EECE 340 – Signals and Systems
Homework # 3

Problem # 1

A realizable system has the frequency response given by:

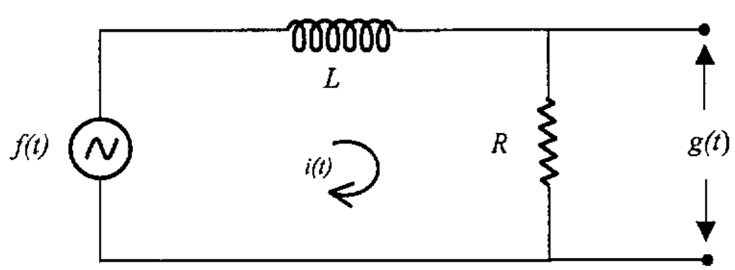
$$H(\omega) = \frac{1}{1 + j\omega}.$$

The system is presented with input $f(t) = 2u(t)$, where $u(t)$ is the unit step function.

a) Determine the system output represented by $g(t)$.
b) Repeat exercise when $H(\omega) = \dfrac{1}{(1 + j\omega)^2}$.

Problem # 2

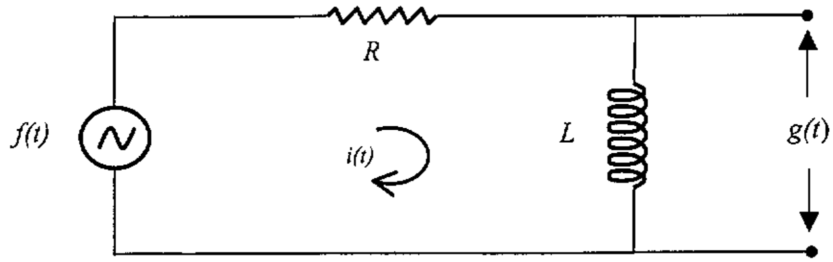
Consider the RL circuit drawn below.



a) Determine first the transfer function, $H(s)$, of the circuit and then obtain the circuit impulse response, $h(t)$.
b) Determine the system output when the input signal $f(t)=u(t)$.

Problem # 3

In this problem, the series RL circuit shown below is considered.



Determine the impulse response, $h(t)$, of the RL circuit.

Problem # 4

Consider the linear, time invariant and causal system whose transfer function, $H(s)$, is given by

$$H(s) = \frac{2s+1}{(s+1)(s+4)}$$

Let this system have an output represented by the voltage drop, $v_0(t)$, across the terminals of some electric device. Determine the initial value of the voltage; i.e., $v_0(0^+)$, when the system is presented with an input $v_i(t)=\delta(t)$.

Problem # 5

Consider the following band-pass signal:

$$f(t) = e^{-2t}\cos(\omega_c t)\,u(t).$$

Let $f(t)$ be present at the input of a linear and time-invariant system with impulse response given by:

$$h(t) = 2A\cos(\omega_c t)\,u(t).$$

Determine the output, $g(t)$, of the system.

Problem # 6

The signal $f(t) = e^{-2t}\ u(t)$ has a Laplace transform given by:

$$F(s) = \frac{1}{(s+2)}\quad for\ \sigma > -2.$$

Use *F(s)* to determine first the Fourier transform of $g(t) = 2e^{-2t}\cos(\omega_c t)\, u(t)$ and then the Laplace transform, G(s), of g(t).

Problem # 7

Consider a signal f(t) with Laplace transform given by:

$$F(s) = \frac{s-2}{(s-2)^2+\omega_c^2} = \frac{s-2}{(s-2-j\omega_c)(s-2+j\omega_c)}\quad for\ \sigma > 2.$$

Determine *f(t)* using the inverse Laplace formula or partial fraction expansion.

Problem # 8

Consider a linear and time-invariant system with impulse response given by:

$$h(t) = 2e^{-t}u(t) + e^{2t}u(-t)$$

Determine the ROC of the system transfer function, *H(s)*, and specify if the system is or is not stable.

Problem # 9

Consider the following differential equation representing a linear and time-invariant system:

$$\frac{d^2g(t)}{dt^2} + 2\frac{dg(t)}{dt} + g(t) = 2f(t)$$

In this differential equation, *f(t)* represents the system input and g(t) the system output.

Determine the transfer function, *H(s)*, of the system and then obtain the system output, *g(t)*, when the input *f(t)=u(t)*. Use the Laplace transform analysis and assume that the system is causal.

Problem # 10

Consider the LTI system whose transfer function is given by

$$H(s) = \frac{1}{(s+1)(s-2)}$$

Assume that the system is unstable and non-causal. Determine the impulse response of the system by applying the inverse Laplace transform formula and contour integration in the s-plane.

EECE 340 – Signals and Systems
Homework #3 Solution

Problem #1

(a) $H(\omega) = \dfrac{1}{1+j\omega}$ is the frequency response of causal system.

$f(t) = 2u(t)$ is the system input, which is also causal.

$\Rightarrow$ The one-sided Laplace applies.

$$G(s) = H(s)F(s) = \frac{2}{s(s+1)} = \frac{2}{s} - \frac{2}{s+1}$$

$\Rightarrow$ the system output is: $g(t) = 2(1-e^{-t})u(t)$.

(b) $$G(s) = H(s)F(s) = \frac{2}{s(s+1)^2}$$

$$= \frac{A}{s} + \frac{Bs+C}{(s+1)^2}$$

This gives $A=2$, $B=-2$, $C=-4$.

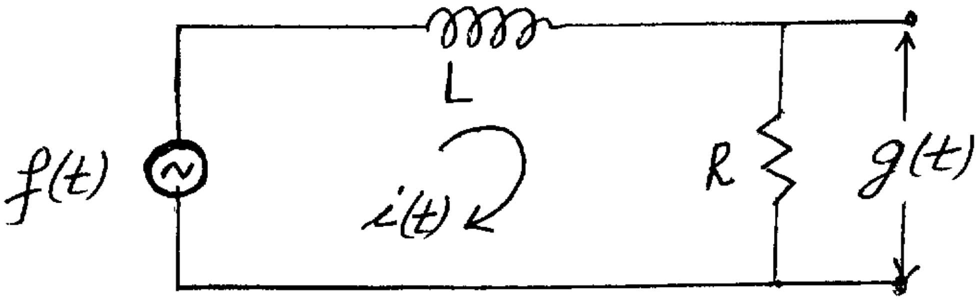
Hence, $$G(s) = \frac{2}{s} - \frac{4}{(s+1)^2} - \frac{2s}{(s+1)^2}$$

and

$$g(t) = \left[2 - 4te^{-t} - 2\frac{d}{dt}(te^{-t})\right]u(t)$$

$$= \left[2 - 2e^{-t} - 2te^{-t}\right]u(t)$$

Problem #2



(a) $F(s) = LsI + RI = (Ls+R)I$

$G(s) = RI$

$$\Rightarrow H(s) = \frac{G(s)}{F(s)} = \frac{R}{Ls+R} = \frac{R/L}{s+R/L}$$

$H(s)$ as above is the transfer function of the RL circuit with input $f(t)$ and output $g(t)$.

The circuit impulse response is:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{R}{L} e^{-(R/L)t} \mu(t).$$

$h(t)$ is the circuit output when the input is $\delta(t)$; Hence, $h(t)$ must be zero for $t<0$ since the system is a physical or causal system.

Note that $H(s)$ can also be obtained by applying the Laplace transform to the circuit differential equation under zero initial conditions.

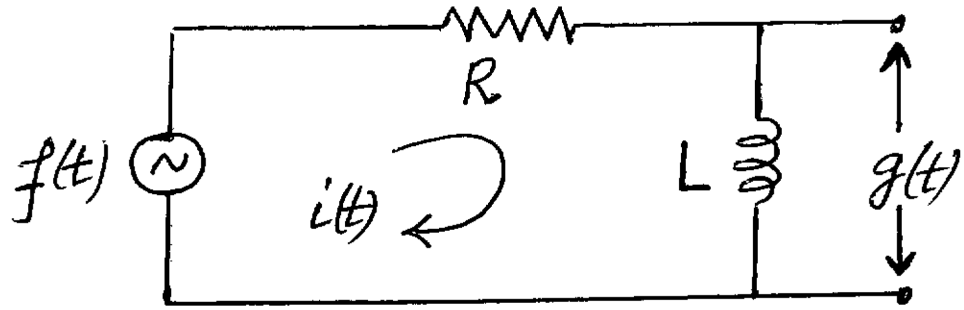
(b) $f(t) = u(t)$

$$G(s) = H(s)F(s) = \frac{R/L}{s(s + R/L)}$$

$$= -\frac{1}{s + R/L} + \frac{1}{s}$$

$$\Rightarrow g(t) = \underbrace{-e^{-(R/L)t}u(t)}_{\substack{\text{transient response} \\ \text{(natural response)}}} + \underbrace{u(t)}_{\substack{\text{steady state response} \\ \text{(forced response)}}}$$

<u>Problem # 3</u>



$$F(s) = RI + LsI = (R + Ls)I$$

$$G(s) = LsI$$

$$\Rightarrow H(s) = \frac{G(s)}{F(s)} = \frac{Ls}{Ls + R} = \frac{s}{s + R/L}$$

$$h(t) = \frac{d}{dt}\left[e^{-(R/L)t}u(t)\right]$$

$$= -\frac{R}{L}e^{-(R/L)t}u(t) + e^{-(R/L)t}\delta(t)$$

$$= -\frac{R}{L}e^{-(R/L)t}u(t) + \delta(t)$$

Problem # 4

$$H(s) = \frac{2s+1}{(s+1)(s+4)}$$

$v_i(t) = \delta(t) \Rightarrow v_o(t) = h(t)$. Hence, $v_o(0^+) = h(0^+)$.

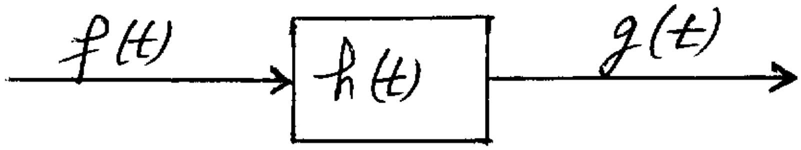
Now, since the system is causal, then we can determine $h(t)$ using the partial fraction expansion of $H(s)$ and the Laplace transform tables (one-sided Laplace).

$$H(s) = -\frac{1/3}{(s+1)} + \frac{7/3}{(s+4)}$$

$$\Rightarrow h(t) = -\frac{1}{3}e^{-t}u(t) + \frac{7}{3}e^{-4t}u(t).$$

$$v_o(0^+) = h(0^+) = -\frac{1}{3} + \frac{7}{3} = 2.$$

Problem # 5



$$f(t) = e^{-2t}\cos(\omega_c t)\,u(t)$$

$$h(t) = 2A\cos(\omega_c t)\,u(t)$$

To determine $g(t)$ we can use the Laplace transform or convolution method. Here, we will use convolution.

To simplify convolation, we consider the low-pass versions of $f(t)$ and $h(t)$ first, convolve them and then band-pass version through $\cos(\omega_c t)$ multiplication.

Hence, $g_L(t) = f_L(t) * h_L(t)$ where

$f_L(t) = e^{-2t}u(t)$ and $h_L(t) = 2Au(t)$.

$\Rightarrow g_L(t) = A(1-e^{-2t})u(t)$,

and $g(t) = A(1-e^{-2t})\cos(\omega_c t)u(t)$.

Problem #6

$f(t) = e^{-2t}u(t)$ has $F(s) = \frac{1}{s+2}$, $\sigma > -2$.

$g(t) = 2e^{-2t}\cos(\omega_c t)u(t)$.

$G(\omega) = \frac{1}{2\pi}\mathcal{F}\{2e^{-2t}u(t)\} * \mathcal{F}\{\cos(\omega_c t)\}$.

$\mathcal{F}\{2e^{-2t}u(t)\} = 2F(s)\Big|_{s=j\omega} = \frac{2}{j\omega+2}$

$$\Rightarrow G(\omega) = \frac{1}{\pi}\frac{1}{j\omega+2} * \pi\left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right]$$

$$= \frac{1}{j(\omega-\omega_c)+2} + \frac{1}{j(\omega+\omega_c)+2}$$

$$G(s) = G(\omega)\Big|_{j\omega=s} = \frac{1}{s-j\omega_c+2} + \frac{1}{s+j\omega_c+2}$$

$$= \frac{s+j\omega_c+2+s-j\omega_c+2}{(s-j\omega_c+2)(s+j\omega_c+2)} = \frac{2(s+2)}{s^2+4s+4+\omega_c^2}$$

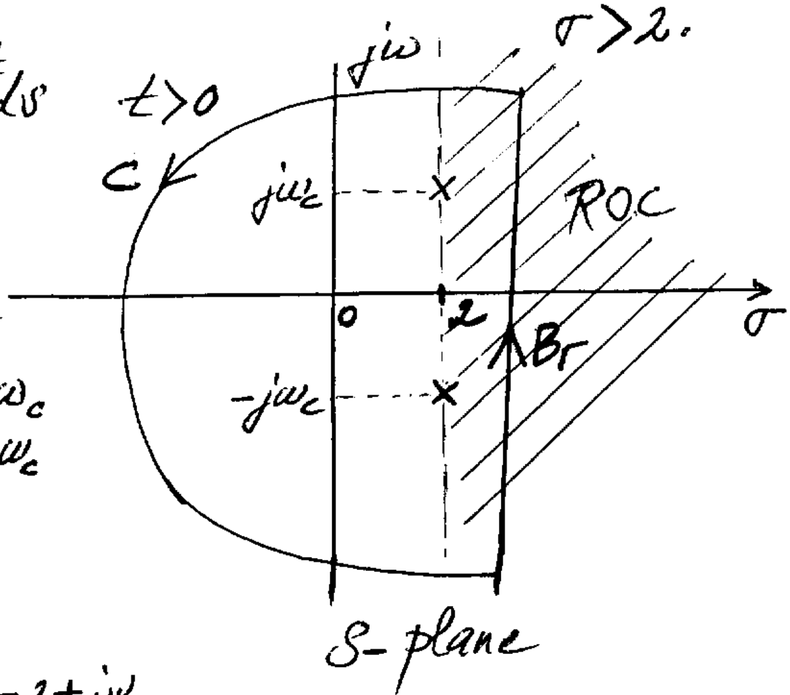
$2(s+2)$

Problem # 7

$$F(s) = \frac{s-2}{(s-2)^2+\omega_c^2} = \frac{s-2}{(s-2-j\omega_c)(s-2+j\omega_c)};$$

For $t>0$,

$$f(t) = \frac{1}{2\pi j}\int_{Br+C} F(s)e^{st}\,ds \quad t>0$$



$= \sum$ residues at poles $s=2+j\omega_c$, $s=2-j\omega_c$

$$= \left.\frac{(s-2)e^{st}}{(s-2+j\omega_c)}\right|_{s=2+j\omega_c} + \left.\frac{(s-2)e^{st}}{(s-2-j\omega_c)}\right|_{s=2-j\omega_c}$$

$$= \frac{j\omega_c e^{(2+j\omega_c)t}}{2j\omega_c} + \frac{-j\omega_c e^{(2-j\omega_c)t}}{-2j\omega_c}$$

$$= \frac{1}{2}e^{2t}e^{j\omega_c t} + \frac{1}{2}e^{2t}e^{-j\omega_c t}$$

$$= e^{2t}\left(\frac{e^{j\omega_c t}+e^{-j\omega_c t}}{2}\right) = e^{2t}\cos(\omega_c t)$$

For $t<0$, $f(t)=0$.

$$\Rightarrow f(t) = e^{2t}\cos(\omega_c t)u(t).$$

Problem # 8

$$h(t) = 2e^{-t}u(t) + e^{2t}u(-t)$$

$$H(s) = 2\int_0^\infty e^{-t}e^{-st}dt + \int_{-\infty}^0 e^{2t}e^{-st}dt$$

$$= 2\int_0^\infty e^{-(s+1)t}dt + \int_{-\infty}^0 e^{-(s-2)t}dt$$

$$= \frac{-2}{s+1}e^{-(s+1)t}\Big|_0^\infty - \frac{1}{s-2}e^{-(s-2)t}\Big|_{-\infty}^0$$

$$= \frac{2}{s+1} - \frac{1}{s-2} = \frac{(s-5)}{(s+1)(s-2)}$$

with $\sigma+1>0$ and $\sigma-2<0$

$\Rightarrow$ ROC is such that $-1<\sigma<2$

The region of convergence of $H(s)$ contains the imaginary axis; i.e., $\sigma=0$. Hence, the system is stable.

Problem # 9

$$\frac{d^2g(t)}{dt^2} + 2\frac{dg(t)}{dt} + g(t) = 2f(t)$$

$$s^2G(s) + 2sG(s) + G(s) = 2F(s)$$

$$\Rightarrow G(s)\left[s^2+2s+1\right] = 2F(s)$$

$$\Rightarrow H(s) = \frac{G(s)}{F(s)} = \frac{2}{s^2+2s+1} = \frac{2}{(s+1)^2}$$

For $f(t) = u(t) \Rightarrow G(s) = H(s)F(s) = \dfrac{2}{s(s+1)^2}$

Problem #10

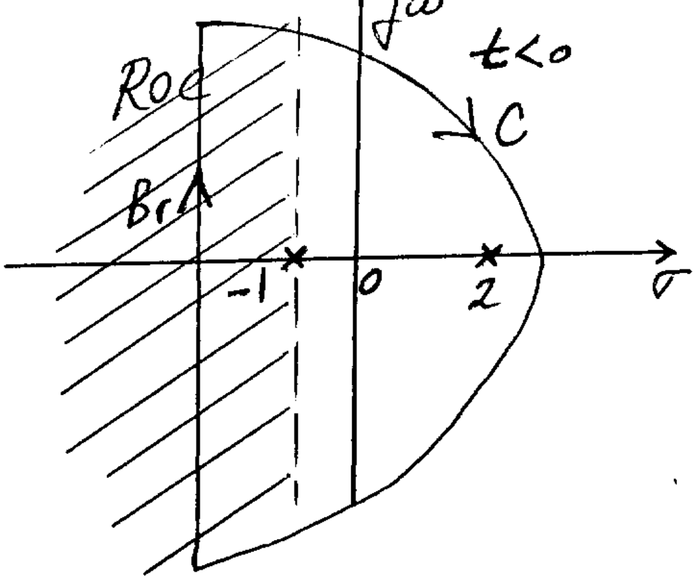
$$H(s) = \frac{1}{(s+1)(s-2)}$$

$H(s)$ is the transfer function of an LTI system that is assumed unstable and non-causal.

Hence, the ROC of $H(s)$ is a vertical strip in the s-plane that is free of poles, does not contain the $j\omega$ axis and it is not a right half s-plane.

$$h(t) = \frac{1}{2\pi j}\int_{Br+C} H(s)e^{st}\,dt$$

$= -\sum$ residues of $H(s)e^{st}$ at poles $s=-1$ and $s=2$.



$$= -\frac{e^{st}}{(s-2)}\bigg|_{s=-1} - \frac{e^{st}}{(s+1)}\bigg|_{s=2}$$

$$= \left[\frac{1}{3}e^{-t} - \frac{1}{3}e^{2t}\right]u(-t).$$

$h(t) = 0$ for $t > 0$.