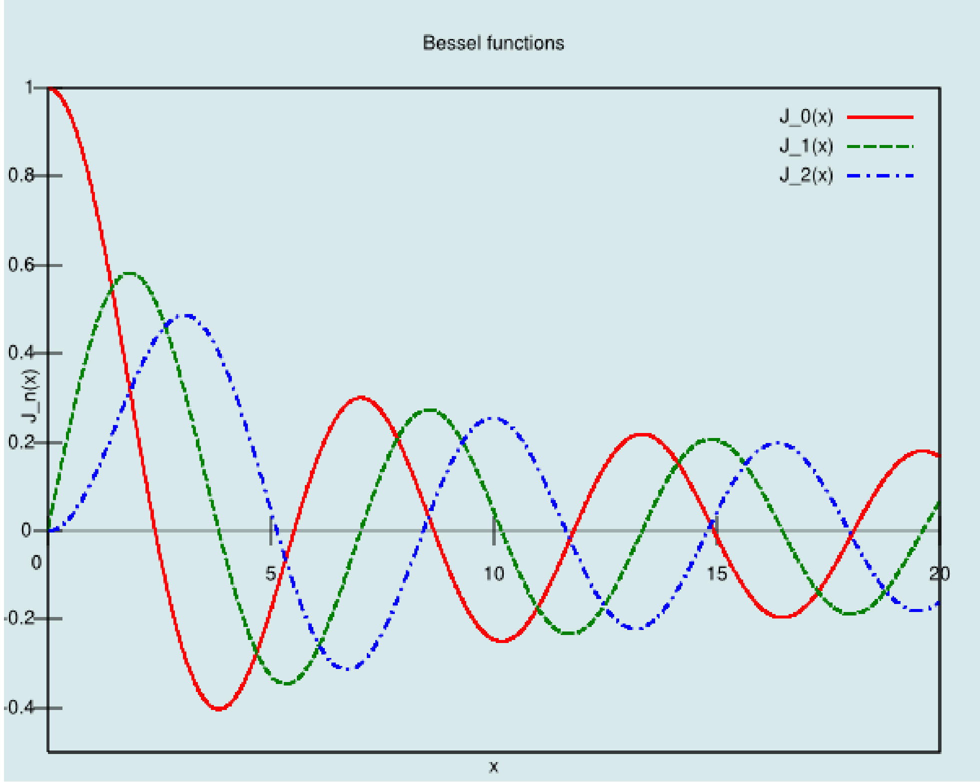
$$s(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) \cos[\omega_c t + n\omega_m t]$$

or

$$S(\omega) = A_c \pi \sum_{n=-\infty}^{+\infty} J_n(\beta)[\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

Properties of the Bessel Function

1. $J_n(\beta)$ is a real-valued signal
2. For n even $J_n(\beta) = J_{-n}(\beta)$
3. For n odd $-J_n(\beta) = J_{-n}(\beta)$
4. For small β, $J_0(\beta) = 1, J_1(\beta) \approx \beta/2, J_{-1}(\beta) \approx -\beta/2.$
5. $\sum_{n=-\infty}^{+\infty} J_n^2(\beta) = 1$ (Parseval's Theorem)



Transmission Bandwidth of FM waves

- In theory, FM wave contains an infinite number of side-bands. However, The magnitude of the higher-order side-bands becomes negligible and the power is contained within a finite bandwidth.
- A side-band is significant if its magnitude is equal to or exceeds 1 % of the unmodulated carrier; $|J_n(\beta)|>0.01$. This number can be obtained from the table for different values of β.

Transmission Bandwidth of FM waves

1. As β becomes very large, this number n can be approximated by β. In this case,

$$B_T = 2n\omega_m = 2\beta\omega_m = 2.\frac{\Delta\omega}{\omega_m}.\omega_m = 2\Delta\omega$$

2. For small β, the Bessel functions of significant magnitude are $J_0(\beta)$ and $J_1(\beta)$. Therefore, the transmission bandwidth is:

$$B_T = 2\omega_m$$

Transmission Bandwidth of FM waves

3. Intermediate cases.

One such rule was proposed by J.R. Carson and it is:

$$B_T = 2(\Delta\omega + \omega_m) = 2.\omega_m(1+\beta)$$

Carson's rule approaches the correct limit for both very large and very small β. It is widely used and gives convenient approximation. The maximum error occurs when β is in the neighbourhood of 1. Also Carson's rule applies for a band-limited signal of bandwidth W. In this case,

$$B_T = 2(\Delta\omega + W) = 2W(1+\frac{\Delta\omega}{W});$$

$\frac{\Delta\omega}{W}$ is denoted by D and called deviation ratio.

Transmission Bandwidth of FM waves