Chapter 5

Fourier Representation of Signals

Introduction

The study of signals and systems using sinusoidal representations is termed *Fourier analysis*, after Joseph Fourier (1768-1830) for his development of the theory. Fourier methods have widespread application beyond signals and systems, being used in every branch of engineering and science. There are four distinct Fourier representations, each applicable to a different class of signals, determined by the periodicity properties of the signal and whether the signal is discrete and or continuous in time

The 4 types of Fourier transforms

- Continuous Periodic Signals: Fourier Series (FS)
- Continuous non-periodic Signals : Fourier Transform (FT)
- Discrete-time Periodic Signals: Discrete Fourier Series (DTFS)
- Discrete-time Non-Periodic Signals: Discrete Fourier Transform (DTFT)

Fourier series

- Let f(t) be a periodic signal with period T_0 and frequency ω_0. By using a Fourier series expansion, f(t) can be written as follows:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

- Where

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t).dt$$

$$a_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(t).\cos n\omega_0 t.dt \qquad n = 1,2,3,\ldots$$

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(t).\sin n\omega_0 t.dt \qquad n = 1,2,3,\ldots$$

The exponential Fourier series

- The Fourier Series expansion of the periodic signal can also be written in an exponential form as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n . e^{jn\omega_0 t}$$

- F_n are denoted by the complex Fourier Series coefficients.

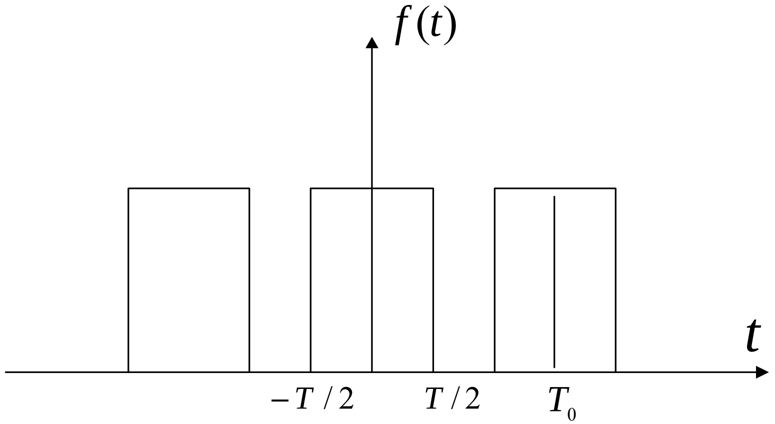
Determination of F_n

- Prove that

$$F_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jn\omega_0 t} dt$$

Example

Derive the exponential Fourier Series of the following function:



Answer:

$$F_n = \frac{A}{n.\pi}\sin\left(n\pi\frac{T}{T_0}\right) = \frac{AT}{T_0}\frac{\sin\left(n\pi\frac{T}{T_0}\right)}{\left(n\pi\frac{T}{T_0}\right)}$$

Parseval's theorem for power signals

- The average power developed across 1-ohm resistance is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 .dt \quad \text{watts} = \frac{1}{T}\int_{-T/2}^{T/2} f(t).f^*(t).dt$$

- Using the exponential Fourier series representation of f(t), we obtain

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \{\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \sum_{m=-\infty}^{\infty} F_m^* e^{-jm\omega_0 t}\} dt; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \sum_{m=-\infty}^{\infty} F_m^* \int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t} dt$$

Parseval's theorem for power signals

- But

$$\int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t} dt = \begin{cases} 0 & n \neq m \\ T & n = m \end{cases} \Rightarrow$$

- then,

$$P = \frac{T}{T} \sum_{n=-\infty}^{\infty} |F_n|^2 \Rightarrow P = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

- This relation is known as Parseval's theorem for periodic signals. Last equation illustrates the fact that the power in a periodic function is distributed over discrete frequencies that are harmonically related to one another

Example

- $f(t)=2\sin(100t)$

The Fourier Transform

Given a non-periodic (Aperiodic) signal f(t), the Fourier transform of f(t), if it exists, is given by:

$$F(\omega) = FT\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

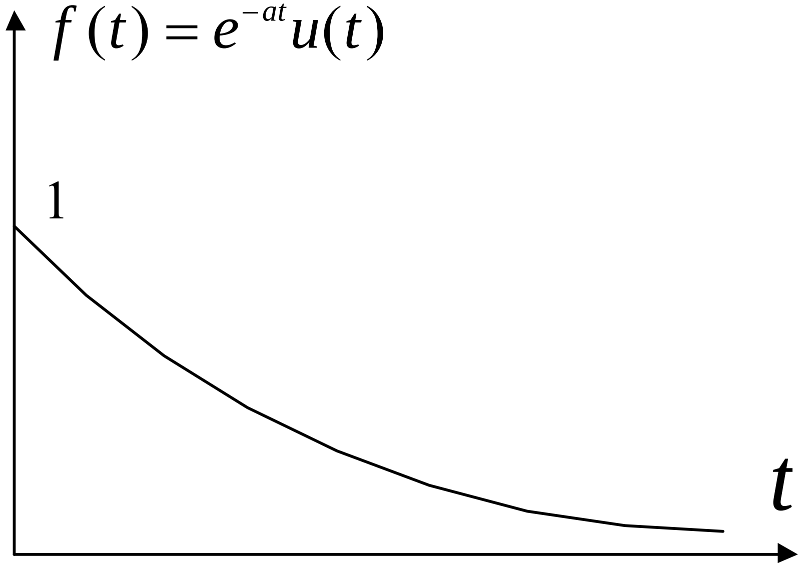
The Fourier Transform Pairs

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad \text{or} \quad f(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi f t} df$$

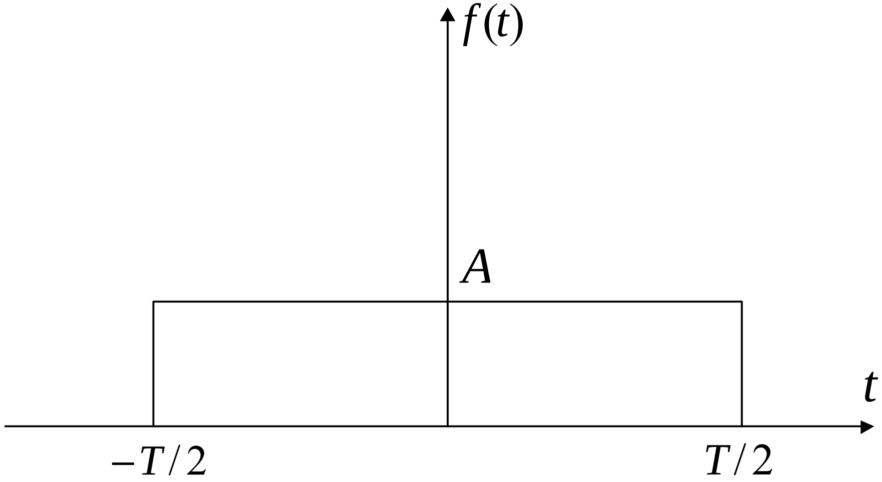
Example

$$f(t) = e^{-at}u(t) \qquad a > 0$$



Example

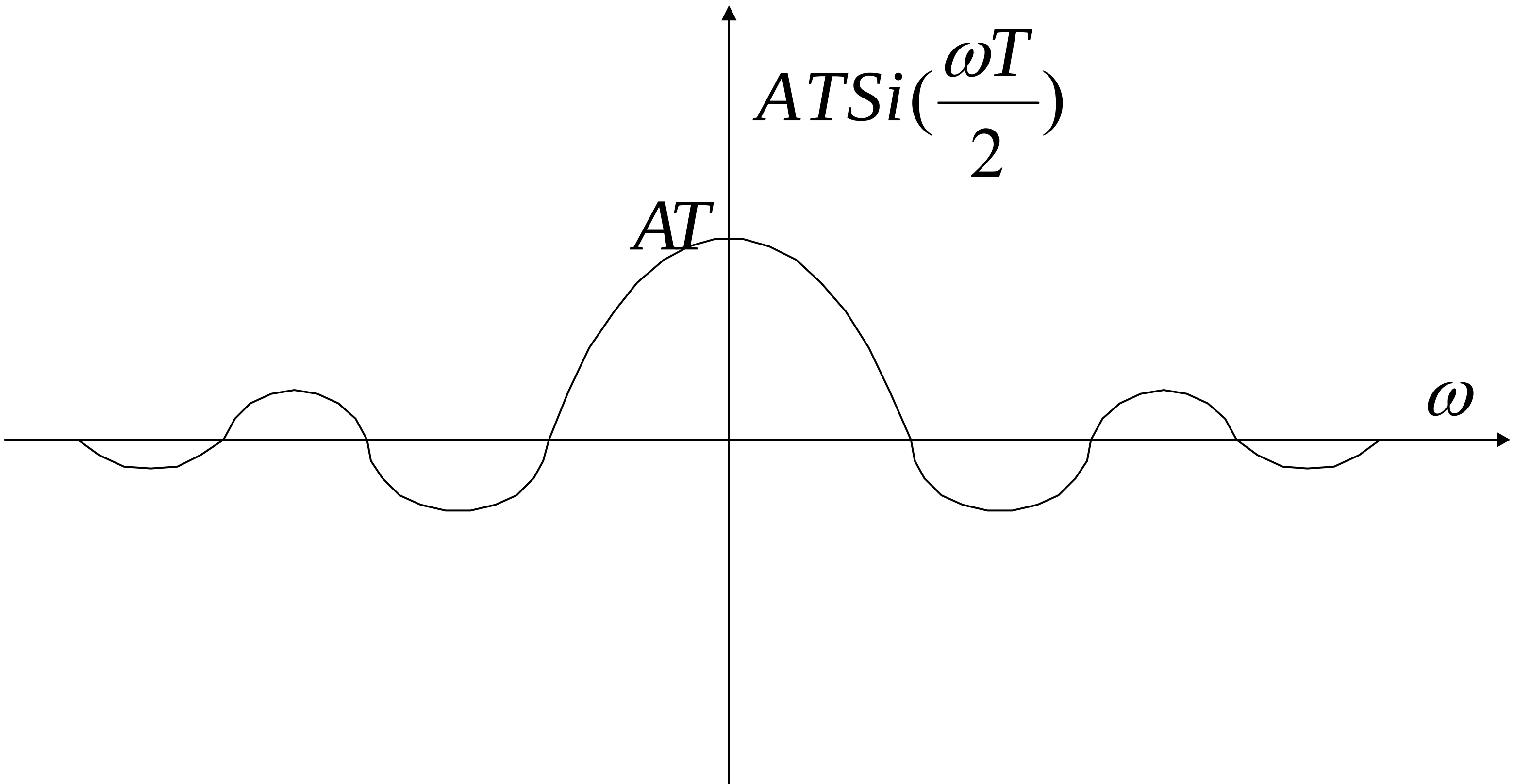
f(t)=A rect (t/T)



Answer:

$$F(\omega) = \frac{2A}{\omega}\sin\left(\omega\frac{T}{2}\right) = AT\frac{\sin(\omega T/2)}{(\omega T/2)} = ATSa(\omega T/2)$$

The Sa function



Existence of The Fourier Transform

1. f(t) has a finite number of maxima and minima in any finite time interval
2. f(t) has only a finite number of discontinuities in any finite time interval.
3. f(t) is absolutely integrable

Parseval's Theorem for Energy Signals

The energy delivered to 1 Ohm resistor by an energy signal f(t) is given by

$$E = \int_{-\infty}^{\infty} \left| f(t) \right|^2 dt = \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

where

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$f^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega$$

Parseval's Theorem for Energy Signals

- Replacing the results of f*(t) into the energy equation, we obtain

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t}d\omega\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|F(\omega)\right|^2 d\omega$$

Parseval's Theorem for Energy Signals

- Therefore, we can conclude

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\omega = \int_{-\infty}^{\infty}\left|f(t)\right|^2 dt$$

Example: The impulse signal δ(t)

$$F.T\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1$$

Example: Complex Exponential

$$f(t) = e^{j\omega_0 t}$$

$$I.F.T\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega$$

$$I.F.T\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}F.T\{e^{j\omega_0 t}\}$$

$$\delta(\omega-\omega_0) = \frac{1}{2\pi}F.T\{e^{j\omega_0 t}\}$$

$$F.T\{e^{j\omega_0 t}\} = 2\pi\delta(\omega-\omega_0)$$

Example: Complex Exponential

- From Previous example

$$F.T\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

- We can conclude

$$F.T\{e^{-j\omega_0 t}\} = 2\pi\delta(\omega + \omega_0)$$

- Moreover, for $\omega_0=0$

$$F.T\{1\} = 2\pi\delta(\omega)$$

Example: The sinusoidal functions

$$\mathrm{F.T}\{\cos\omega_0 t\} = \mathrm{F.T}\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right)$$

$$= \mathrm{F.T}\left(\frac{1}{2}e^{j\omega_0 t}\right) + \mathrm{F.T}\left(\frac{1}{2}e^{-j\omega_0 t}\right)$$

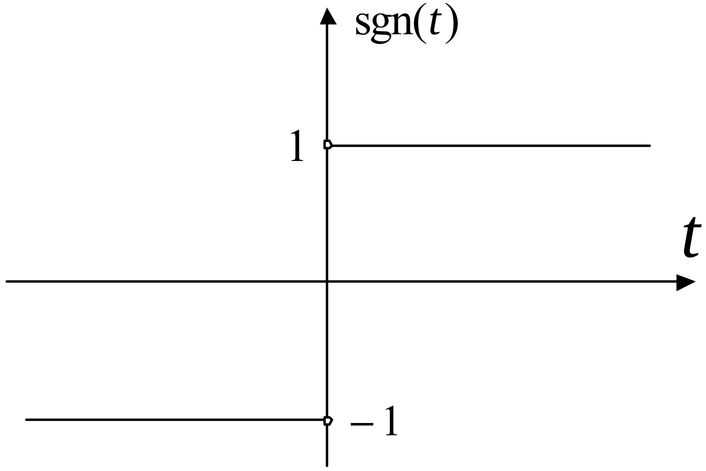
$$= \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$F.T\{\sin\omega_0 t\} = \frac{\pi}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

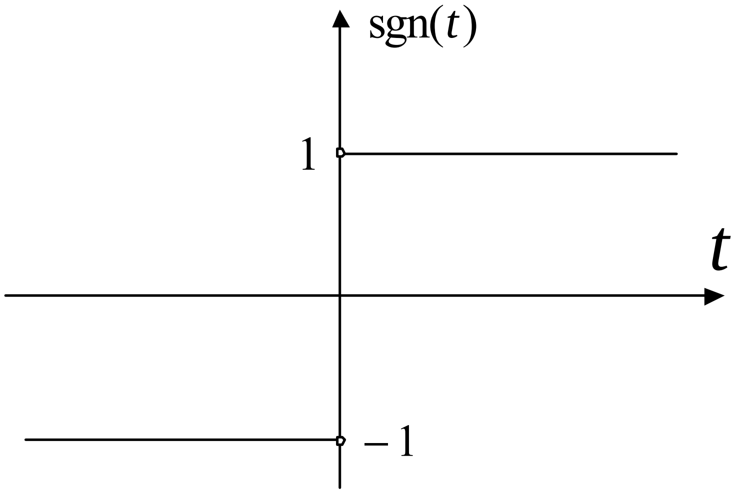
The signum function

- The signum function f(t)=sign(t) is defined as:

$$f(t)=\begin{cases} 1 & t>0 \\ 0 & t=0 \\ -1 & T<0 \end{cases}$$



Properties of the signum function



- 1.Average value=0
- 2.Piece-wise continuous
- 3.Not absolutely integrable.

Fourier Transform of sign(t)

Since sign(t) is not absolutely integrable, the following approach is used. Let $g(t) = e^{-a|t|}\text{sign}(t)$

The Fourier transform of g(t) is given by:

$$F.T\{g(t)\} = \int_{-\infty}^{\infty} e^{-a|t|}\,\text{sgn}\,t.e^{-j\omega t}dt = -\int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{-1}{a - j\omega} + \frac{1}{a + j\omega}$$

Therefore, the Fourier transform of f(t)=sign(t) is simply obtained by taking the limit of the Fourier transform of g(t) as a goes to zero.

$$F.T\{\text{sign}\ (t)\} = \frac{2}{j\omega}$$

Example: The Unit-Step function u(t)

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$\begin{aligned}
F.T\{u(t)\} &= F.T\left(\frac{1}{2}\right) + F.T\left(\frac{1}{2}\,\mathrm{sign}(t)\right) \\
&= \frac{1}{2}F.T\{1\} + \frac{1}{2}F.T\{\mathrm{sgn}(t)\} \\
&= \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\frac{2}{j\omega} = \pi\delta(\omega) + \frac{1}{j\omega}
\end{aligned}$$

The FT of a Periodic Function

- Let f(t) be a periodic function of frequency ω_0. The function f(t) can be written using the complex Fourier representation as:

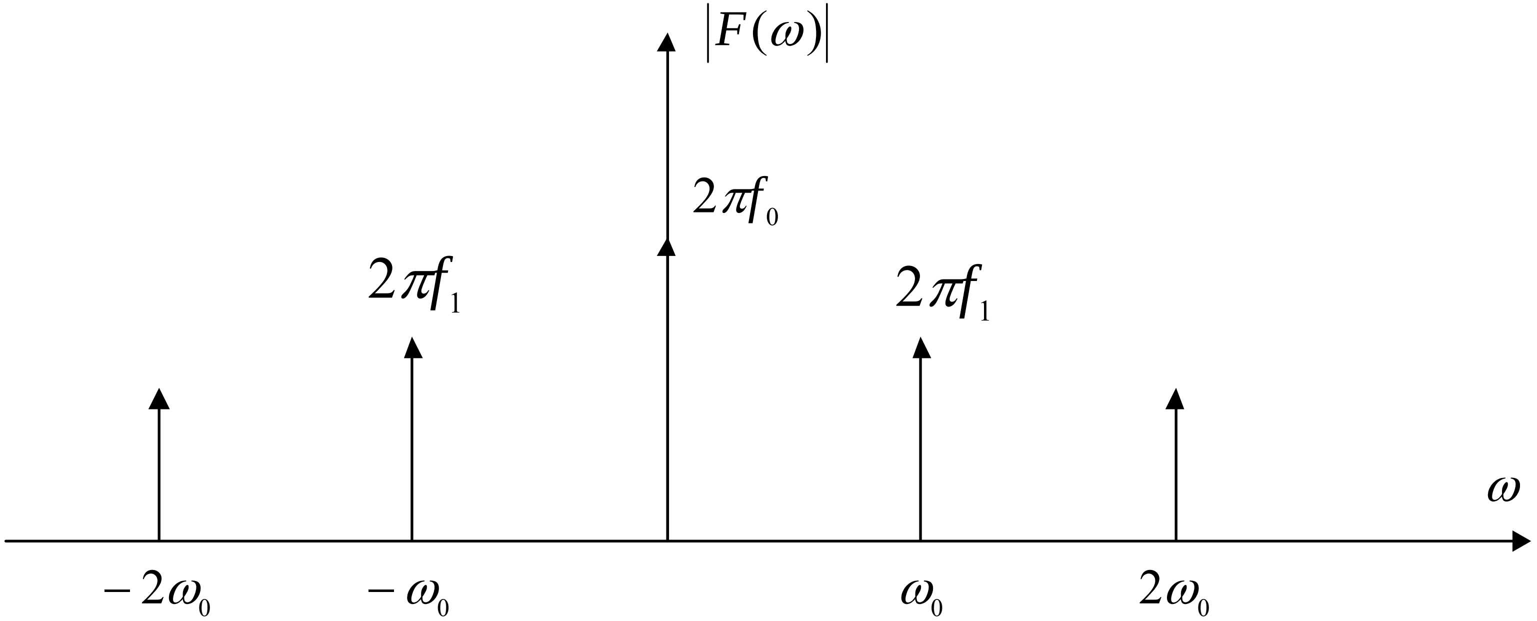
$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

- The Fourier transform of f(t) is:

$$F.T\{f(t)\} = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega 0 t} e^{-j\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} F_n \int_{-\infty}^{\infty} e^{-j(\omega - n\omega_0)t} dt = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0)$$

The FT of a Periodic Function



Properties of the Fourier transform

1. Linearity

$$f_1(t) \leftrightarrow F_1(\omega)$$

$$f_2(t) \leftrightarrow F_2(\omega)$$

$$F.T\{a_1.f_1(t) + a_2.f_2(t)\} \leftrightarrow a_1.F_1(\omega) + a_2.F_2(\omega)$$

Properties of the Fourier transform

2. Complex conjugate

$$f(t) \leftrightarrow F(\omega)$$

$$f^{*}(t) = F^{*}(-\omega)$$

Properties of the Fourier transform

3. Duality

$$f(t) \leftrightarrow F(\omega)$$

$$F(t) \leftrightarrow 2\pi f(-\omega)$$

Properties of the Fourier transform

4. Scaling

$$f(t) \leftrightarrow F(\omega)$$

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Properties of the Fourier transform

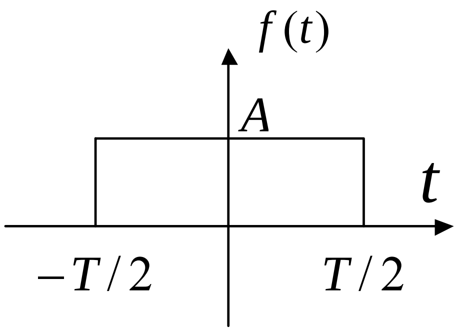
5. Time Shifting

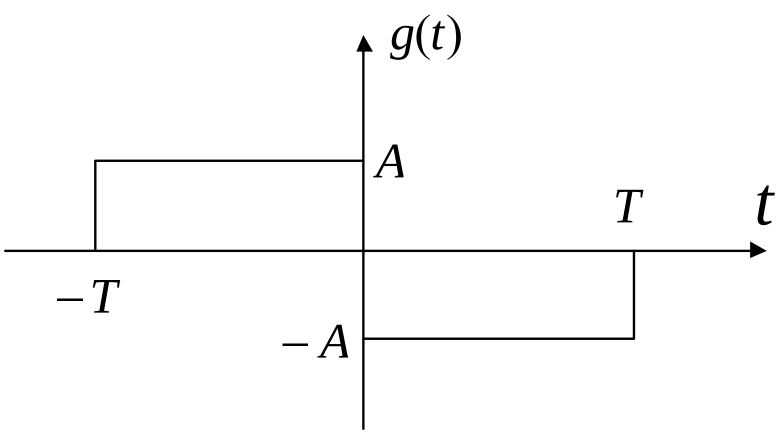
$$f(t) \leftrightarrow F(\omega)$$

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(\omega)$$

Example

- Determine the FT of g(t) as a function of that of f(t)



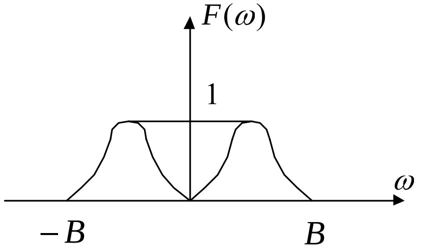


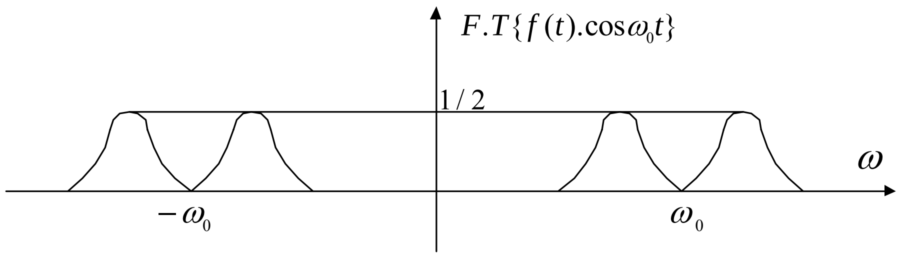
Properties of the Fourier transform

6. Frequency Shifting (VIP)

$$f(t) \leftrightarrow F(\omega)$$

$$f(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]$$





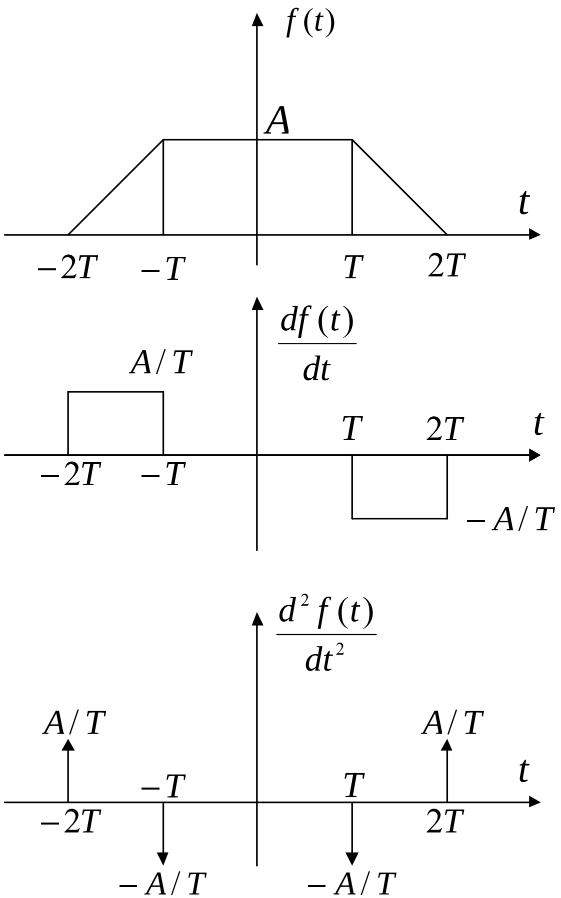
Properties of the Fourier transform

7. Differentiation

$$f(t) \leftrightarrow F(\omega)$$

$$F.T\left(\frac{df(t)}{dt}\right) \leftrightarrow j\omega F(\omega)$$

Example



$$g(t) = \frac{A}{T}\{\delta(t-2T) - \delta(t-T) - \delta(t+T) + \delta(t+2T)\}$$

$$G(\omega) = (j\omega)^2 F(\omega) = \frac{A}{T}\{e^{-j\omega 2T} - e^{-j\omega T} - e^{j\omega T} + e^{j\omega 2T}\}$$

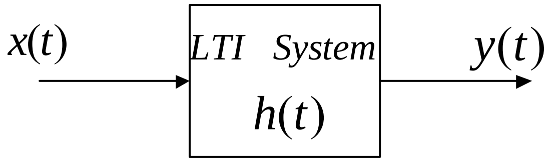
Properties of the Fourier transform

8. Integration

$$F.T\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \frac{F(\omega)}{j\omega} + \pi.F(0)\delta(\omega)$$

Properties of the Fourier transform

9. Time Convolution



h(t) impulse response of the system, x(t) is the input, and y(t) is the output. The input-output relationship is given by:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Properties of the Fourier transform

$$y(t) = x(t) * h(t)$$

$$\mathrm{Y}(\omega) = \mathrm{X}(\omega)\mathrm{H}(\omega)$$