EECE 340- Signals and Systems
Homework # 2

Problem # 1

Consider the instantaneous sampling of the following sinusoidal signal:

$$f(t) = 2\cos(\omega_m t)$$

It is desired to produce a DSB-SC signal from *f(t)* by sampling this signal and then using an ideal BPF of bandwidth equal to $2\omega_m$. The desired carrier frequency of the DSB-SC signal is 500KHz, which is considered much bigger than f_m. Determine the smallest sampling period that can be used to obtain the DSB-SC signal.

Problem # 2

Determine the spectrum $F(\omega)$ of the signal given below with ϕ being a constant phase.

$$f(t) = A\cos(\omega_0 t + \phi).$$

Problem # 3

Consider the following DSB-SC signal generated by the transmitter of a communication system:

$$s(t) = f(t)\cos(\omega_0 t).$$

The signal *f(t)* is the information signal with bandwidth equal to *10KHz*. Due to time delay caused by signal propagation between transmitter and receiver, the signal at the receiver input is given by

$$s_r(t) = f(t)\cos(\omega_0 t + \phi).$$

Assume that ϕ is independent of time (constant). Determine the minimum bandwidth, BW, of the band pass filter that needs to be used in the receiver to pick $s_r(t)$. Use the result of Problem # 11 and the correspondence between time domain multiplication and frequency domain convolution to obtain the spectrum of $s_r(t)$.

Problem # 4

Consider the following signal:

$$f(t) = 4\ rect\left(\frac{t}{4 \times 10^{-4}}\right).$$

Let $f(t)$ be present at the input of a LPF of bandwidth B Hz. The LPF output is sampled at the rate of $(1/T)$ samples/sec. Assume that the LPF is ideal and the sampling is instantaneous. Let B Hz be also the bandwidth of $f(t)$ defined by the first zero-crossing of the spectrum of $f(t)$ with the frequency axis. Determine the sampling rate that permits the LPF output signal to be retrieved from its sampled version by ideal low-pass filtering.

Problem # 5

Consider the following signal:

$$f(t) = 2\ rect\left(\frac{t - 10^{-5}}{2 \times 10^{-5}}\right).$$

Let $f(t)$ be sampled at the rate of 50,000 samples/sec to obtain the discrete time signal $f(n)$. Determine the magnitude of the Fourier transform of $f(n)$; i.e., $|F(\omega)|$.

Problem # 6

Consider the periodicity of the Fourier transform, $F(\omega)$, of a discrete time signal, $f(n)$, and the frequency shift property of $F(\omega)$; i.e., $f(n)e^{j\omega_0 nT} \leftrightarrow F(\omega - \omega_0)$. Determine all the values of ω_0 that make $F(\omega-\omega_0)=F(\omega)$.

Problem # 7

Consider a DSB-SC signal $p(t)$ given as follows:

$$p(t) = A_c f(t) \cos(\omega_c t).$$

The signal $f(t)$ is of low-pass type and having no frequency components above BHz. The DSB-SC signal is present at the input of an ideal BPF having a center frequency ω_c, bandwidth equal to 2B Hz and frequency response $H(\omega)$. The height of the magnitude frequency response of the BPF is 1 and its phase frequency response is equal to zero.

(a) Determine the output of the BPF.
(b) Draw the block diagram of a system that can be used to demodulate the DSB-SC signal.

Problem # 8

Consider the following low-pass signal:

$$f(t) = e^{-2t}\, u(t).$$

The signal *f(t)* is sampled at a rate equal to *1/T* , where *T* is the sampling period. Plot the magnitude spectrum of the sampled signal and determine the bandwidth of an ideal LPF that picks the frequency components in *f(t)* within its half-power bandwidth.

Problem # 9

Let a discrete-time signal *f(n)* be given by:

$$f(n) = e^{-2n}\, u(n) = a^n\, u(n)$$

The signal *f(n)* is obtained by sampling $f(t) = e^{-2t}\, u(t)$ at $t{=}nT$.

(a) Apply the Discrete-Time Fourier transform; i.e., $F(\omega) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega nT}$, to *f(n)* and determine *F(ω)*.

(b) Determine and plot the magnitude spectrum of *f(n)*; i.e., $|F(\omega)|$, and say if this is consistent with the plot you obtained in Problem 8.

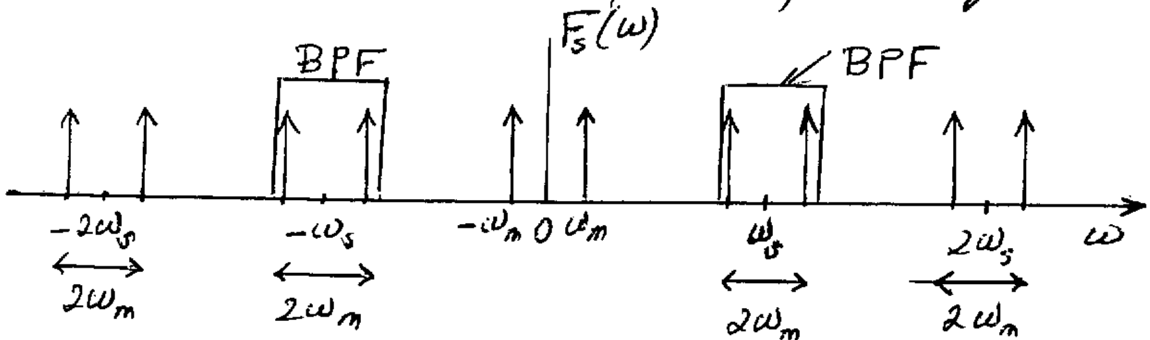
EECE 340 – Homework # 2 Solution
Signals and Systems

Problem #1

The spectrum of the instantaneously sampled version of $f(t)$ is given by:

$$F_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s),$$

where $F(\omega) = 2\pi\,\delta(\omega - \omega_m) + 2\pi\,\delta(\omega + \omega_m)$, $\omega_s = 2\pi/T_s$ and T_s is the period of sampling.



The smallest sampling period can be obtained from the largest sampling rate, $\frac{1}{T_s}$. Hence, since the largest $\frac{1}{T_s} = 500\,KHz$, as determined from the above figure, then the smallest

$$T_s = \frac{1}{500 \times 10^3} = 2 \times 10^{-6}\ \text{secs or } 2\,\mu s.$$

With the above sampling frequency, the BPF having a center frequency 500 KHz and bandwidth $2\omega_m$ picks the desired DSB-SC signal.

Of course, the sampling rate $\frac{1}{T_s}$ can be smaller than 500 KHz. For example, 250, 125, 62.5 KHz, ... can also be used. But, $\frac{1}{T_s}$ must be greater than $2\omega_m$.

Problem # 2

The following signal is considered:

$$f(t) = A\cos(\omega_0 t + \phi)$$

ϕ is a constant phase.

$f(t)$ can be written as follows:

$$f(t) = \frac{A}{2}\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right]$$

$$= \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

$$F(\omega) = \frac{A}{2} e^{j\phi} \times 2\pi\delta(\omega-\omega_0) + \frac{A}{2} e^{-j\phi} \times 2\pi\delta(\omega+\omega_0)$$

$$= A\pi e^{j\phi}\delta(\omega-\omega_0) + A\pi e^{-j\phi}\delta(\omega+\omega_0)$$

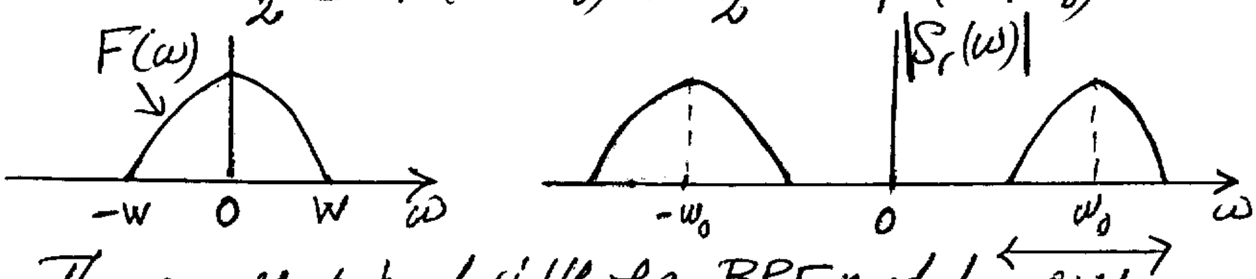
Problem # 3

$$S_r(t) = f(t)\cos(\omega_0 t+\phi)$$

$$S_r(\omega) = \frac{1}{2\pi} F(\omega) * \mathcal{F}\{\cos(\omega_0 t+\phi)\}$$

$$= \frac{1}{2} F(\omega) * \left[e^{j\phi}\delta(\omega-\omega_0) + e^{-j\phi}\delta(\omega+\omega_0)\right]$$

$$= \frac{1}{2} e^{j\phi} F(\omega-\omega_0) + \frac{1}{2} e^{-j\phi} F(\omega+\omega_0)$$



The smallest bandwidth of an BPF needed

Problem # 4

$$f(t) = 4\,\mathrm{rect}\left[\frac{t}{4\times 10^{-4}}\right]$$

$$f(t) \longrightarrow \boxed{\begin{array}{c}\text{Ideal}\\ \text{LPF}\\ B\text{ Hz}\end{array}} \xrightarrow{f_o(t)} \times \xrightarrow{f_{os}(t)} \boxed{\begin{array}{c}\text{Ideal}\\ \text{LPF}\end{array}} \xrightarrow{f_o(t)}$$

$f(t)$ is a rectangular pulse having a duration equal to $T_d = 4\times 10^{-4}$ secs. Hence, the spectrum of $f(t)$; i.e., $F(\omega)$ is a Sa function. The first zero crossing of $F(\omega)$ with the ω axis; i.e., the bandwidth of $f(t)$ is given by $B = \frac{1}{T_d} = \frac{1}{4\times 10^{-4}} = 2500$ Hz.

Hence, the first LPF produces a bandlimited signal, $f_o(t)$, having a bandwidth equal to $B = 2500$ Hz.

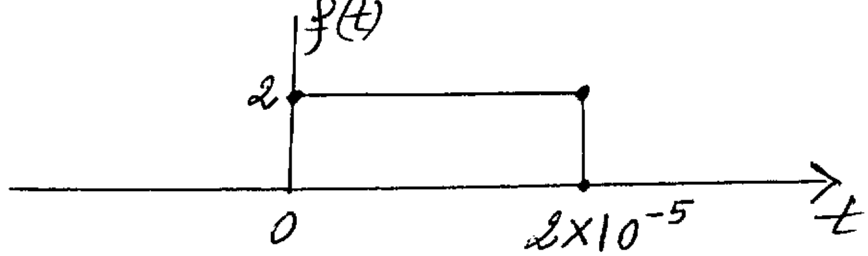
Now, in order to be able to retrieve $f_o(t)$ from its sampled version; i.e., $f_{os}(t)$, by ideal low-pass filtering, we apply the Nyquist condition:

$$\text{Sampling rate} = \frac{1}{T} \geqslant 2B = 2\times 2500$$

$$= 5000 \text{ samples/sec.}$$

Problem #5

$$f(t) = 2\,\mathrm{rect}\left[\frac{t - 10^{-5}}{2\times 10^{-5}}\right]$$



$$\frac{1}{T} = 50{,}000 \text{ samples/sec.} \Rightarrow T = 2\times 10^{-5} \text{ secs.}$$

Hence, $f(n) = \begin{cases} 2, & n = 0,1 \\ 0, & \text{elsewhere.} \end{cases}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega n T} = f(0) + f(1) e^{-j\omega T}$$

$$= 2\left(1 + e^{-j\omega T}\right) = 4 e^{-j\frac{\omega T}{2}} \left[\frac{e^{j\frac{\omega T}{2}} + e^{-j\frac{\omega T}{2}}}{2}\right]$$

$$= 4 e^{-j\frac{\omega T}{2}} \cos\left(\frac{\omega T}{2}\right).$$

$$\Rightarrow |F(\omega)| = 4\left|\cos\frac{\omega T}{2}\right|.$$

Alternatively, the following can be written:

$$F(\omega) = 2\left(1 + e^{-j\omega T}\right) = 2\left(1 + \cos(\omega T) - j\sin(\omega T)\right)$$

$$|F(\omega)| = 2\left[\left(1 + \cos(\omega T)\right)^2 + \sin^2(\omega T)\right]^{1/2}$$

$$= 2\left[2\left(1 + \cos(\omega T)\right)\right]^{1/2} = 2\left[4\cos^2\frac{\omega T}{2}\right]^{1/2}$$

$$= 4\left|\cos\frac{\omega T}{2}\right|$$

Problem # 6

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega nT} \text{ is periodic with period } \frac{2\pi}{T}.$$

Hence, any frequency shift of $F(\omega)$ by an integer multiple of its period; i.e., by $\frac{2k\pi}{T}$, where $k=0,\pm1,\pm2,\ldots$ provides $F(\omega)$.

Therefore, $F(\omega-\omega_0)=F(\omega)$ if $\omega_0=\frac{2k\pi}{T}$, where $k=0,\pm1,\pm2,\pm3,\ldots$

Checking can also be done as follows:

$$F(\omega-\omega_0) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega nT} e^{j\omega_0 nT}$$

$$= \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega nT} \underbrace{e^{j\frac{2k\pi}{\cancel{T}} n\cancel{T}}}_{=1 \text{ for } k=0,\pm1,\pm2,\ldots}$$

$$= F(\omega).$$

Problem # 7 $\quad p(t) = A_c f(t) \cos(\omega_c t)$; DSB-SC.

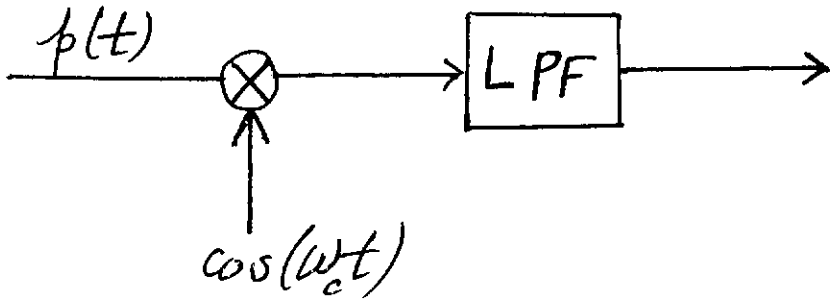
(a) The ideal BPF has the same bandwidth $(2B)$ and center frequency (ω_c) as the DSB-SC signal spectrum. With $|H(\omega)|=1$ and $\angle H(\omega)=0$ within the BPF bandwidth, then the BPF output is the same as the DSB-SC

Also, within the $2B$ bandwidth, we can write,

$$P_o(\omega) = P(\omega)\,H(\omega)$$
$$= P(\omega)\,|H(\omega)|\,e^{j\angle H(\omega)}$$
$$= P(\omega) \times 1 \times e^{j0} = P(\omega).$$

$\Rightarrow$ The BPF output $p_o(t) = p(t) = A_c f(t) \cos(\omega_c t)$.

(b)



$$p(t)\cos(\omega_c t) = A_c f(t) \cos^2(\omega_c t)$$
$$= \frac{1}{2} A_c f(t) + \frac{1}{2} A_c f(t) \cos(2\omega_c t)$$

A LPF of bandwidth B Hz picks $\frac{1}{2} A_c f(t)$.

We need $2f_c - B \geq B$ or $f_c \geq B$.

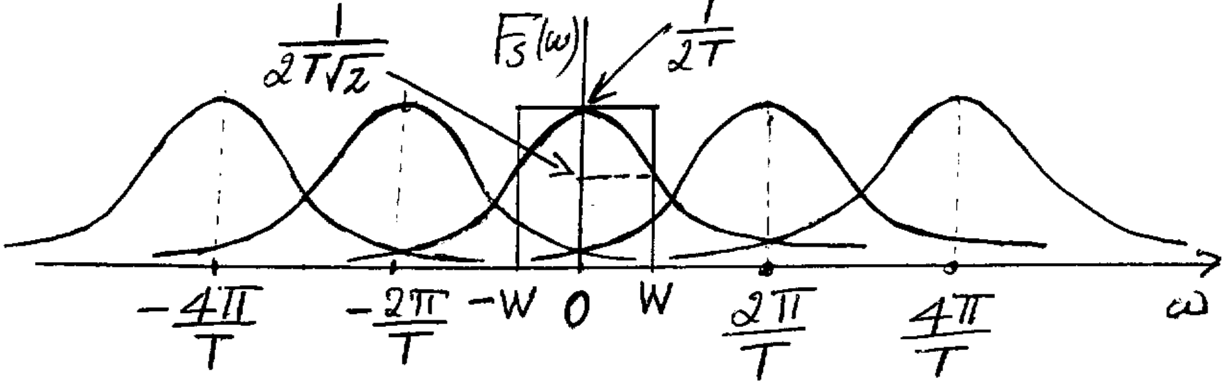
In practice, the carrier frequency is usually much larger than the bandwidth of the message signal $f(t)$. For speech signals, for example, B is in the order of 4 KHz, while f_c is in the order of 1000's of KHz or even in the GHz.

Problem # 8

$$f(t) = e^{-2t}u(t)$$

$$F(\omega) = \frac{1}{2+j\omega} \Rightarrow |F(\omega)| = \frac{1}{\sqrt{\omega^2+4}}$$

$$|F_s(\omega)| = \frac{1}{T}\sum_{n=-\infty}^{\infty} |F(\omega - n\omega_s)| \; ; \; \omega_s = \frac{2\pi}{T}.$$



Let W be the bandwidth of the LPF that picks the frequency components in f(t) within its half-power bandwidth. W is such that:

$$\frac{1}{\sqrt{W^2+4}} = \frac{1}{2\sqrt{2}} \Rightarrow W^2+4 = 8 \Rightarrow W = 2 \text{ rad/sec}$$

$$\text{or } B = \frac{1}{\pi} \text{ Hz.}$$

Problem #9

$$f(n) = e^{-2n}\mu(n) = a^n \mu(n)$$

$$\Rightarrow a = e^{-2}.$$

(a)
$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega nT}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega nT} = \sum_{n=0}^{\infty} \left[a e^{-j\omega T}\right]^n$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N-1} \left[a e^{-j\omega T}\right]^n$$

$$= \lim_{N\to\infty} \left[\frac{1-(a e^{-j\omega T})^N}{1-a e^{-j\omega T}}\right]$$

$$= \frac{1}{1-a e^{-j\omega T}} \quad \text{since } 0 < a = e^{-2} < 1.$$

(b)
$$|F(\omega)| = \frac{1}{\left[(1-a\cos(\omega T))^2 + (a\sin\omega T)^2\right]^{1/2}}$$

This plot is consistent with $|F_s(\omega)|$ as in Prob. 8.

$$= \frac{1}{\left[1+a^2-2a\cos(\omega T)\right]^{1/2}}$$